1. A rectangle, sides length x and y, has perimeter 16 cm and area 15 cm^2.

   (a) Show that $x + y = 8$, and write down a second equation linking x and y.

   (b) Solve to find the lengths x and y.

   In (a), the second equation is "area equals 15". Then, for (b), we now have a pair of simultaneous equations. On these, use substitution to produce a quadratic equation in one variable, and solve by factorising.

   (a) The formula for the area of a rectangle tells us that $xy = 15$.

   (b) We rearrange this to $y = \dfrac{15}{x}$ and substitute:

$$\begin{aligned} x + \frac{15}{x} &= 8 \\ \implies x^2 - 8x + 15 &= 0 \\ \implies (x-3)(x-5) &= 0 \\ \implies x = 3, 5, \quad y &= 5, 3. \end{aligned}$$

   Hence, the lengths are 3 cm and 5 cm.

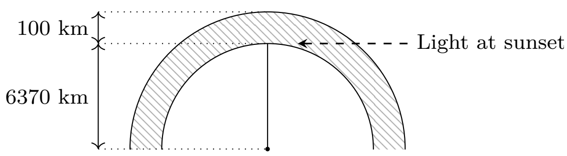
2. Solve the equation $\dfrac{2x-1}{2x+1} = 3$.

   Multiply both sides by $2x + 1$, then gather the x's together on one side.

   We first multiply up by the denominator, and then gather like terms:

$$\begin{aligned} \frac{2x-1}{2x+1} &= 3 \\ \implies 2x - 1 &= 3(2x+1) \\ \implies 2x - 1 &= 6x + 3 \\ \implies -4 &= 4x \\ \implies x &= -1. \end{aligned}$$

3. In this question, model Earth as a sphere of radius 6370 km, and the atmosphere as extending to a constant height of 100 km above the surface.



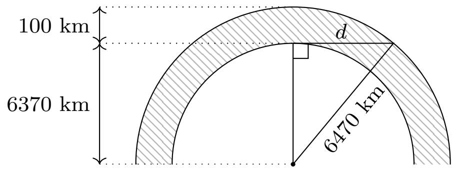
   (a) Explain how you know that, at sunset, light arrives at right angles to the Earth's radius.

   (b) Show that, at sunset, sunlight travels through around 11 times more atmosphere on its way to us than it does when the sun is overhead.

   Form a right-angled triangle by drawing a radius from the centre of the Earth to the point at which the light at sunset enters the atmosphere. Then use Pythagoras's theorem.

   (a) At the moment of sunset, light arrives from the horizon, i.e. tangentially to the ground. And the tangent to a sphere such as Earth, exactly like the tangent to a circle, is perpendicular to the radius.

   (b) The radius of the atmosphere is approximately $6370 + 100 = 6470$ km, which, using part (a), forms the hypotenuse of a right-angled triangle:



   Hence, Pythagoras' theorem gives the distance travelled by the light d as

$$\begin{aligned} d &= \sqrt{6470^2 - 6370^2} \\ &= 1133 \text{ km} \\ &\approx 11 \times 100 \text{ km}. \end{aligned}$$

4. Simplify the following, leaving your answers in fully factorised form:

   (a) $a + b \times c - d + d + c \times b - a$,

   (b) $ab(c-d) - bc(a-d) + ad(b-c)$.

   Consider BIDMAS (brackets, indices, division, multiplication, addition, subtraction) carefully.

   (a) $a + b \times c - d + d + c \times b - a \equiv 2bc$.

   (b)
$$\begin{aligned} &ab(c-d) - bc(a-d) + ad(b-c) \\ &\equiv abc - abd - abc + bcd + abd - acd \\ &\equiv bcd - acd \\ &\equiv cd(b-a). \end{aligned}$$

5. A circle is given by $x^2 + 4x + y^2 - 6y = 87$.

   (a) Express this as $(x-a)^2 + (y-b)^2 = r^2$.

   (b) Hence, write down the centre and the radius of the circle.

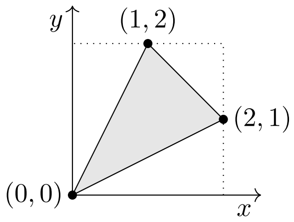
   In (a), complete the square on $x^2 + 4x$ and $y^2 - 6y$, and then gather the constant terms. In (b), Pythagoras tells us that $(x-a)^2 + (y-b)^2 = r^2$ has centre (a, b) and radius r.

(a) Completing the square for both x and y,

$$\begin{aligned} &x^2+4x+y^2-6y=87 \\ \implies &(x+2)^2-4+(y-3)^2-9=87 \\ \implies &(x+2)^2+(y-3)^2=100. \end{aligned}$$

(b) The centre is $(-2,3)$; the radius is 10.

6. A triangle has vertices as depicted.



(a) Show that the perimeter is $\sqrt{2}+2\sqrt{5}$.

(b) By considering the square with dotted edges, or otherwise, find the area of the triangle.

In (a), use Pythagoras's theorem to find each side length. Then, in (b), subtract the area of the three unshaded triangles from the area of the square.

(a) By Pythagoras, the edge lengths are given by $\sqrt{1^2+1^2}$, $\sqrt{1^2+2^2}$ and $\sqrt{1^2+2^2}$. These sum to $\sqrt{2}+2\sqrt{5}$.

(b) The dotted square has area 4, and the three unshaded right-angled triangles within it have areas $1, 1, \frac{1}{2}$. Therefore, the area of the shaded triangle is $4-1-1-\frac{1}{2}=\frac{3}{2}$. This is often the most efficient way of finding the area of a triangle (or other polygon) whose vertices are given as coordinates.

7. Find the equation of the straight line through the points $(-2,-7)$ and $(1,2)$, in the form $y=mx+c$.

First, find the gradient m with a "rise over run" triangle, and then substitute in a point on the line, e.g. $(1,2)$, to find the $+c$. Alternatively, use the formula $y-y_0=m(x-x_0)$ and simplify.

The gradient is $\frac{\Delta y}{\Delta x}=\frac{9}{3}=3$. So the equation of the line is $y=3x+c$. Substitute in $(1,2)$, and we get $2=3+c$, which gives $c=-1$. So the equation of the line is $y=3x-1$.

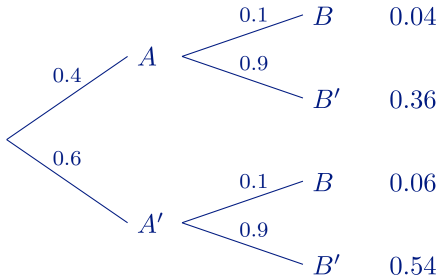
8. Independent events A and B have $P(A)=0.4$ and $P(B)=0.1$. Using a tree diagram, or otherwise, find the following probabilities:

(a) $P(A\cap B)$,

(b) $P(A\cup B)$.

If A and B are independent, then the probability of B doesn't change with information about A. So $P(B \mid A)$ and $P(B \mid A')$, i.e. the probability of B conditioned on whether A has happened or not, are both equal to $P(B)$.

Since A and B are independent, the second set of branches are identical.



Summing the relevant branches gives

(a) $P(A\cap B)=0.04$,

(b) $P(A\cup B)=0.04+0.36+0.06=0.46$.

9. Simplify the following expression, where $x>a,b$:

$$\sqrt{\frac{x-a}{x-b}}\times\sqrt{\frac{a-x}{b-x}}.$$

Notice that the numerators are negatives of each other, and likewise the denominators. Cancelling the minus signs gives squares, which can then be square rooted.

Since $(x-a)\equiv-(a-x)$, we get

$$\begin{aligned} &\sqrt{\frac{x-a}{x-b}}\times\sqrt{\frac{a-x}{b-x}} \\ \equiv\ &\sqrt{\frac{x-a}{x-b}}\times\sqrt{\frac{-(x-a)}{-(x-b)}} \\ \equiv\ &\sqrt{\frac{x-a}{x-b}}\times\sqrt{\frac{x-a}{x-b}} \\ \equiv\ &\frac{x-a}{x-b}. \end{aligned}$$

The condition $x>a,b$ ensures that $x-a$ and $x-b$ are both positive. This ensures no division by zero, and also that the square roots are well defined (positive inputs).

10. This problem was written down on a clay tablet in Babylon, 4000 years ago: "At a non-compound interest rate of $\frac{1}{60}$/month, find the doubling time."

"Non-compound" means the addition of the same amount every month. So, this problem requires no consideration of scale factors, other than the fact that doubling requires an increase of 1 on top of the original quantity 1.

The "doubling time" means an increase of 1 on top of the original 1. At a rate of $\frac{1}{60}$ per month, this will take 60 months, or 5 years.

11. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $x^2y^2 = 0 \implies x^2 = 0$,
(b) $x^2 = 0 \implies x^2y^2 = 0$.

The squares are not relevant to the logic, and can be ignored. Consider the possibility that only one of x or y is zero.

The first statement is false. A counterexample is $x = 1, y = 0$, for which $x^2y^2 = 0$ but $x^2 \neq 0$.

12. Integers a and b are such that their difference is 100 and their product is 5964. By setting up and solving a pair of simultaneous equations, find all possible values of a and b.

The equations are $a - b = 100$, $ab = 5964$. Using substitution, these generate a quadratic equation.

The equations are $a - b = 100$ and $ab = 5964$. We rearrange the former to give $a = 100 + b$, and substitute, yielding

$$\begin{aligned} &(100+b)b = 5964 \\ \implies &b^2 + 100b - 5964 = 0 \\ \implies &b = \frac{-100 \pm \sqrt{100^2 + 4 \cdot 5964}}{2} \\ &\quad = 42, -142. \end{aligned}$$

So the integers are $(142, 42)$ or $(-42, -142)$.

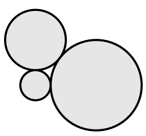
13. By listing them explicitly, show that there are six possible rearrangements of AABB.

List them alphabetically, starting with AABB and ending up with BBAA.

In alphabetical order, the solutions are

AABB	BAAB
ABAB	BABA
ABBA	BBAA

14. Three circles of radius 1, 2 and 3, with centres at P, Q, R, are all tangent to each other, as shown.



Show that PQR is a right-angled triangle.

The implication in Pythagoras's theorem goes both ways. So, it can be used to find lengths, or, in this case, to prove that a triangle is right-angled.

Summing the radii, the distances (PQ, QR, RP), are $(3, 4, 5)$. This is a Pythagorean triple with $3^2 + 4^2 = 5^2$, so the triangle is right-angled.

15. A quadratic equation is given as $f(x) = 0$, where $f(x) = x^2 + kx + 8$, for some constant $k \in \mathbb{Z}$. You are given that $(x - 4)$ is a factor of $f(x)$.

(a) Use the factor theorem to find k.
(b) Hence, solve the quadratic equation.

In (a), the factor theorem states that $x = a$ is a root of a polynomial $f(x)$ exactly when $(x - a)$ is a factor. So, substitute $x = 4$.

(a) By the factor theorem, since $(x-4)$ is a factor, $x = 4$ is a root. Substituting $x = 4$, then, gives $24 + 4k = 0$, so $k = -6$.
(b) Solving with this k, $x^2 - 6x + 8 = 0$
$$\begin{aligned} &\implies (x-2)(x-4) = 0 \\ &\implies x = 2, 4. \end{aligned}$$

16. Solve $\dfrac{(x+3)(x-1)}{(x+2)(x+6)} = 0$.

A fraction is zero only if its numerator is zero.

A fraction can only equal zero if its numerator equals zero. In this case, this is at $x = 1, -3$. These would only fail to satisfy the equation if they also made the denominator zero. This is not the case. So, the solution is $x = 1, -3$.

17. State, with a reason, whether $y = x + 2$ intersects the following lines:

(a) $y = x - 2$,
(b) $y = 2 - x$.

If in doubt, sketch the lines.

(a) No. The lines are parallel and non-identical.
(b) Yes. The lines are not parallel.

18. Simplify $p^{\log_p q} \times q^{\log_q p}$.

The definition of a logarithm is as follows: $\log_p q$ translates as "the index you need to raise p by to get q". So, raise p by this, and what do you get?

Each factor simplifies by definition. The factor $p^{\log_p q}$ is literally "raise p by whatever power you need to raise p by in order to get q". You get q. Therefore $p^{\log_p q} \times q^{\log_q p} = q \times p = pq$.

19. A parabola passes through the points $(-6, 0)$, $(-4, 0)$ and $(0, 48)$.

(a) Explain why the parabola must have the form $y = a(x + 4)(x + 6)$.

(b) Determine the value of a.

In (a), use the factor theorem: when $x = p$ is a root, $(x - p)$ must be a factor. In (b), substitute the point $(0, 48)$ into the equation of the curve.

(a) By the factor theorem, since the y value of the curve is zero at -4 and -6, it has factors $(x+4)$ and $(x+6)$. Since it is a parabola, it can have no more algebraic factors than those two. It can, however, have a constant factor a.

(b) Substituting $x = 0, y = 48$, when get $24a = 48$, so $a = 2$.

20. In a statistical study, the elements of the sample have units ms^{-1}. Give the units of

(a) the mean,

(b) the inter-quartile range,

(c) the variance,

(d) the standard deviation.

In each case, consider whether there are squares in the formulae, i.e. whether the summary statistic is in the same units as the original data.

The mean, IQR and standard deviation all have the original units ms^{-1}. The variance is a squared measure, however, so has units m^2s^{-2}.

21. Write down the equation of the locus of points equidistant from $y = 4x + 6$ and $y = 4x + 10$.

These are a pair of parallel lines, so the points equidistant from them form a third parallel line halfway between the two.

The given lines are parallel, so the locus must be parallel to and halfway between them. Hence, it has equation $y = 4x + 8$.

22. The variable p is inversely proportional to the square of x and proportional to the square root of y. When $x = 1$, $y = 5$. Find y in terms of x.

Translate the proportionality statements into equations with constants of proportionality, and then eliminate p.

We know that $p \propto \frac{1}{x^2}$ and $p \propto \sqrt{y}$. Hence, $p = \frac{a}{x^2}$ and $p = b\sqrt{y}$. Equating the two instances of p and then squaring gives

$$\frac{a}{x^2} = b\sqrt{y} \implies \frac{a^2}{x^4} = b^2 y.$$

We can combine the constants of proportionality into a new constant k, to write

$$y = \frac{k}{x^4}.$$

Substituting $x = 1, y = 5$ yields $y = \dfrac{5}{x^4}$.

23. Simplify the following, where $n \in \mathbb{N}$, giving your answers in standard form:

(a) $5 \times 10^n + 6 \times 10^n$,

(b) $5 \times 10^n \times 6 \times 10^n$.

In (a), use the fact that $11 = 1.1 \times 10$. In (b), combine the powers of ten using index laws.

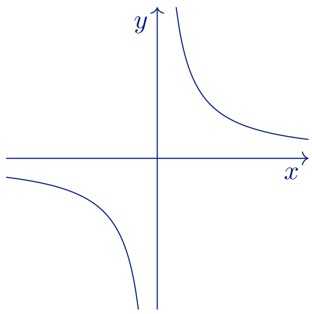
(a)
$$\begin{aligned} & 5 \times 10^n + 6 \times 10^n \\ = \; & 11 \times 10^n \\ = \; & 1.1 \times 10^{n+1} \end{aligned}$$

(b)
$$\begin{aligned} & 5 \times 10^n \times 6 \times 10^n \\ = \; & 30 \times 10^{2n} \\ = \; & 3 \times 10^{2n+1} \end{aligned}$$

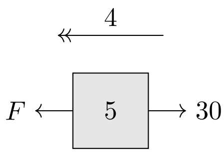
24. Sketch the graph $xy = 1$.

Rearrange to $y = \frac{1}{x}$ if that form is more familiar. The graph is one of inverse proportion, in both the positive and negative quadrants.

The graph is the standard hyperbola:



25. An object is modelled in the force diagram below, with mass given in kilograms, forces in Newtons, and acceleration in ms^{-2}.



Show that $F = 50$ Newtons.

Use Newton II: $F = ma$, remembering that the F in $F = ma$ is "resultant force in the direction of the acceleration."

The resultant force in the direction of the acceleration is $F = 30$, so the equation of motion is $F - 30 = 5 \times 4$. Hence, $F = 20 + 30 = 50$ N.

26. Two dice are rolled together. By drawing a 6×6 table of the outcomes (a possibility space), or otherwise, find the probability that the sum of the two scores is 8.

The possibility space is a 6×6 grid. Count up the successful outcomes to give a fraction over 36.

The possibility space is:

	1	2	3	4	5	6
1						
2						✓
3					✓	
4				✓		
5			✓			
6		✓				

Therefore, since all 36 outcomes are equally likely, the probability is $\frac{5}{36}$.

27. A linear function is $f(x) = ax + b$, for $a, b \in \mathbb{R}$. Prove that $f(k - x) + f(k + x)$ is independent of x for any $k \in \mathbb{R}$.

Substitute the inputs $(k - x)$ and $(k + x)$ into $f(x) = ax + b$, and simplify. To show that the result is "independent of x" means showing that it contains no reference to x.

Simplifying the given function:

$$\begin{aligned} & f(k-x) + f(k+x) \\ \equiv\ & a(k-x) + b + a(k+x) + b \\ \equiv\ & ak - ax + ak + ax + 2b \\ \equiv\ & 2ak + 2b. \end{aligned}$$

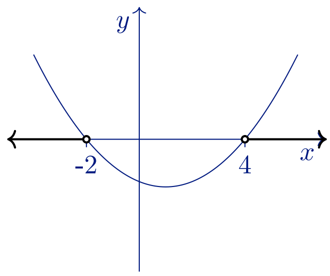
This is independent of x.

28. For the inequality $x^2 - 2x - 8 > 0$, the boundary equation is $x^2 - 2x - 8 = 0$. By solving it and sketching $y = x^2 - 2x - 8$, determine the set of values of x which satisfy the inequality.

You are looking for the set of x values for which the parabola $y = x^2 - 2x - 8$ is above the x axis.

Solving $x^2 - 2x - 8 = 0$ gives $x = -2, 4$.

So $y = x^2 - 2x - 8$ is a positive parabola passing through $(-2, 0)$ and $(4, 0)$:



The inequality is satisfied by x values in the region outside of and not including the roots. In interval set notation, this can be expressed as the union $(-\infty, -2) \cup (4, \infty)$.

29. The first-principles differentiation of the parabola $y = x^2$ is set up in the following limit:

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

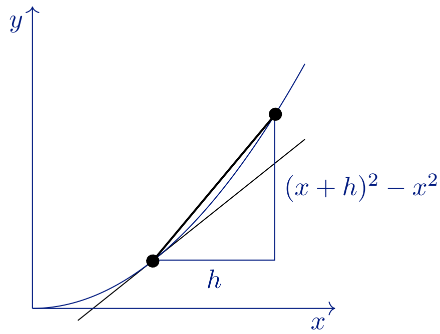
(a) With reference to a sketch, explain why this limit gives the gradient of the curve at the point (x, x^2).

(b) Expand and simplify the numerator.

(c) Hence, show that $\dfrac{dy}{dx} = 2x$.

The fraction inside the limit is the gradient of a chord of $y = x^2$. The limit asks "Towards which value does the gradient of the chord tend?"

(a) The fraction of which we are taking the limit is the gradient of a chord of the parabolic curve $y = x^2$, drawn from (x, x^2) to $(x+h, (x+h)^2)$:



As $h \to 0$, the mobile right-hand point moves closer and closer to the fixed left-hand point, and the gradient of the chord gets closer and closer to the gradient of the tangent, which is the definition of "the gradient of the curve" at (x, x^2).

(b) Simplifying,

$$\begin{aligned} & (x+h)^2 - x^2 \\ \equiv\ & x^2 + 2xh + h^2 - x^2 \\ \equiv\ & 2xh + h^2. \end{aligned}$$

(c) $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{(x+h)^2 - x^2}{h}$

$= \lim_{h\to 0} \dfrac{2xh + h^2}{h}$

$= \lim_{h\to 0} 2x + h$

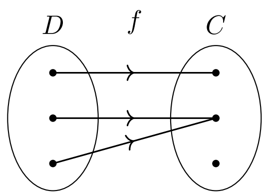
$= 2x$ as required.

30. Complete the square to write $4x^2+24x+54$ in the form $a(x+b)^2+c$, where $a, b, c \in \mathbb{Z}$.

It can be easiest to begin with $4(x^2+6x)+54$. Complete the square on x^2+6x, then multiply by 4 afterwards.

$$\begin{aligned}\text{Completing the square,}\quad & 4x^2+24x+54 \\ \equiv\ & 4(x^2+6x)+54 \\ \equiv\ & 4(x+3)^2 - 4\cdot 9 + 54 \\ \equiv\ & 4(x+3)^2 + 18.\end{aligned}$$

31. A mapping f maps a domain D to a codomain C, according to the following scheme:



State, with a reason, whether $f : D \mapsto C$ is

(a) a well-defined function,

(b) invertible.

Functions can be many-to-one, and do not have to map to all elements of the codomain. Invertible functions, on the other hand, must be one-to-one. In (b), ask whether you could determine a unique input in D, given a specific output in C.

(a) The mapping f is a well-defined function, because every input in D maps to precisely one output in C.

(b) f is not invertible, for two reasons, each of which would be sufficient individually to stop f being one-to-one, hence invertible.

i. Element #3 in C has no input mapping to it, so an inverse would have nothing to map that element back to,

ii. Element #2 in C has two domain elements mapping to it (f is many-to-one). Hence, an inverse would be one-to-many, i.e. not a well-defined function.

32. Solve the inequality $3-2x > 4$, giving your answer in set notation.

Since this is a linear inequality, it can be solved algebraically, without any need to sketch graphs. Rearrange to make x the subject, remembering to flip the inequality sign when dividing by -2.

Since the inequality is linear, we can solve by direct algebraic manipulation, with a reversal of the direction of the inequality when dividing by -2:

$$\begin{aligned} & 3 - 2x > 4 \\ \implies\ & -2x > 1 \\ \implies\ & x < -\tfrac{1}{2} \\ \implies\ & x \in (-\infty, -\tfrac{1}{2}).\end{aligned}$$

33. Three integers are as follows: twice the first is the second, thrice the second is the third, and all three add to eighteen. Find the integers.

Call the first number x, and set up and solve a single equation in x.

The numbers are $x, 2x, 6x$, so $x + 2x + 6x = 18$. Solving for x gives $x = 2$. Therefore the numbers are $2, 4, 12$.

34. (a) Calculate $5x^2 - 12x + 4\,\big|_{x=2}$.

(b) Hence, show that $(x-2)$ is a linear factor of the expression $5x^2 - 12x + 4$, giving the name of the theorem you use.

In (a), the bar notation means "evaluated at", so $f(x)\,\big|_{x=5}$ means plug $x = 5$ into $f(x)$. In (b), you need the factor theorem.

(a) Substituting $x = 2$ into the expression gives

$$5x^2 - 12x + 4\,\big|_{x=2} = 0.$$

(b) Since $x = 2$ is a root of $5x^2 - 12x + 4$, we know that $(x-2)$ is a factor, by the factor theorem.

35. Variables a, b, c are linked by $a = b^2$ and $b = c + 1$. Express c in terms of a.

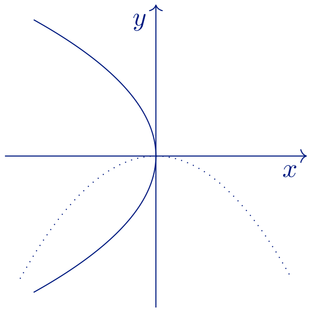
Substitute for b.

Substituting for b gives $a = (c+1)^2$. Rearranging this gives $c = \pm\sqrt{a} - 1$.

36. Sketch $x = -y^2$.

First, sketch the graph $y = -x^2$, then think about the effect of swapping x and y.

The graph is a reflection of $y = -x^2$ (dotted) in the line $y = x$:



37. A car of mass 800 kg sets off from rest along a straight horizontal road. The driving force is 1000 N, and there is a resistance force of 400 N.

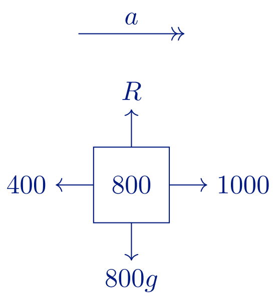
    (a) Draw a force diagram.

    (b) Determine the acceleration.

    (c) Find the displacement after 20 seconds.

    In (b), use $F = ma$. For part (c), having found the acceleration, the relevant *suvat* equation is $s = ut + \frac{1}{2}at^2$.

    (a) The force diagram is



    (b) Horizontal $F = ma$ gives $1000 - 400 = 800a$, so $a = 0.75$ ms^{-2}.

    (c) Substituting this acceleration into the formula $s = ut + \frac{1}{2}at^2$, the displacement is given by $s = 0 \cdot 20 + \frac{1}{2} \cdot 0.75 \cdot 20^2 = 150$ m.

38. By listing the outcomes, or otherwise, find the probability that tossing three coins yields exactly two heads.

    There are eight outcomes in the possibility space.

    The possibility space is

HHH	THH
HHT	THT
HTH	TTH
HTT	TTT

    Since the eight outcomes are equally likely, the probability of two heads is $\frac{3}{8}$.

    Alternatively, using the language of binomial random variables, $X \sim B(3, \frac{1}{2})$, and therefore

$$P(X = 2) = {}^3C_2 \times \tfrac{1}{2}^1 \times \tfrac{1}{2}^2 = \tfrac{3}{8}.$$

39. The graph $y = x^2 - x - 6$ crosses the x axis twice.

    (a) Determine the x coordinates of these roots.

    (b) Find $\dfrac{dy}{dx}$.

    (c) Show that, at its roots, the gradient of the parabola is ± 5.

    (d) Explain why the gradients of a quadratic graph $y = f(x)$ at its x axis intercepts must always be of the form $\pm k$.

    In (a), set the quadratic to zero and solve for x. Substitute these to get ± 5 in (c). In (d), consider the symmetry of a parabola.

    (a) Setting $y = 0$, $x^2 - x - 6 = 0$
$$\implies (x-3)(x+2) = 0$$
$$\implies x = -2, 3.$$

    (b) $y = x^2 - x - 6$
$$\implies \frac{dy}{dx} = 2x - 1.$$

    (c) $2x - 1\,\big|_{x=-2,3}$
$$= 2 \times (-2) - 1, \quad 2 \times 3 - 1$$
$$= -5, 5.$$

    (d) Every parabola has a line of symmetry; if there are two roots, this line must lie exactly halfway between them. Hence, the tangents at the roots are also reflections, and thus their gradients must be $\pm k$.

40. Rearrange the following to make x the subject:

$$y = \frac{x}{x+a}.$$

    Multiply by the denominator $x + a$, gather the x's onto one side, take out a common factor of x, then divide to get x on its own.

    Rearranging, $y = \dfrac{x}{x+a}$
$$\implies (x+a)y = x$$
$$\implies xy + ay = x$$
$$\implies ay = x - xy$$
$$\implies ay = x(1-y)$$
$$\implies x = \frac{ay}{1-y}, \qquad y \neq 1.$$

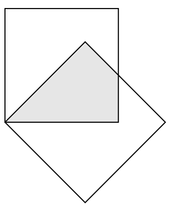
41. A student solves $9x^2 - 1 > 0$, and gets $\frac{1}{3} < x < -\frac{1}{3}$. Explain the notational error and correct it.

Each inequality comparing x to a number is correct, but the combination of the two inequalities isn't.

Each individual inequality comparing x with a number is correct, i.e. the original inequality is satisfied when $\frac{1}{3} < x$ or when $x < -\frac{1}{3}$. But these can't happen together. So the grouping of the inequalities into one simultaneous inequality is incorrect notation. By proxy, it claims that $\frac{1}{3} < -\frac{1}{3}$, which isn't true.

Such an inequality, consisting of two *alternative* regions, should be written as two alternatives, i.e. "$x < -\frac{1}{3}$ or $x > \frac{1}{3}$". Alternatively, set notation is even clearer: $x \in (-\infty, -\frac{1}{3}) \cup (\frac{1}{3}, \infty)$.

42. Two squares of side length 1 are drawn with one vertex in common, and with an edge of one along the diagonal of the other, as below.



Show that the area of the shaded kite is $\sqrt{2} - 1$.

Using the hypotenuse in a $(1, 1, \sqrt{2})$ Pythagorean triangle, so that the side lengths of the kite are 1 and $\sqrt{2} - 1$. Split the kite into two symmetrical right-angled triangles to find its area.

The diagonals of the squares have length $\sqrt{2}$, by Pythagoras. These are split into 1 (the side length of the other square) and what remains, which then has length $\sqrt{2}-1$. So, by symmetry, the short sides of the shaded kite also have length $\sqrt{2} - 1$. The area of the kite is then that of two triangles:

$$A = 2 \times \frac{1}{2} \times 1 \times (\sqrt{2} - 1) = \sqrt{2} - 1. \quad \text{Q.E.D.}$$

43. Solve, to 3sf, the equation $3^{x+1} = 7$.

Take logarithms of both sides, base 3.

$$\begin{aligned} \text{Taking logs,} \quad & 3^{x+1} = 7 \\ \implies \quad & x + 1 = \log_3 7 \\ \implies \quad & x = \log_3 7 - 1 \\ & \quad = 0.771 \quad \text{(3sf)}. \end{aligned}$$

44. The equation $ax^2 + bx + y^2 + 4y = c$ is a circle which passes through the origin and is centred on a point satisfying $y = x$. Find the constants a, b, c.

Use the fact that

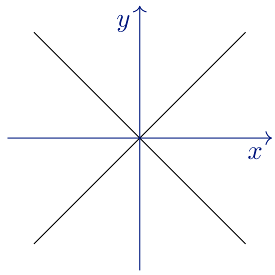
- the curve is a circle to find a,
- the centre is on $y = x$ to find b,
- the circle goes through the origin to find c.

In the equation of a circle, the x^2 and y^2 terms must have the same coefficient, so $a = 1$. Then, since the centre is on $y = x$, $b = 4$. (You could complete the square to verify this.) And, because $(0, 0)$ must satisfy the equation, we get $c = 0$.

45. Sketch the graph $y^2 = x^2$.

Square root both sides, remembering a $\pm$.

Square rooting, we get $y = \pm x$ which has two parts, $y = x$ and $y = -x$. These are both straight lines, so the graph is a cross:



46. Two students are calculating the displacement, over a given period of time, of a particle whose velocity is $v = \frac{1}{2}t + 3$, using areas under a velocity-time graph. The first student calculates the area using the formula for the area of a trapezium; the second sets up the following (correct) integral:

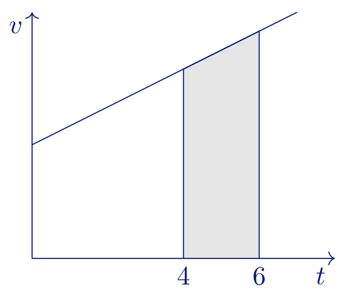
$$\Delta x = \int_{t=4}^{t=6} \tfrac{1}{2}t + 3 \, dt = 11.$$

(a) Sketch a velocity-time graph, and shade the region whose area is Δx.

(b) Use the first student's method to verify that $\Delta x = 11$.

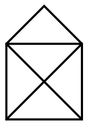
Δx, which is change in x, is another way of expressing the displacement s.

(a) The velocity-time graph is:



(b) $$\begin{aligned}\Delta x &= \tfrac{1}{2}(a+b)h \\ &= \tfrac{1}{2}\left[(\tfrac{4}{2}+3)+(\tfrac{6}{2}+3)\right]\cdot 2 \\ &= 11.\end{aligned}$$

47. A simple game consists of drawing the following shape, whose base has length 1 and which consists of horizontal, vertical and 45° lines, without taking the pen off the paper.



Find the total length of the line drawn.

The roof diagonals have length $\frac{\sqrt{2}}{2}$.

By Pythagoras, the square's diagonals have length $\sqrt{2}$, so the line consists of four unit segments, and six segments of length $\frac{\sqrt{2}}{2}$. Hence, the total length is $4+6\frac{\sqrt{2}}{2}=4+3\sqrt{2}$.

48. Solve the equation $(x^2+1)^7(3x-2)=0$.

You don't need to multiply this out, and shouldn't try! It is already in the form you want. Use the fact that, if two factors multiply to give zero, then one of them must be zero.

This is already factorised, so we can solve directly:

$$\begin{aligned}&(x^2+1)^7(3x-2)=0 \\ \Longrightarrow\ &(x^2+1)^7=0 \text{ or } (3x-2)=0 \\ \Longrightarrow\ &x=\tfrac{2}{3}, \text{ since } x^2+1=0 \text{ has no real roots.}\end{aligned}$$

49. An object of mass m kg has exactly two forces acting on it, whose magnitudes are $2m$ and $3m$ N. Give the minimum and maximum possible values for the magnitude of the acceleration.

Consider the *directions* of the two forces, relative to each other, which would give maximum or minimum acceleration.

Maximum or minimum acceleration occurs if the forces are parallel or antiparallel, which gives the resultant force as either $5m$ or m Newtons. Hence, $a_{\max}=5$ ms^{-2}, and $a_{\min}=1$ ms^{-2}.

50. Give, with a reason, the formula for the exterior angle θ, defined in radians, of a regular n-gon.

Consider one full journey around the perimeter, divided up equally between n vertices.

One circuit of the perimeter of the n-gon, via n vertices, involves turning 2π radians. Hence, each exterior angle (the angle turned through at each vertex) is $\frac{2\pi}{n}$ radians.

51. By completing the square twice, once for x and once for y, show that $x^2+6x+y^2-3y=0$ is a circle, and determine its centre and exact radius.

Put the equation in the form $(x-a)^2+(y-b)^2=r^2$.

Completing the square twice gives

$$\begin{aligned}&x^2+6x+y^2-3y=0 \\ \Longrightarrow\ &(x+3)^2-9+(y-\tfrac{3}{2})^2-\tfrac{9}{4}=0 \\ \Longrightarrow\ &(x+3)^2+(y-\tfrac{3}{2})^2=\tfrac{45}{4}.\end{aligned}$$

So the equation is a circle, with centre $(-3,\frac{3}{2})$ and radius $\frac{3\sqrt{5}}{2}$.

52. Find the following sums:

(a) $\displaystyle\sum_{r=1}^{10}1$, (b) $\displaystyle\sum_{r=0}^{n}1$, (c) $\displaystyle\sum_{r=1}^{n}a$.

To simplify, it's often helpful to write such sums out longhand, e.g. (a) is $\underbrace{1+1+\ldots+1}_{10 \text{ times}}$.

(a) $\underbrace{1+1+\ldots+1}_{10 \text{ times}}=10$.

(b) $\underbrace{1+1+\ldots+1}_{n+1 \text{ times}}=n+1$.

(c) $\underbrace{a+a+\ldots+a}_{n \text{ times}}=an$.

53. By first finding the roots of the equation using an analytic method such as the quadratic formula, explain why the decimal search method would be likely to fail to find a root of $42x^2-71x+30=0$.

Having solved to find the roots, sketch the graph of $y=42x^2-71x+30=0$ to visualise the problem.

The roots are

$$\begin{aligned}x&=\frac{71\pm\sqrt{71^2-4\cdot 42\cdot 30}}{2\cdot 42} \\ &=\frac{70}{84} \text{ or } \frac{72}{84} \\ &=0.833\ldots \text{ or } 0.857\ldots\end{aligned}$$

These roots both lie between 0.8 and 0.9, so, unless a decimal search starts at at least 2dp accuracy, it will fail to spot a sign change: the value of the quadratic will be positive at both 0.8 and 0.9.

54. A bird has velocity $5\mathbf{i}-12\mathbf{j}$ ms^{-1}. Find its speed.

Since $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors, we can use Pythagoras. The velocity triangle is right-angled, with perpendicular lengths 5 and 12.

Since $\mathbf{i}$ and $\mathbf{j}$ are the standard perpendicular unit vectors, $v = \sqrt{5^2 + 12^2} = 13 \text{ ms}^{-1}$.

55. An inequality is given by $-x^2 - 6x \geq 0$.

(a) Solve the boundary equation $-x^2 - 6x = 0$.

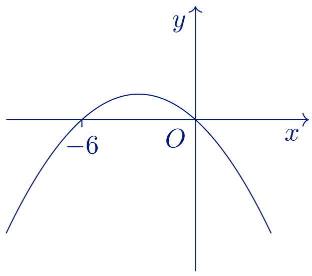
(b) Sketch the graph $y = -x^2 - 6x$.

(c) Hence, solve the inequality, giving your answer in interval set notation.

In (c), consider the x values for which the curve $y = -x^2 - 6x$ is above the x axis.

(a) $-x(x+6) = 0 \implies x = 0, -6$,

(b) Graph of $y = -x^2 - 6x$:



(c) We require x values such that the graph is above or on the x axis, which are the values between and including the roots. We can write this in interval set notation as

$$-x(x+6) \geq 0 \implies x \in [-6, 0].$$

56. Explain why the discriminant $\Delta = b^2 - 4ac$ gives the number of real roots of a quadratic equation.

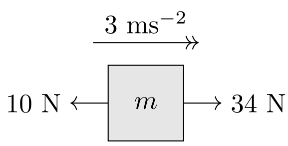
Consider the role of Δ in the quadratic formula.

The quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $\Delta > 0$, then the formula gives two distinct roots. If $\Delta = 0$, the square root is zero, so the formula gives one root. If $\Delta < 0$, there is no real value of $\sqrt{\Delta}$, so the equation has no real roots.

57. An object is modelled as below:



Solve to find the mass m.

Use $F = ma$, and solve for m algebraically.

$$\begin{aligned} F = ma \text{ gives } \; 34 - 10 &= 3m \\ \implies m &= \tfrac{24}{3} = 8 \text{ kg.} \end{aligned}$$

58. Write down the number of ways of rearranging:

(a) ABC,

(b) AAB.

For small numbers of items, you can list them explicitly if you can't remember the formula.

Quoting standard results:

(a) $3! = 6$,

(b) ${}^3C_1 = 3$.

59. If $f(x) = 4x(1 - x^2)$, find $f'(x)$.

Multiply out before differentiating.

We multiply out and differentiate term by term:

$$\begin{aligned} & f(x) = 4x(1 - x^2) = 4x - 4x^3 \\ \implies & f'(x) = 4 - 12x^2. \end{aligned}$$

60. Express a right angle in radians.

There are 2π radians in a circle, or, equivalently, π radians in a semicircle or along a straight line.

There are π radians in a semicircle. A right-angle is half of a semicircle, giving $\frac{1}{2} \times \pi = \frac{\pi}{2}$ radians.

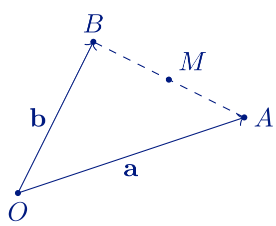
61. Points A, B have position vectors $\mathbf{a}$, $\mathbf{b}$, relative to an origin O. M is the midpoint of AB. Give the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$:

(a) $\overrightarrow{AB}$,

(b) $\overrightarrow{OM}$,

Draw a sketch with A and B at arbitrary locations. (This is often helpful in vector problems.)

Sketching, with A and B in arbitrary locations:



(a) The journey $\overrightarrow{AB}$ is the same as: back from A to O then forwards from O to B. Therefore,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ &= \mathbf{b} - \mathbf{a}.\end{aligned}$$

(b) The position vector of a midpoint is the mean of the position vectors: $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.

62. The equation $(x-p)(x-q)(x-\frac{1}{2}q) = 0$ has roots at $x = 2, 3, 4$. Find the values of p and q.

Use the factor theorem: if $x = \alpha$ is a root of a polynomial, then $(x-\alpha)$ is a factor.

By the factor theorem, the roots of the equation are $p, q, \frac{1}{2}q$. These must be $2, 3, 4$ in some order. So we require $\frac{1}{2}q = 2, p = 3, q = 4$.

63. In any given year, the probability of my birthday falling on a Tuesday is $\frac{1}{7}$. Explain, using formal language, what is wrong with this statement: "The probability of my birthday falling on a Tuesday in two consecutive years is $\frac{1}{7} \times \frac{1}{7} = \frac{1}{49}$."

Consider independence: whether knowing that a birthday falls on a Tuesday this year affects the probability that it does next year too.

The day of the week on which a birthday falls is not independent in any two given years. In fact, a birthday can't fall on the same day in consecutive years because a year is 52 weeks and 1 day (or 2 days in a leap year). So, the relevant probability is actually zero.

64. Solve the following quadratic in x^2:

$$x^4 - x^2 - 72 = 0.$$

Factorise as $(x^2 + ...)(x^2 - ...) = 0$.

This biquadratic (quadratic in x^2) equation can be factorised as follows:

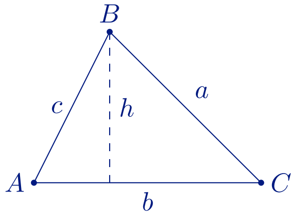
$$\begin{aligned}& x^4 - x^2 - 72 = 0\\ \Longrightarrow\ & (x^2-9)(x^2+8) = 0\\ \Longrightarrow\ & x^2 = 9 \text{ or } -8.\end{aligned}$$

The latter has no real roots, so $x = \pm 3$.

65. By dropping a suitable perpendicular, prove the area formula $A_\triangle = \frac{1}{2}ab\sin C$ for triangles.

Sketch a triangle ABC, and drop a perpendicular from the point B to the side AC. Find the height in terms of length a and angle C.

The sketch is



Using the right-hand triangle, the perpendicular height h is given by $h = a\sin C$. Hence, we get

$$\begin{aligned}A_\triangle &= \tfrac{1}{2}\text{base} \times \text{height}\\ &= \tfrac{1}{2}b \times a\sin C\\ &= \tfrac{1}{2}ab\sin C, \text{ as required.}\end{aligned}$$

66. You are given that invertible functions f and g have $f(1) = 2$, $f(2) = 3$, $g(2) = 3$ and $g(3) = 4$. Write down the values of

(a) $gf(2)$,

(b) $f^{-1}g^{-1}(4)$,

(c) $g^{-1}f^2(1)$.

In (a), gf means "apply f then g"; in (b), f^{-1} is the inverse of f, mapping back from output to input; in (c), f^2 is the application of f twice in succession. N.B. this latter notation doesn't carry over to e.g. $\sin^2 x$, which means $(\sin x)^2$; the squared trig functions are so common they have their own notation.

(a) $gf(2) = g(3) = 4$,

(b) $f^{-1}g^{-1}(4) = f^{-1}(3) = 2$,

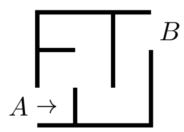
(c) $g^{-1}f^2(1) = g^{-1}f(2) = g^{-1}(3) = 2$.

67. Evaluate $\log_a a^2 + \log_b b^3$.

Use the fact that $\log_x y$ stands for "the power you need to raise x by to get y".

By definition of a logarithm, $\log_a a^2 + \log_b b^3$ reads: "the power you need to raise a by to get a^2, added to the power you need to raise b by to get b^3. This is $2 + 3 = 5$.

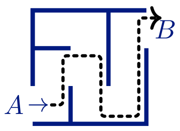
68. A robot is navigating the maze below, starting at point A. When it hits a wall, it turns 90° left or right, choosing the direction at random.



Find the probability that it navigates to B along the shortest route available to it.

Work out the number of turns involved in the shortest possible route. To take the shortest route, the correct direction must be chosen at each, meaning that the probability is simply $\frac{1}{2}^n$.

The shortest route through the maze has 6 turns:



The probability of choosing the right direction at every turn is given by $\frac{1}{2}^6 = \frac{1}{64}$.

69. Solve the equation $|2x - 1| = 9$.

Rewrite as $\pm(2x - 1) = 9$.

Solving, $|2x - 1| = 9$

$$\begin{aligned} &\implies 2x - 1 = 9, -9 \\ &\implies x = 5, -4. \end{aligned}$$

70. Write down the equations of any vertical asymptotes of the following curve, in which a, b, c, d are distinct real numbers:

$$y = \frac{(x+a)(x-b)}{(x+c)(x-d)}.$$

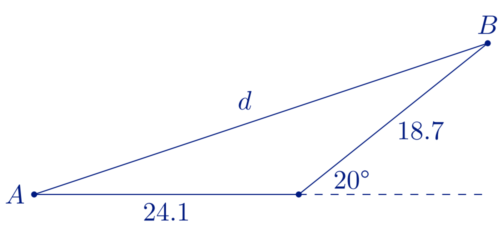
Vertical asymptotes are lines of the form $x = k$, at which the value of a function or expression tends to either plus or minus infinity. Such a vertical asymptote appears where there is division by zero in the algebra.

Division by zero occurs at $x = -c$ and $x = d$, so these are the vertical asymptotes of the curve.

71. A ship sets out from port A. Having travelled 24.1 nautical miles, it corrects course by 20°, arriving at port B after a further 18.7 nautical miles. Find the extra distance it travelled above the minimum.

Draw a clear diagram and use the cosine rule.

Approximately to scale, we have



Using the cosine rule:

$$\begin{aligned} d^2 &= 24.1^2 + 18.7^2 - 2 \cdot 24.1 \cdot 18.7 \cos 160° \\ &= 1777.4825... \end{aligned}$$

Hence, $d = 42.16$. This gives the extra distance as $24.1 + 18.7 - 42.16 = 0.63979... = 0.640$ nautical miles, to 3sf.

72. "The line $x = 1$ is tangent to the curve $y = x^2 - x$." True or false?

Sketch the graphs.

False, because $x = 1$ is parallel to the y axis. No polynomial $y = f(x)$ is ever parallel to the y axis.

73. The quadratic function $f(x) = ax^2 + 4x + 15$, where a is a constant, has range $[7, \infty)$.

(a) Explain how you know that a must be positive.

(b) By differentiating, or otherwise, show that the minimum value of $f(x)$ is at $x = -\frac{2}{a}$.

(c) Hence, show that $a = \frac{1}{2}$.

In (a), consider whether $y = 7$ is a minimum or a maximum. In (b), to find the minimum point, set $f'(x) = 0$ or complete the square. In (c), solve the resulting equation.

(a) Since the range (set of attained outputs) has a minimum but not a maximum, the parabola must be positive.

(b) For the vertex (stationary point) $f'(x) = 0$, which gives $2ax + 4 = 0$, so $x = -\frac{2}{a}$.

(c) Substituting $x = -\frac{2}{a}$ into $f(x)$ and equating to 7 gives

$$\begin{aligned} &a\left(-\tfrac{2}{a}\right)^2 + 4\left(-\tfrac{2}{a}\right) + 15 = 7 \\ \implies &\tfrac{4}{a} - \tfrac{8}{a} = -8 \\ \implies &\tfrac{4}{a} = 8 \\ \implies &a = \tfrac{1}{2}. \end{aligned}$$

74. In one case, write down the derivative with respect to x; in the other, explain why no derivative exists.

(a) $x = 4$,

(b) $y = 4$.

Sketch the lines, and consider the fact that a derivative with respect to x is a rate of change *as x changes*. In only one of these equation is x allowed to change.

(a) $x = 4$ can have no derivative with respect to x, as x cannot vary. On an (x, y) graph, the equation defines a vertical line. The gradient of such as line involves division by zero, and cannot be defined.

(b) The equation $y = 4$ describes a horizontal line, so its derivative is well defined. The gradient is everywhere zero: $\frac{dy}{dx} = 0$.

75. Solve $(3x - 7)(x^2 + 1)(x^2 - 4) = 0$.

Use the factor theorem, and consider the three equations it generates one by one.

By the factor theorem, either $(3x - 7) = 0$, or $(x^2+1) = 0$, or $(x^2-4) = 0$. These have, in order, a root at $x = \frac{7}{3}$, no real roots, roots at $x = \pm 2$. So, the solution set is $x \in \{-2, 2, \frac{7}{3}\}$.

76. Evaluate $\left[x^2 + x + 1\right]_0^4$.

$\left[F(x)\right]_a^b \equiv F(b) - F(a)$.

The notation $\left[F(x)\right]_a^b$ means $F(b) - F(a)$. So,

$$\begin{aligned} & \left[x^2 + x + 1\right]_0^4 \\ = \; & (16 + 4 + 1) - (0 + 0 + 1) \\ = \; & 20. \end{aligned}$$

77. A set of data $\{x_i\}$ is coded according to the formula $y_i = ax_i + b$. Write down the mean $\bar{y}$ and variance s_y^2 of the new set $\{y_i\}$, in terms of $a, b, \bar{x}$ and s_x^2.

Remember that the mean is a measure of central tendency, while the variance is a squared measure of spread.

The mean, as a measure of central tendency, is affected by both a and b: $\bar{y} = a\bar{x} + b$. The variance, however, as a measure of spread, is unaffected by the shift $+b$. It is a squared measure, so $s_y^2 = a^2 s_x^2$.

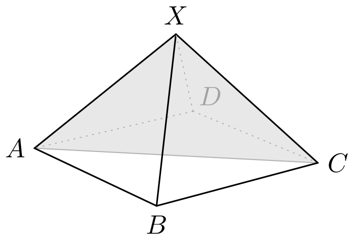
78. By starting with the right-hand side, prove that

$$\frac{1}{pq} \equiv \frac{1}{p(p+q)} + \frac{1}{q(p+q)}.$$

Put the fractions of the right-hand side over a common denominator.

The RHS is
$$\begin{aligned} & \frac{1}{p(p+q)} + \frac{1}{q(p+q)} \\ \equiv \; & \frac{q}{pq(p+q)} + \frac{p}{pq(p+q)} \\ \equiv \; & \frac{p+q}{pq(p+q)} \\ \equiv \; & \frac{1}{pq} \qquad \text{Q.E.D.} \end{aligned}$$

79. The square-based pyramid shown below is formed of eight edges of unit length.



Show that triangle AXC is right-angled.

Use Pythagoras.

Pythagoras on $\triangle ABC$ tells us that $AC = \sqrt{2}$. Hence, triangle AXC has sides of length $(1, 1, \sqrt{2})$. By Pythagoras, $\triangle AXC$ is right-angled.

80. Rationalise the denominator of $\dfrac{3}{\sqrt{7} - 2}$.

Multiply top and bottom of the fraction by $\sqrt{7}+2$.

To produce a difference of two squares, we multiply numerator and denominator by the conjugate of the denominator:

$$\begin{aligned} & \frac{3}{\sqrt{7}-2} \\ = \; & \frac{3(\sqrt{7}+2)}{(\sqrt{7}-2)(\sqrt{7}+2)} \\ = \; & \frac{3\sqrt{7}+6}{7-4} \\ = \; & \sqrt{7} + 2. \end{aligned}$$

81. Disprove the following statement: "If the decimal expansion of a number does not terminate, then the number is irrational."

Look for a counterexample: a rational number with a non-terminating decimal expansion.

Any recurring decimal, such as $\frac{1}{9} = 0.1111...$, is a counterexample.

82. Verify that the function $f(x) = \sin \frac{a}{b} x$ and its derivative $f'(x) = \frac{a}{b} \cos \frac{a}{b} x$ satisfy the equation

$$(af(x))^2 + (bf'(x))^2 = a^2.$$

Substitute into the LHS, and use a Pythagorean identity.

Substituting into the LHS of the identity, we get

$$\begin{aligned}
&(af(x))^2 + (bf'(x))^2 \\
\equiv\ &(a \sin \tfrac{a}{b} x)^2 + (b \tfrac{a}{b} \cos \tfrac{a}{b} x)^2 \\
\equiv\ &a^2 \sin^2 \tfrac{a}{b} x + a^2 \cos^2 \tfrac{a}{b} x \\
\equiv\ &a^2 (\sin^2 \tfrac{a}{b} x + \cos^2 \tfrac{a}{b} x) \\
\equiv\ &a^2, \quad \text{by the Pythagorean identity.}
\end{aligned}$$

83. Find the probability that, when a coin is tossed three times, all three tosses show the same result.

Count successful outcomes and total outcomes, and use the rule, for equally likely outcomes:

$$p = \frac{\text{successful}}{\text{total}}.$$

There are $2^3 = 8$ outcomes, of which HHH and TTT are successful. So the probability is $\frac{2}{8} = \frac{1}{4}$.

84. Variables x and y take the following values:

x	1	3	6
y	0	4	9

Show that the relationship cannot be linear.

Consider the rate of change of y with respect to x.

The rate of change of y with respect to x is $\frac{4-0}{3-1} = 2$ and then $\frac{9-4}{6-3} = \frac{5}{3}$. In a linear relationship, these rates would be the same.

85. Solve the equation $(2^x + 1)(2^x - 8) = 0$.

Use the factor theorem.

By the factor theorem, $2^x = -1, 8$. The former has no real roots, the latter has $x = 3$.

86. Two runners are planning a 1 km race, and are deciding on a time handicap for the faster runner. On average, A runs at 5 ms^{-1}, while B runs at 4 ms^{-1}. Find the time delay for A which would have them finish the race, on average, together.

Find the time taken for each to run the race.

A runs the race in $\frac{1000}{5} = 200$ seconds, while B runs it in $\frac{1000}{4} = 250$ seconds. Hence, if B starts 50 seconds before A, they will finish, on average, together.

87. Simplify $\sqrt{\dfrac{x-p}{x-q}} \div \sqrt{\dfrac{q-x}{p-x}}$.

Remember that $x - p \equiv -(p - x)$.

Simplifying,
$$\begin{aligned}
&\sqrt{\frac{x-p}{x-q}} \div \sqrt{\frac{q-x}{p-x}} \\
\equiv\ &\sqrt{\frac{x-p}{x-q}} \times \sqrt{\frac{p-x}{q-x}} \\
\equiv\ &\sqrt{\frac{x-p}{x-q}} \times \sqrt{\frac{x-p}{x-q}} \\
\equiv\ &\frac{x-p}{x-q}.
\end{aligned}$$

88. Determine which of the points $(3, 3)$ and $(5, 1)$ is closer to the point $(1, 0)$.

Use Pythagoras to compare the squared distances.

The squared distances to $(0, 1)$ are $2^2 + 3^2 = 13$ and $4^2 + 1^2 = 17$. So $(3, 3)$ is closer.

89. Simplify the following expressions:

(a) $\log_a \sqrt[3]{a}$,

(b) $\log_{a^2} a$,

(c) $\log_{a^3} a^2$.

Remember the definition of a logarithm: $\log_a b$ is "what you have to raise a by to get b".

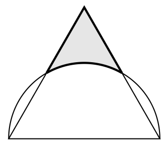
By the definition of a logarithm, $\log_a b$ is "what you have to raise a by to get b". This gives

(a) $\log_a \sqrt[3]{a} = \log_a a^{\frac{1}{3}} = \frac{1}{3}$.

(b) $\log_{a^2} a = \frac{1}{2}$.

(c) $\log_{a^3} a^2 = \frac{2}{3}$.

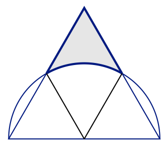
90. An arrowhead design is constructed from a semicircle and an equilateral triangle of side length l. The arrowhead has perimeter P.



Show that $P = l(1 + \frac{1}{3}\pi)$.

Draw in the radii from the centre of the semicircle to the points of intersection.

The radii to the points of intersection produce equilateral triangles of side length $\frac{1}{2}l$:



So, the arc boundary of the shaded arrowhead subtends an angle of 60° at the centre, with radius $\frac{1}{2}l$. So, its length is $\frac{1}{6} \times 2\pi(\frac{1}{2}l) = \frac{1}{3}\pi l$. The other two sides have length $\frac{1}{2}l$, giving $P = l(1 + \frac{1}{3}\pi)$.

91. If $2x + 3y = 5$, find $\dfrac{dy}{dx}$.

$\frac{dy}{dx}$ is the gradient. You don't need calculus.

The straight line is $y = -\frac{2}{3}x + \frac{5}{3}$, with gradient $m = -\frac{2}{3}$. Hence, $\frac{dy}{dx} = -\frac{2}{3}$.

92. The function $g(x) = 6x^2 - x - 35$ is defined over the real numbers.

(a) Solve $g(x) = 0$,

(b) Sketch $y = g(x)$,

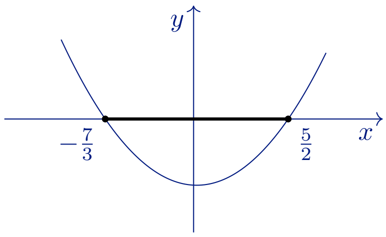
(c) Hence, give the solution set of $g(x) \leq 0$.

Add your answers to (a) to the sketch in (b), and consider the x values for which the graph is below the x axis.

(a) $6x^2 - x - 35 = 0$

$$\implies (3x + 7)(2x - 5) = 0$$
$$\implies x = -\tfrac{7}{3}, \tfrac{5}{2}.$$

(b) Sketch of $y = 6x^2 - x - 35$, with solution set for part (c) marked:



(c) The solution set is between and including the roots, $\{x \in \mathbb{R} : -\frac{7}{3} \leq x \leq \frac{5}{2}\}$, or equivalently, and more succinctly, $[-\frac{7}{3}, \frac{5}{2}]$.

93. Give 2.6 radians to the nearest degree.

There are 2π radians in 360°.

$2.6 \times \frac{360°}{2\pi} = 149°$ (0dp)

94. Give counterexamples to the following statements:

(a) $x \in \mathbb{Z} \implies x \in \mathbb{N}$,

(b) $x \in \mathbb{Q} \implies x \in \mathbb{Z}$,

(c) $x \in \mathbb{R} \implies x \in \mathbb{Q}$.

Find any element of the former set that is not an element of the latter.

(a) Any negative integer, e.g. -1.

(b) Any fraction, e.g. $\frac{1}{2}$.

(c) Any irrational number, e.g. $\sqrt{2}$.

95. A straight line has parametric vector equation

$$\mathbf{r} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} t.$$

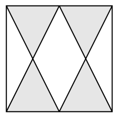
Find the Cartesian equation of the line, giving your answer in the form $ax + by = c$, for $a, b, c \in \mathbb{Z}$

Find two points with $t = 0$ and $t = 1$.

The gradient of the line is $\frac{5}{2}$, since increasing t by 1 adds 2 to x and 5 to y. So the line is $y = \frac{5}{2}x + c$. At $t = 0$, the line is at $x = -3$, $y = 1$, so substitute these values to find c. This gives $1 = \frac{5}{2}(-3) + c$, so $c = \frac{17}{2}$. Hence

$$\begin{aligned} &y = \tfrac{5}{2}x - \tfrac{17}{2} \\ \implies &2y = 5x + 17 \\ \implies &-5x + 2y = 17. \end{aligned}$$

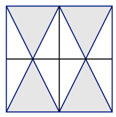
96. A design is drawn on a square, connecting vertices and midpoints of edges as below.



Find the fraction of the total area that is shaded.

No calculation is needed: join the midpoints with lines and count triangles.

Adding two lines:



Counting congruent triangles, we can see that half of the total area is shaded.

97. Two sets of data are to be combined. For $i = 1, 2$, they contain n_i data, with mean $\bar{x}_i$. Determine a formula for the mean of the combined set of data.

Calculate $\sum x$ for each set, then add.

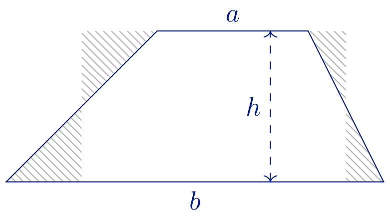
For each set, $\sum x$ is given by $n_i\bar{x}_i$. So the combined average is the sum of these, over the total number of data $n_1 + n_2$. This is

$$\bar{x}_{\text{total}} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}.$$

98. Prove that the area of a trapezium is $\frac{1}{2}(a+b)h$.

Dissect and rearrange a general trapezium to make a rectangle.

A trapezium can be dissected at the midpoints of its non-parallel sides and rearranged as follows:



Since the dissection is at the midpoints, the length of the rectangle formed is the average of a and b. Hence, $A = \frac{1}{2}(a+b)h$.

99. Give counterexamples to the following:

(a) $a < b \implies a^2 < b^2$,

(b) $a < b \implies \dfrac{1}{a} < \dfrac{1}{b}$.

In each case, find a pair of integers that satisfies the first inequality but doesn't satisfy the second.

(a) $-3 < -2$, but $9 \not< 4$.

(b) $\frac{1}{3} < \frac{1}{2}$, but $3 \not< 2$.

100. Two forces of magnitudes 6 and 8 Newtons act on an object of mass 10 kg. Find the acceleration if those forces act

(a) in the same direction,

(b) perpendicular to each other.

Use Pythagoras for (b).

(a) $6 + 8 = 10a$ gives $a = 1.4$ ms^{-2}.

(b) Since the forces are perpendicular, their magnitudes add according to Pythagoras. $6^2 + 8^2 = 10^2$, so $a = 1$ ms^{-2}.

101. Show that $0.\dot{4}\dot{2} = \dfrac{14}{33}$.

Let $x = 0.\dot{4}\dot{2}$, then find $100x - x$.

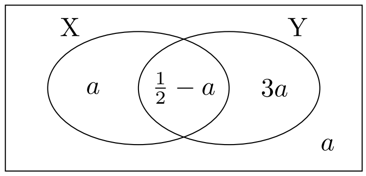
Let $x = 0.\dot{4}\dot{2}$. Then $99x = 100x - x = 14$. Hence, $x = \frac{42}{99} = \frac{14}{33}$.

102. The parabola $y = ax^2 + bx + c$ and the line $y = k$ intersect twice. At the points of intersection, the parabola has gradients m_1 and m_2. Prove that $m_1 + m_2 = 0$.

A parabola $y = ax^2 + bx + c$ has a line of symmetry.

A parabola $y = ax^2 + bx + c$ has a line of symmetry parallel to the y axis. Two points with the same y value $y = k$ must be reflections of each other in this line. The tangents at these points are reflections of each other, so their gradients are negatives. Hence, $m_1 + m_2 = 0$.

103. The probabilities of events X and Y are given in terms of $a \in \mathbb{R}$ as follows:



Find a.

Use the fact that probabilities sum to 1.

Since the probabilities must sum to 1, we know that $a + (\frac{1}{2} - a) + 3a + a = 1$. This gives $4a = \frac{1}{2}$, so $a = \frac{1}{8}$.

104. Determine whether the point $(2, 3)$ lies inside, on, or outside the circle $x^2 + y^2 = 10$.

Test the squared distance to the origin.

The squared distance from the point $(2, 3)$ to the origin is 13. For points on the circle, that value is 10. So the point lies outside the circle.

105. Evaluate $p^{\log_p 3} \times q^{2\log_q 2}$.

Use a log rule on the exponent of the second factor, before simplifying using the original definition of a logarithm.

The law of logarithms $n \log_a b \equiv \log_a(b^n)$ allows us to write

$$\begin{aligned} & p^{\log_p 3} \times q^{2\log_q 2} \\ =\ & p^{\log_p 3} \times q^{\log_q 2^2} \\ =\ & 3 \times 4 \\ =\ & 12. \end{aligned}$$

106. Show that $\mathbf{r} = -\frac{7}{25}\mathbf{i} + \frac{24}{25}\mathbf{j}$ is a unit vector.

Use Pythagoras: a unit vector has length 1.

Since $(7, 24, 25)$ is a Pythagorean triple, this vector has unit length.

107. A projectile is thrown from the ground at 12.5 m/s horizontally and 19.6 m/s vertically.

(a) Write down the two key assumptions of the projectile model, re acceleration and projectile size.

(b) Using $s = ut + \frac{1}{2}at^2$ for the vertical motion, show that the projectile lands after 4 seconds.

(c) Hence, show that the range is 50 metres.

Treat the horizontal and vertical components of the motion separately, but use the same t value in both.

(a) A projectile is modelled as (i) a particle, which (ii) accelerates downwards at g.

(b) At landing, the vertical displacement is zero, so $0 = 19.6t - \frac{1}{2}gt^2$. This is a quadratic, with roots $t = 0$ (take-off) and $t = \frac{2\cdot 19.6}{g} = 4$ (landing).

(c) In four seconds, the horizontal range is $d = 12.5 \times 4 = 50$ metres.

108. The curve $y = x^2$ has a tangent drawn to it at $x = 1$. By differentiating, show that this tangent has equation $y = 2x - 1$.

Use the formula $y = x^n \implies \frac{dy}{dx} = nx^{n-1}$.

The derivative is $\frac{dy}{dx} = 2x$. Evaluating this at $x = 1$ yields a gradient of 2. So the tangent has equation $y = 2x + c$. At $x = 1$, $y = 1$, so $1 = 2 \cdot 1 + c$, and $c = -1$. Hence, the tangent has equation $y = 2x - 1$.

109. If $y = 2^x$, write 8^x in terms of y.

Express 8 as a power of 2.

$8^x = (2^3)^x = (2^x)^3 = y^3$.

110. The small-angle approximation for $\sin\theta$, for θ in radians, is $\sin\theta \approx \theta$. Find the percentage error in this approximation at

(a) $\theta = 0.1$ radians,

(b) $\theta = 0.5$ radians.

Put your calculator in radian mode, and substitute the values.

For angles in radians

(a) $\dfrac{0.1 - \sin 0.1}{\sin 0.1} = 0.167\%$ (3sf),

(b) $\dfrac{0.5 - \sin 0.5}{\sin 0.5} = 4.29\%$ (3sf).

111. The equation $(3x-1)(2x+a)(x+a) = 0$ has roots $x = 3$ and $x = 6$. Write down the value of a.

Use the factor theorem.

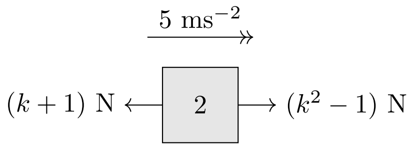
Since $(2x + a)$ and $(x + a)$ are factors, $x = -a$ and $x = -\frac{a}{2}$ are roots. These must be 6 and 3 respectively, so $a = -6$.

112. By setting up a boundary equation $f(x) = 0$ and sketching the graph $y = f(x)$, solve the inequality $x^2 - 5x < 0$, giving your answer in set notation.

Find the set of x values for which the parabola $y = x^2 - 5x$ is below the x axis.

The boundary equation $x^2 - 5x = 0$ has roots at $x = 0$ and $x = 5$. Since $y = x^2 - 5x$ is a positive parabola, we require x values between and not including these roots. So $x^2 - 5x < 0 \implies x \in (0, 5)$.

113. Forces act on an object as depicted:



Solve to find the positive value of k.

Use $F = ma$ to set up a quadratic in k.

$$F = ma \text{ gives } \begin{aligned} & k^2 - 1 - (k+1) = 10 \\ \implies & k^2 - k - 12 = 0 \\ \implies & (k-4)(k+3) = 0 \\ \implies & k = -3, 4. \end{aligned}$$

The positive value of k is 4.

114. The lines $x + y = 1$, $2x - 5y = 9$ and $2x - y = k$ are concurrent. Find k.

"Concurrent" means all intersecting at the same point. So find the intersection of the first two, then find k such that the third line passes through that point.

The intersection of the first two lines is at $(2, -1)$. If the third line is to pass through this point, then $2 \cdot 2 - (-1) = k$, so $k = 5$.

115. Find $\displaystyle\int 6x^2 + 5\,dx$.

Use the rule for integration $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$.

Using the rule for integration $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$, we get

$$\int 6x^2 + 5\,dx = 2x^3 + 5x + c.$$

116. A *perfect number* is a positive integer which is equal to the sum of all its positive divisors except itself. So 6 is perfect, because $6 = 1+2+3$. Show that 28 is a perfect number.

Find the prime factors of 28 first.

$28 = 2^2 \cdot 7$, so the sum of its factors is $1+2+4+7+14 = 28$. Hence, 28 is a perfect number.

117. Giving your answer in radians, find the remaining interior angle in a pentagon with interior angles

$$\frac{4\pi}{10}, \frac{5\pi}{10}, \frac{6\pi}{10}, \frac{7\pi}{10}.$$

In radians, the sum of the interior angles of an n-gon is given by $\pi(n-2)$.

In radians, the sum of the interior angles of an n-gon is given by $\pi(n-2)$. So

$$\frac{4\pi}{10} + \frac{5\pi}{10} + \frac{6\pi}{10} + \frac{7\pi}{10} + \theta = 3\pi.$$

This yields $\theta = \frac{8}{10}\pi$.

118. State, with a reason, whether the following hold:

(a) $y = x^2 \implies y = x^{\frac{1}{2}}$,
(b) $y = x^3 \implies y = x^{\frac{1}{3}}$.

Consider positives and negatives.

(a) No, because positive numbers have two real square roots.

(b) Yes, because each number has exactly one real cube root.

119. By factorising, solve the following quadratic in 3^x:

$$(3^x)^2 - 6 \cdot (3^x) - 27 = 0.$$

Write in the form $(3^x + ...)(3^x - ...) = 0$, and use the factor theorem.

$$\begin{aligned} & (3^x)^2 - 6 \cdot (3^x) - 27 = 0 \\ \implies & (3^x + 3)(3^x - 9) = 0 \\ \implies & 3^x = -3 \text{ or } 9. \end{aligned}$$

The former has no roots, the latter has $x = 2$.

120. Find the intersections of $y = 5x^2 - 20x + 40$ and $y = 2x^2 + 15x + 52$.

Solve simultaneously, by substituting for y.

For intersections, we require

$$5x^2 - 20x + 40 = 2x^2 + 15x + 52,$$

which is $3x^2 - 35x - 12 = 0$. This factorises to $(x-12)(3x+1) = 0$, so $x = 12, -\frac{1}{3}$. Subbing these values back in yields $(12, 520)$ and $(-\frac{1}{3}, \frac{425}{9})$.

121. A woman of mass 55 kg is standing in a lift, which is accelerating upwards at 2 ms^{-2}.

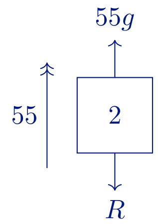
(a) Draw a force diagram for the woman.

(b) Find the magnitude of the force exerted by the lift floor on her feet.

(c) Find the magnitude of the force exerted by the woman on the lift.

For (c), consider Newton III.

(a) Force diagram:



(b) $$\begin{aligned} &\updownarrow: R - 55g = 55 \cdot 2 \\ &\implies R = 55(2+g) = 649 \text{ N}. \end{aligned}$$

(c) This is the Newton III pair of part (b), so also 649 N.

122. Show that the following is the equation of a straight line:

$$x^2 + y^2 = (x-1)^2 + (y-1)^2.$$

Multiply out and simplify.

Multiplying out and simplifying yields $x + y = 1$, which is the equation of a straight line.

123. A rectangle, sides length x and y, has perimeter 22 cm and area 28 cm^2. Find the lengths of its edges.

Set up equations and solve simultaneously.

We know that $2(x+y) = 22$ and $xy = 28$. Solving simultaneously yields side lengths 4 and 7 cm.

124. Prove that $\dfrac{1}{b(ab-1)} + \dfrac{1}{b} \equiv \dfrac{a}{ab-1}$.

Put the LHS over a common denominator.

$$\begin{aligned} & \frac{1}{b(ab-1)} + \frac{1}{b} \\ \equiv & \frac{1}{b(ab-1)} + \frac{ab-1}{b(ab-1)} \\ \equiv & \frac{1+ab-1}{b(ab-1)} \\ \equiv & \frac{ab}{b(ab-1)} \\ \equiv & \frac{a}{ab-1} \qquad \square \end{aligned}$$

125. Show that, if the relationship between x and y is quadratic, of the form $y = ax^2 + bx$, then $\frac{y}{x}$ and x are related linearly.

Variables $\frac{y}{x}$ and x "related linearly" means $\frac{y}{x} = px + q$, for some constants p, q.

Dividing by x gives $\frac{y}{x} = ax + b$, which is a linear relationship between $\frac{y}{x}$ and x.

126. State, with a reason, whether getting four of a kind is likelier if four cards are picked

(a) with replacement,

(b) without replacement.

Consider the changing numerators.

With replacement. Without replacement, the probability is $1 \times \frac{3}{51} \times ...$, but with replacement it is $1 \times \frac{4}{52} \times ...$, and $\frac{4}{52} > \frac{3}{51}$.

127. Show that $(x-2+\sqrt{11})(x-2-\sqrt{11})$ is a quadratic with integer coefficients.

Multiply out and see what happens!

$$\begin{aligned} & (x-2+\sqrt{11})(x-2-\sqrt{11}) \\ = \; & x^2 + (-2+\sqrt{11}-2-\sqrt{11})x + (4-11) \\ = \; & x^2 - 4x - 7. \end{aligned}$$

128. (a) Show that $\frac{1}{2}x^2 + 1 = x$ has no real roots.

(b) Sketch $y = \frac{1}{2}x^2 + 1$ and $y = x$ together.

(c) Hence, prove that the outputs of the function $f(x) = \frac{1}{2}x^2 + 1$ always exceed its inputs.

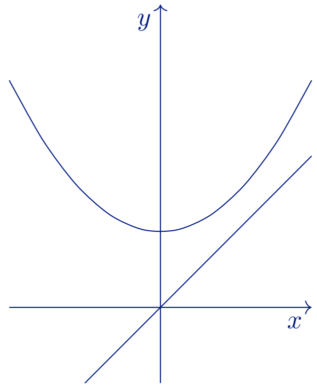
(a) Put in the form $ax^2 + bx + c = 0$ and consider the quadratic discriminant.

(b) ...

(c) Remember that output values are plotted as y values.

(a) Rearranging to $x^2 - 2x + 2 = 0$ gives a discriminant $\Delta = b^2 - 4ac = 4 - 4 \cdot 2 = -4$. Since $\Delta < 0$, the equation has no real roots.

(b) We know that $y = \frac{1}{2}x^2 + 1$ is a positive quadratic, symmetrical about the y axis, crossing at $y = 1$. Furthermore, from part (a), we know that it does not intersect $y = x$. So, the sketch is



(c) Since the parabola is always above the line $y = x$, its y values are always greater than its x values. So the outputs of the function are always greater than its inputs.

129. You are given that the straight lines $ax + y = 1$ and $11 - 2x + 3y = 0$ intersect at the point $(1, b)$. Find the constants a and b.

Substitute $(1, b)$ into both lines, and solve simultaneously.

Since $(1, b)$ is on both lines, we know that

$$\begin{aligned} & a + b = 1, \\ & 11 - 2 + 3b = 0. \end{aligned}$$

The second equation gives us $b = -3$, then the first gives us $a = 4$.

130. If $3y - 1 = 2x^{\frac{3}{2}}$, show that $\frac{dy}{dx} = \sqrt{x}$.

Differentiate both sides, or rearrange to the form $y = ...$ then differentiate.

Differentiating both sides, we get

$$\begin{aligned} & 3y - 1 = 2x^{\frac{3}{2}} \\ \Longrightarrow \; & 3\frac{dy}{dx} = 2 \cdot \frac{3}{2} x^{\frac{1}{2}} \\ \Longrightarrow \; & \frac{dy}{dx} = x^{\frac{1}{2}} \\ \Longrightarrow \; & \frac{dy}{dx} = \sqrt{x}. \end{aligned}$$

131. The quadratic $x^2 + 5x + k$ has $(x+2)$ as a factor. Find the value of k.

Use the factor theorem.

By the factor theorem, $x = -2$ is a root. So $(-2)^2 + 5(-2) + k = 0$. Hence $k = 6$.

132. A circle C_1 is given by $x^2 + x + y^2 + y = 0$.

(a) Find the centre and radius of C_1.

(b) Show that $y = -x$ is tangent to C_1.

(c) Give the equation of C_2, the circle produced when C_1 is reflected in $y = -x$.

(d) Sketch both circles and the tangent line on the same set of axes.

(a) Complete the square for x and y.

(b) Use a symmetry argument.

(c) Consider the centre and radius of the new circle.

(d) ...

(a) Completing the square gives

$$(x + \tfrac{1}{2})^2 + (y + \tfrac{1}{2})^2 = \frac{1}{2}.$$

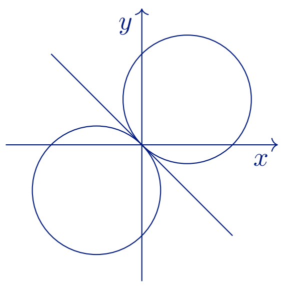
The centre is $(-\frac{1}{2}, -\frac{1}{2})$ and the radius is $\frac{1}{\sqrt{2}}$.

(b) Since the circle goes through O and is symmetrical in x and y, $y = -x$ must be a tangent.

(c) The new circle has centre $(\frac{1}{2}, \frac{1}{2})$ and the same radius, so its equation is

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{2}.$$

(d) Sketch:



133. Prove that the product of two odd numbers is odd.

Write the numbers as $2a+1$ and $2b+1$, for $a, b \in \mathbb{Z}$.

Two odd numbers can be written as $2a + 1$ and $2b + 1$, for $a, b \in \mathbb{Z}$. Their product is

$$\begin{aligned} & (2a + 1)(2b + 1) \\ \equiv\ & 4ab + 2a + 2b + 1 \\ \equiv\ & 2(2ab + a + b) + 1. \end{aligned}$$

Since $(2ab + a + b)$ is an integer, this is one more than an even number, and must be odd. Q.E.D.

134. Write down the roots of $\dfrac{(x+1)(x-1)}{(x+3)(x-3)} = 0$.

A fraction can only be zero when...

A fraction can only be zero when its numerator is zero, so the roots are $x = \pm 1$.

135. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $x^2 = 1 \implies x^5 = 1$,

(b) $x^5 = 1 \implies x^2 = 1$.

Consider $x = -1$.

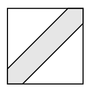
$x = -1$ is a counterexample to (a), as $(-1)^2 = 1$, but $(-1)^5 \neq 1$.

136. By first setting up and solving a boundary equation, determine the integer value of x for which $3x^2 + 6x + 1 < 0$.

Use the quadratic formula, and give the roots as decimals.

Solving $3x^2 + 6x + 1 = 0$, by the quadratic formula, gives $x = -0.184, -1.816$. We require a positive quadratic to be less than zero, so $x \in (-1.816, -0.184)$. The only integer in this interval is $x = -1$.

137. A set of tiles has the following pattern on each.



Four such tiles are laid together, in a 2 by 2 grid. Find the probability that

(a) the four stripes form a square,

(b) the four stripes are all parallel.

Count up outcomes and calculate $p = \frac{\text{successful}}{\text{total}}$.

There are $2^4 = 16$ outcomes, from the two orientations of each tile.

(a) Only one outcomes gives a square, so $\frac{1}{16}$.

(b) Two outcomes are all parallel, so $\frac{1}{8}$.

138. Find the gradient of $y = x^5 - 4x^3 + 10x^2$ at the point with x coordinate 2.

Differentiate as substitute $x = 2$.

$y = x^5 - 4x^3 + 10x^2$ Evaluating this at

$$\implies \frac{dy}{dx} = 5x^4 - 12x^2 + 20x.$$

$x = 2$ gives $\frac{dy}{dx} = 72$.

139. Ten books are placed on a shelf. The two heaviest are chosen to go at the ends. Show that there are 80640 ways of arranging the books.

Place the heavy books first, then consider the rest.

There are 2 ways of placing the heavy books. Then the remaining eight books can go in any order in 8 positions, which means 8! possible orders. So the total number of ways is $2 \times 8! = 80640$.

140. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $(x-1)(x-2) = 0$,
- $x = 1$.

Consider the value $x = 2$. The implication is backwards, so $\Longleftarrow$.

141. Find the distance between the points (a, b) and $(a+3, b-4)$, where a, b are real constants.

Use Pythagoras.

By Pythagoras, the distance is $\sqrt{3^2 + 4^2} = 5$.

142. The quadratic equation $x^2 + ax - 6 = 0$ has distinct roots at $x = 1$ and $x = b$. Find a and b.

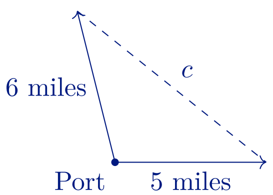
Substitute values and solve simultaneously.

We get two equations by substituting: $a - 5 = 0$ and $b^2 + ab - 6$. The first tells us that $a = 5$, and the second that $b = 1$ or -6. But since $x = 1$ and $x = b$ are distinct, b must be -6.

143. Two submarines leave port simultaneously. One travels on bearing 090° at 15 mph; the other travels on bearing 340° at 18 mph. Determine the distance between the submarines after 20 minutes.

Use the cosine rule.

After twenty minutes, the situation is as follows:



Using the cosine rule,

$$c^2 = 5^2 + 6^2 - 2 \cdot 5 \cdot 6 \cos 110°$$
$$\Longrightarrow c = \pm 9.03.$$

So the distance is 9.03 miles (3sf).

144. A French riddle goes: "A waterlily doubles in size every day. On day 30 it will cover the pond, killing everything else. The gardener decides to act when the pond is half covered. On which day is that?"

...

Day 29.

145. (a) Evaluate $16x^2 - 44x + 6 \big|_{x=\frac{1}{2}}$.

(b) Hence, show that $(2x-1)$ is not a factor of the expression $16x^2 - 44x + 6$, giving the name of the theorem you use.

The factor $(2x-1)$ corresponds to the root $x = \frac{1}{2}$.

(a) $16x^2 - 44x + 6 \big|_{x=\frac{1}{2}} = -12$.

(b) Since $x = \frac{1}{2}$ is not a root of the expression, the factor theorem tells us that $(2x-1)$ cannot be a factor.

146. Find $f'(x)$ for the following functions:

(a) $f(x) = 0$,

(b) $f(x) = 1$,

(c) $f(x) = 1 + x$.

Consider the gradients of the relevant graphs.

(a) $f(x) = 0 \implies f'(x) = 0$,

(b) $f(x) = 1 \implies f'(x) = 0$,

(c) $f(x) = 1 + x \implies f'(x) = 1$.

147. Solve the equation $\dfrac{x+1}{(x-1)^2} = 1$.

Multiply both sides by $(x-1)^2$.

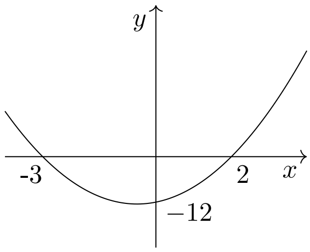
$$\begin{aligned}
&\frac{x+1}{(x-1)^2} = 1 \\
\Longrightarrow\ & x + 1 = (x-1)^2 \\
\Longrightarrow\ & x + 1 = x^2 - 2x + 1 \\
\Longrightarrow\ & 0 = x^2 - 3x \\
\Longrightarrow\ & 0 = x(x-3) \\
\Longrightarrow\ & x = 0, 3
\end{aligned}$$

148. Solve $2 \sin x = \sqrt{3}$, giving all values $x \in [0, 360°)$.

Quote or calculate $\arcsin \frac{\sqrt{3}}{2}$, then generate the other root.

The primary value is $\arcsin \frac{\sqrt{3}}{2} = 60°$. Consulting the unit circle, the other angle (anticlockwise from positive x) which produces the same y value is 120°.

149. Find the equation of the parabola shown below, on which the axes intercepts have been marked, giving your answer in expanded polynomial form.



Use the factor theorem, keeping in mind that a constant factor will also be necessary to ensure the parabola passes through $(0, -12)$

Since the parabola passes through $(-3, 0)$ and $(2, 0)$, it must be of the form $y = a(x + 3)(x - 2)$. Substituting $(0, -12)$ gives $-12 = a \cdot -6$, so $a = 2$. Multiplying out, the equation of the parabola is $y = 2x^2 + 2x - 12$.

150. Two angles of a triangle are 1.42 and 0.46 radians. Find the third, to 2dp.

Angles in a triangle add up to π radians.

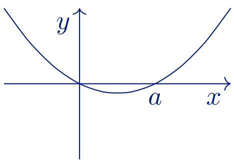
Angles in a triangle add up to π radians. So $1.42 + 0.46 + \theta = \pi$. Hence, $\theta = 1.26$ (2dp).

151. Sketch the following graphs, where $0 < a$,

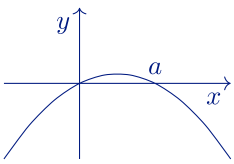
(a) $y = x(x - a)$,
(b) $y = x(a - x)$,
(c) $y = x^2(x - a)$.

Use the factor theorem, and consider the sign of the leading coefficient (coefficient of the highest power).

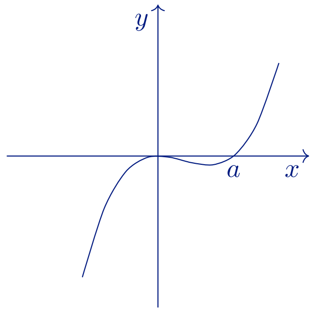
(a) $y = x(x - a)$ is a positive parabola passing through $x = 0$ and $x = a$, so



(b) $y = x(a - x)$ is as above, but reflected in the x axis, since $(a - x)$ is the negative of $(x - a)$.



(c) $y = x^2(x - a)$ is a cubic with a double root (just touches) at $x = 0$ and a single root (crosses) at $x = a$:



152. Solve the equation $3|10 - 3x| - 12 = 0$.

Rearrange to $|10 - 3x| = \ldots$

First, rearrange to $|10 - 3x| = 4$, which gives $10 - 3x = \pm 4$. So $x = 2, \frac{14}{3}$.

153. A rectangular lawn with area 48 m^2 measures 10 metres diagonally. Find the perimeter of the lawn.

Define the dimensions of the lawn as x and y, and set up equations.

Defining x and y to be the dimensions of the lawn, we have $xy = 48$ and $x^2 + y^2 = 100$. Substituting the former into the latter yields

$$\begin{aligned} x^2 + \frac{48^2}{x^2} &= 100 \\ \implies x^4 - 100x^2 + 2304 &= 0 \\ \implies x = \pm 8, \pm 6. \end{aligned}$$

So, the perimeter is $6 + 6 + 8 + 8 = 28$ metres.

154. The numbers a, b, c, d are consecutive terms of an arithmetic progression. Prove that $a + d = b + c$.

Write b, c, d in terms of a and the common difference (usually d, call it k here).

Since this is an AP, we know that $b = a + k$, $c = a + 2k$, $d = a + 3k$, where k is the common difference. This tells us that $a + d = 2a + 3k = b + c$. Q.E.D.

155. Simplify the following sets:

(a) $(-\infty, 1] \cup [-1, \infty)$,
(b) $(-\infty, -1]' \cap [1, \infty)'$.

Sketch the sets on a number line.

(a) The intervals overlap and stretch to both infinities, so their union is $\mathbb{R}$.

(b) We are looking for numbers which are in neither of the named intervals. This is $(-1, 1)$.

156. Find the probability that, if the letters of the word WORD are rearranged at random, they spell the original word.

Consider the number of arrangements of four objects.

There are $4! = 24$ arrangements of any four objects. One of these is successful, so the probability is $\frac{1}{24}$.

157. By completing the square, or otherwise, show that, over the positive reals, the minimum value of the function $f(x) = x(x-5)$ is $-\frac{25}{4}$.

This is the same as asking for the vertex or stationary point of $y = x(x-5)$.

A quadratic function is symmetrical. Since this quadratic function has roots at 0 and 5, its vertex must therefore be at $\frac{5}{2}$. Substituting this in gives

$$x(x-5)\,\big|_{x=\frac{5}{2}} = -\frac{25}{4}.$$

158. By factorising, solve $3x^4 - 14x^3 + 8x^2 = 0$.

Take out a factor of x^2 first.

$$\begin{aligned}
&\implies 3x^4 - 14x^3 + 8x^2 = 0\\
&\implies x^2(3x^2 - 14x + 8) = 0\\
&\implies x^2(3x-2)(x-4) = 0\\
&\implies x = 0, \tfrac{2}{3}, 4.
\end{aligned}$$

159. A chord is drawn to the curve $y = x^2$ at the points where $x = -1$ and $x = 2$. Show that this chord crosses the y axis at $y = 2$.

Find the equation of the chord.

The endpoints of the chord are $(-1, 1)$ and $(2, 4)$. Its equation, therefore, is $y = x + 2$, which crosses the y axis at $(0, 2)$.

160. As x increases, determine which of the following functions reaches an output of 100 first.

$$\begin{aligned}
f(x) &= 4x + 20,\\
g(x) &= 6x.
\end{aligned}$$

Solve $f(x) = 100$ and $g(x) = 100$.

Solving $f(x) = 100$ gives $x = 20$, and solving $g(x) = 100$ gives $x = 16\frac{2}{3}$. So $g(x)$ reaches 100 first as x increases.

161. Simplify the following expressions:

(a) $\sqrt{2} + \sqrt{8} + \sqrt{32}$,

(b) $\dfrac{1}{1-\sqrt{a}} + \dfrac{1}{1+\sqrt{a}}$.

(a) Take out square factors from the surds.

(b) Common denominator.

(a)
$$\begin{aligned}
&\sqrt{2} + \sqrt{8} + \sqrt{32}\\
=\ &\sqrt{2} + 2\sqrt{2} + 4\sqrt{2}\\
=\ &7\sqrt{2}
\end{aligned}$$

(b)
$$\begin{aligned}
&\frac{1}{1-\sqrt{a}} + \frac{1}{1+\sqrt{a}}\\
\equiv\ &\frac{1+\sqrt{a}}{1-a} + \frac{1-\sqrt{a}}{1-a}\\
\equiv\ &\frac{2}{1-a}.
\end{aligned}$$

162. Determine all values $n \in \mathbb{N}$ satisfying ${}^nC_2 = 15$.

Use the definition ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

Using the definition ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, we have

$$\begin{aligned}
&\frac{n!}{2(n-2)!} = 15\\
\implies &\frac{n(n-1)}{2} = 15\\
\implies &n^2 - n - 30 = 0\\
\implies &n = -5, 6.
\end{aligned}$$

Since ${}^nC_r$ is not defined for negative n, the solution is $n = 6$.

163. Find all possible functions $f(x)$, given than

(a) $f'(x) = 0$,

(b) $f'(x) = 3$,

(c) $f'(x) = 2x + 1$.

Integrate each, remembering the $+c$.

(a) $f(x) = \displaystyle\int 0\,dx = c$,

(b) $f(x) = \displaystyle\int 3\,dx = 3x + c$,

(c) $f(x) = \displaystyle\int 2x + 1\,dx = x^2 + x + c$.

164. Show that the line segment defined by $x = 2t$, $y = 2 - 3t$, for $t \in [0, 1]$ is never further than $\sqrt{5}$ from the origin.

Calculate the distances of the endpoints from the origin.

We substitute $t = 0, 1$ to find the endpoints of this line segment, which are $(0, 2)$ and $(2, -1)$. These lie at distances 2 and $\sqrt{5}$ from the origin. Since $\sqrt{5} > 2$, no point on the line segment can lie further from the origin than $\sqrt{5}$.

165. Solve the equation $\dfrac{x}{x+1} - \dfrac{x-1}{x} = \dfrac{1}{2}$.

Get rid of the fractions first.

Multiplying up to get rid of the fractions:

$$\begin{aligned}
&\frac{x}{x+1} - \frac{x-1}{x} = \frac{1}{2} \\
\Longrightarrow\ & 2x^2 - 2(x-1)(x+1) = x(x+1) \\
\Longrightarrow\ & x^2 + x - 2 = 0 \\
\Longrightarrow\ & x = -2, 1.
\end{aligned}$$

166. Three forces, with magnitudes $2, 4, 6$ Newtons, act on an object of mass 10 kg, which is in equilibrium. The smallest force is then removed. Determine the subsequent acceleration of the object.

Consider the change in the resultant force when the smallest force is removed.

The resultant force was zero. With the 2 Newton force removed, it must now have magnitude 2 Newtons. Hence, the acceleration is 0.2 ms^{-2}.

167. Rationalise the denominator of $\dfrac{1}{5 - 2\sqrt{6}}$.

Multiply top and bottom by the conjugate of the bottom.

$$\begin{aligned}
&\frac{1}{5-2\sqrt{6}} \\
=\ & \frac{(5+2\sqrt{6})}{(5-2\sqrt{6})(5+2\sqrt{6})} \\
=\ & \frac{5+2\sqrt{6}}{25-24} \\
=\ & 5 + 2\sqrt{6}.
\end{aligned}$$

168. An equation is given as $4^x + 2^x - 6 = 0$.

(a) Write 4^x in terms of 2^x.

(b) Hence, factorise the equation.

(c) Solve for x.

This is a quadratic in 2^x.

(a) $4^x = (2^x)^2$

(b) $$\begin{aligned} & 4^x + 2^x - 6 = 0 \\ \Longrightarrow\ & (2^x + 4)(2^x - 2) = 0 \end{aligned}$$

(c) $\Longrightarrow x = 1$, since $2^x + 4 = 0$ has no roots.

169. Using a Venn Diagram, or otherwise, determine whether $A' \cup B$ and $A \cap B'$ are mutually exclusive.

They are! Explain why.

The sets are complements of each other, meaning that their union is the universal set, and their intersection is empty. This latter fact is what it means for them to be mutually exclusive.

170. Find, in terms of a, the area of the region enclosed by the lines $y = 2x$, $y = -4x$, and $x = a$.

Write the area as a single integral.

The vertical distance between the two lines is $6x$, so we can express the area as a single integral:

$$\begin{aligned}
A &= \int_0^a 6x\, dx \\
&= \left[3x^2\right]_0^a \\
&= 3a^2.
\end{aligned}$$

171. Consider the function $f(x) = x^3 + 6x^2 - x - 30$.

(a) Show that $(x - 2)$ is a factor.

(b) Express $f(x)$ as the product of three factors.

(c) Hence, solve $f(x) = 0$.

Use the factor theorem (again!).

(a) $f(2) = 2^3 + 6 \cdot 2^2 - 2 - 30 = 0$.
So, by the factor theorem, $(x - 2)$ is a factor,

(b) $f(x) = (x - 2)(x + 3)(x + 5)$,

(c) $x = 2, -3, -5$.

172. *Goldbach's conjecture* is one of the most famous unproved conjectures in mathematics. Goldbach claimed that every even number greater than 2 can be written as the sum of two primes. Verify Goldbach's conjecture up to $n = 20$.

"Verify" means "check that it works". So, find the revelant pairs of primes for $n = 4, 6, ..., 20$.

$4 = 2 + 2$,

$6 = 3 + 3$,

$8 = 3 + 5$,

$10 = 3 + 7$,

$12 = 5 + 7$,

$14 = 7 + 7$,

$16 = 5 + 11$,

$18 = 7 + 11$,

$20 = 7 + 13$.

173. Show that the lines $ax + by = c_1$ and $bx - ay = c_2$ are perpendicular.

Perpendicular lines have gradients which are negative reciprocals.

The gradients of these lines are $m_1 = -\frac{a}{b}$ and $m_2 = \frac{b}{a}$. Multiplying these gives $m_1 m_2 = -1$, so the lines are perpendicular.

174. A linear function has $f(1) = f'(1) = 2$. Find $f(6)$.

A linear function is $f(x) = ax + b$.

A linear function can be expressed as $f(x) = ax+b$. Since $f'(1) = 2$, we know that $a = 2$. Then, substituting $x = 1$ gives $2 = 2 + b$, so $b = 0$. Hence $f(6) = 2 \cdot 6 = 12$.

175. The formula for the sum of the first n terms of an arithmetic sequence is given by

$$S_n = \tfrac{1}{2}n\left(2a + (n-1)d\right).$$

Find the sum of the first n odd integers.

Work out the first term a and the common difference d of the sequence $1, 3, 5, ...,$ and substitute.

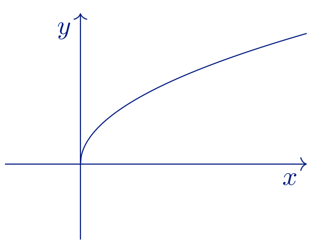
For $1, 3, 5, ...,$ we have $a = 1$ and $d = 2$. So the formula gives

$$\begin{aligned} S_n &= \tfrac{1}{2}n\left(2 + 2(n-1)\right) \\ &= \tfrac{1}{2}n\,(2n) \\ &= n^2. \end{aligned}$$

176. Sketch $y = \sqrt{x}$.

Consider the curve $x = y^2$.

The square root graph $y = \sqrt{x}$ is the top half of the parabola $x = y^2$:



177. Two dice are rolled. State which, if either, of the following events has the greater probability:

- two sixes,
- a five and a six.

178. A triangle has perimeter 12. Find all possible values for the length of the shortest side of the triangle, giving your answer as a set.

Consider an equilateral triangle and a very thin isosceles triangle.

In an isosceles triangle with two sides just shorter than 6, the shortest side can be as small as required. At the other end of the spectrum is an equilateral triangle with sides of length 4. So the set of possible values for the length of the shortest side is $(0, 4]$.

179. Determine whether the line $y = 2x - 1$ intersects the curve $y = x^2 - 2x + 3$.

Solve simultaneously.

We set $2x - 1 = x^2 - 2x + 3$ and rearrange to $x^2 - 4x + 4 = 0$. This has one repeated root $x = 2$, so the line does intersect the circle: it is tangent to it.

180. True or false?

(a) $x^2 = 1 \iff x = \pm 1$,

(b) $x^3 = 1 \iff x = \pm 1$,

(c) $x^4 = 1 \iff x = \pm 1$.

The odd and even cases are different.

(a) True.

(b) False, as $(-1)^3 \neq 1$.

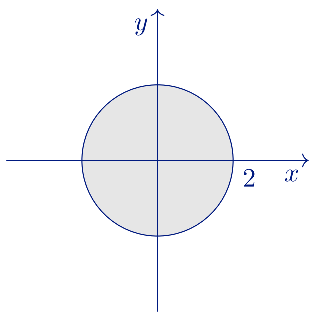
(c) True.

.

181. By sketching the boundary equation $x^2 + y^2 = 4$, shade the region of the (x, y) plane which satisfies the inequality $x^2 \leq 4 - y^2$.

The boundary equation is a circle.

The boundary equation is a circle with radius 2, centred at the origin. So the region we require is all points inside and on the circle:



182. Simplify $\dfrac{2\sqrt{2}}{4-\sqrt{8}}$.

Rationalise the denominator.

$$\begin{aligned}&\frac{2\sqrt{2}}{4-\sqrt{8}}\\ =\ &\frac{2\sqrt{2}(4+\sqrt{8})}{16-8}\\ =\ &\frac{8\sqrt{2}+8}{16-8}\\ =\ &\sqrt{2}+1.\end{aligned}$$

183. Three forces, with magnitudes 45, 60, and 75 N, act on an object, which remains in equilibrium. Find the angle between the 45 and 60 N forces.

Use a triangle of forces.

The object is in equilibrium under the action of three forces. So, those forces, drawn as vectors, must form a closed triangle of forces. Furthermore, since $45^2+60^2=75^2$, this triangle is right-angled. The angle between the shorter sides is 90°.

184. The parabola $x=y^2+1$ and the line $y=mx+c$ intersect at $y=-2$ and $y=3$.

(a) Set up simultaneous equations in m and c.

(b) Solve to find m and c.

Find the x coordinates of the points of intersection using the equation of the parabola. Then substitute.

(a) The points of intersection are $(5,-2)$ and $(10,3)$. So $-2=5m+c$ and $3=10m+c$.

(b) Eliminating c gives $m=1$, $c=-7$.

185. Find $\dfrac{dy}{dx}$ for the following graphs:

(a) $y=(x-1)(x-2)$,

(b) $y=(\sqrt{x}-1)(x-2)$.

Multiply out first.

(a)
$$\begin{aligned}&y=x^2-3x+2\\ \Longrightarrow\ &\frac{dy}{dx}=2x-3.\end{aligned}$$

(b)
$$\begin{aligned}&y=x^{\frac{3}{2}}-x+2x^{\frac{1}{2}}+2\\ \Longrightarrow\ &\frac{dy}{dx}=\tfrac{3}{2}x^{\frac{1}{2}}-1+x^{-\frac{1}{2}}.\end{aligned}$$

186. Ten names are put into a hat. Find the probability that the first three drawn out are the first three alphabetically, in alphabetical order.

Multiply probabilities for the first three names.

The probability that the first is the first alphabetically is $\frac{1}{10}$, followed by $\frac{1}{9}$ and $\frac{1}{8}$. So the overall probability is $\frac{1}{10}\times\frac{1}{9}\times\frac{1}{8}=\frac{1}{720}$.

187. A pentagon has sides given, in top-to-tail order around the perimeter, by the vectors $\mathbf{a},\mathbf{b},\mathbf{c},-\mathbf{a},\mathbf{d}$. Show that $\mathbf{b}+\mathbf{c}+\mathbf{d}=0$.

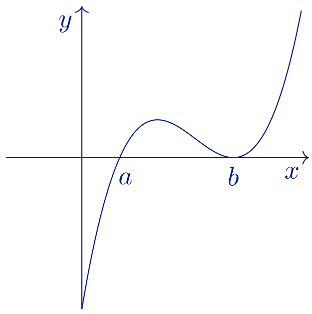
Consider the sum of all five vectors.

Since the vectors, added top-to-tail, form a closed perimeter, their sum must be zero. The $\mathbf{a}$ and $-\mathbf{a}$ sides cancel, and we are left with the required result.

188. A cubic graph has equation $y=(x-a)(x-b)^2$, where $0<a<b$. Sketch the graph, labelling the axis intercepts in terms of a and b.

Consider the single root at $x=a$ and double root at $x=b$.

The cubic has a single root at $x=a$ and a double root at $x=b$. Its leading coefficient is positive, so the graph is as below:



189. In each case, state, with a reason, whether the value of the limit is zero, one or infinity.

(a) $\displaystyle\lim_{x\to\infty}\frac{3^x}{2^x+1}$,

(b) $\displaystyle\lim_{x\to\infty}\frac{3^x}{3^x+1}$,

(c) $\displaystyle\lim_{x\to\infty}\frac{3^x}{4^x+1}$.

If in doubt, plug some large numbers in!

(a) ∞, as the function approaches $\left(\frac{3}{2}\right)^x$.

(b) 1, as the function approaches $\left(\frac{3}{3}\right)^x$

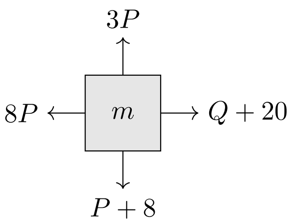
(c) 0, as the function approaches $\left(\frac{3}{4}\right)^x$.

190. By locating a double root, show that $y = 2x - 1$ is a tangent to $y = x^2$.

Solve simultaneously, and factorise the resulting quadratic equation.

Solving simultaneously, we have $x^2 = 2x - 1$, which factorises as $(x-1)^2 = 0$. Since $x = 1$ is a double root, the line must be tangent to the curve at this point.

191. Four forces keep an object in equilibrium:



Solve to find P and Q.

Plug into $F = ma$ horizontally and vertically, and solve simultaneously.

$$\begin{aligned} \updownarrow \quad &: 3P - (P+8) = 0 \\ \leftrightarrow \quad &: Q + 20 - 8P = 0. \end{aligned}$$

The first equation gives $P = 4$, then $Q = 12$.

192. A rectangle has perimeter $8\sqrt{2}$ cm and area 6 cm^2. By setting up simultaneous equations and solving, show that the diagonals are $\sqrt{20}$ cm long.

Name the dimensions x and y.

The simultaneous equations are $x + y = 4\sqrt{2}$ and $xy = 6$. Substituting for y gives $x(4\sqrt{2} - x) = 6$, which multiplies out to $x^2 - 4\sqrt{2}x + 6 = 0$. This factorises as $(x - 3\sqrt{2})(x - \sqrt{2}) = 0$, so the dimensions of the rectangle are $3\sqrt{2}$ and $\sqrt{2}$. Using Pythagoras, the diagonals are $\sqrt{18+2} = \sqrt{20}$ cm long.

193. Determine which two of the three points $(3,0)$, $(4,-2)$, $(1,-3)$ are closest to each other.

Calculate distances (or squared distances) using Pythagoras.

$(3,0)$ and $(4,-2)$ are closest: the squared distances are $5, 10, 13$.

194. Events P and Q are independent, with $P(A) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{12}$. Find the following probabilities:

(a) $P(B)$,

(b) $P(A \cup B)$.

Independence tells you that $P(A \cap B) = P(A) \times P(B)$.

By independence $P(A) \times P(B) = \frac{1}{12}$, so

(a) $P(B) = \frac{1}{3}$,

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2}$.

195. A graph has $\frac{dy}{dx} = 4x - 1$, and passes through the point $(0,3)$. Find y in terms of x.

Integrate to find y.

$$\begin{aligned} y &= \int 4x - 1 \, dx \\ &= 2x^2 - x + c. \end{aligned}$$

Substituting $(0,3)$ gives $y = 2x^2 - x + 3$.

196. Use the discriminant Δ to show that the equation $41x - 5 = 100x^2$ has no real roots.

$\Delta = b^2 - 4ac$.

Rearranging gives $100x^2 - 41x + 5$. The discriminant is $\Delta = 41^2 - 4 \cdot 5 \cdot 100 = -319 < 0$, so the equation has no real roots.

197. Describe the quadrilaterals which have vertices at the following points:

(a) O, $(1,0)$, $(2,1)$, $(2,2)$.

(b) O, $(2,-1)$, $(2,1)$, $(3,0)$.

Sketch the quadrilaterals.

(a) Parallelogram.

(b) Kite.

198. Determine the values of p and q which make this equation an identity:

$$1 + \frac{px}{1-x} = \frac{q}{1-x}.$$

Multiply by $(1-x)$, and compare coefficients.

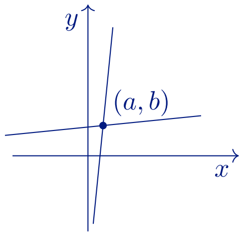
We multiply by $(1-x)$ to get $1 - x + px \equiv q$. Comparing coefficients of x, $p = 1$; comparing constant terms, $q = 1$.

199. On the same axes, for positive constants a, b, sketch the graphs

(a) $\dfrac{y-b}{x-a} = 10$, (b) $\dfrac{y-b}{x-a} = \dfrac{1}{10}$.

These are straight lines.

These are straight lines through a general point (a,b), with reciprocal gradients 10 and $\frac{1}{10}$.



200. One radian is defined as the angle subtended at the centre of a unit circle by an arc of unit length. Use this definition to prove that there are 2π radians in a revolution.

Use scale factors. The circumference of a unit circle is 2π. The length scale factor from an arc of unit length to the circumference is, therefore, $\frac{2\pi}{1}$. This also scales the angle, so the circumference subtends an angle of 2π radians.

201. The number of tails X when four coins are tossed follows a binomial distribution $X \sim B(4, \frac{1}{2})$. Show, without quoting a binomial formula, that $P(X = 2) = \frac{3}{8}$.

Use $p = \frac{\text{successful}}{\text{total}}$.

There are sixteen total outcomes, of which six are successful, six being the number of different orders of HHTT. So $p = \frac{6}{16} = \frac{3}{8}$.

202. A kite has diagonals length 6 and 8. Find its area.

Draw a diagram, and consider triangles.

The kite can be considered as two back-to-back triangles, base 8 and height 3. The overall area is $A = 2 \times \frac{1}{2} \cdot 8 \cdot 3 = 24$.

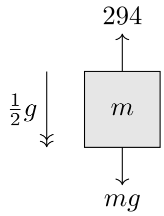
203. State, with a reason, whether the following gives a well-defined function:

$$f : \begin{cases} \mathbb{R} \mapsto \mathbb{R} \\ x \mapsto \dfrac{1}{x}. \end{cases}$$

Are there any values in $\mathbb{R}$ which cannot be reciprocated?

This is not a well-defined function, as the reciprocal of 0 is not defined, and 0 is in the proposed domain R.

204. A particle has forces acting on it as modelled in the force diagram shown. Forces are in Newtons, and acceleration in ms^{-2}.



Show that $m = 60$ kg.

Solve for m in $F = ma$.

$F = ma$ gives $mg - 294 = \frac{1}{2}mg$, which simplifies to $\frac{1}{2}mg = 294$. So $m = \frac{2 \cdot 294}{9.8} = 60$ kg.

205. Simplify $\dfrac{x^4 - 1}{x^2 - 1}$.

The numerator is a difference of two squares.

$$\frac{x^4 - 1}{x^2 - 1} \equiv \frac{(x^2 + 1)(x^2 - 1)}{x^2 - 1} \equiv x^2 + 1.$$

206. "The x axis is tangent to the curve $y = x^2 - 6x + 9$." True or false?

Factorise.

$x^2 - 6x + 9 \equiv (x - 3)^2$. The presence of a double root means that the curve is tangent to the x axis at $x = 3$.

207. The sum of the first n integers is given by the function $S(n) = \frac{1}{2}n(n + 1)$. Verify this result, by showing that $S(n - 1) + n = S(n)$.

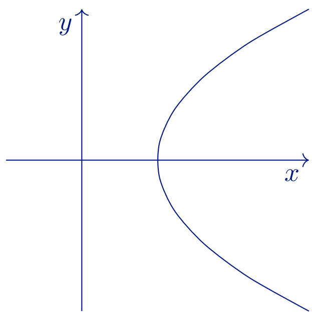
Take the LHS $S(n - 1) + n$, and simplify by factorising.

Consider
$$\begin{aligned} & S(n-1) + n \\ \equiv\ & \frac{1}{2}(n-1)n + n \\ \equiv\ & n\left(\frac{1}{2}(n-1) + 1\right) \\ \equiv\ & \frac{1}{2}n(n+1) \\ \equiv\ & S(n). \end{aligned}$$

208. Sketch the curve $x = y^2 + 1$.

First, sketch $y = x^2 + 1$, then switch x and y.

This is a parabola with vertex at $(1, 0)$:



209. Mutually exclusive events A, B have probabilities p_1 and p_2. Describe the relationship between the events if

(a) $p_1 = p_2$,

(b) $p_1 + p_2 = 1$.

"Mutually exclusive" means the events cannot both happen.

(a) Equally likely.

(b) Complementary.

210. The straight line segment $x = 2t$, $y = 1 - t$, for $t \in [0, 5]$ is reflected in the line $y = x$. Write down, without doing any calculations, the equation of the new line, in the same form.

Consider that reflection in $y = x$ sends point (a, b) to point (b, a).

Reflection in $y = x$ changes all the x's for y's and vice versa. So the new line is $x = 1 - t$, $y = 2t$, for $t \in [0, 5]$.

211. A curve has gradient formula $\dfrac{dy}{dx} = 3x^2 + 1$.

(a) Show that $y = ax^3 + bx + c$, where a and b are constants to be determined.

(b) The curve passes through $(-1, 5)$. Find c.

Integrate and substitute.

(a)
$$\frac{dy}{dx} = 3x^2 + 1$$
$$\Longrightarrow y = x^3 + x + c, \text{ so } a = 1, b = 0.$$

(b) Substituting $(-1, 5)$ gives $c = 7$.

212. An AP has n^{th} term u_n. Show that $w_n = u_{n-1} + u_n$ is also arithmetic.

Substitute the general formula $u_n = a + (n-1)d$ for the n^{th} term of an AP.

Substituting $u_n = a + (n-1)d$, we get

$$\begin{aligned} w_n &= a + (n-2)d + a + (n-1)d \\ &= 2a + 2nd - 3d \\ &= (2a - d) + (n-1)(2d) \end{aligned}$$

This is an arithmetic progression with first term $(2a - d)$ and common difference $2d$.

Alternatively (and less algebraically), since u_{n-1} is itself an arithmetic sequence, both terms in the definition of w_n increase by d as n increases by 1. Hence, w_n has a common difference of $2d$, and is an AP.

213. Show that the following claim is not true: "It is impossible for an object to remain in equilibrium under the action of five forces of magnitude F."

Construct a counterexample to the claim.

If the five forces act symmetrically along the directions of the sides of a regular pentagon, then the resultant force will be zero ("pentagon of forces").

214. A student suggests that the following is an identity for some suitable choice of constants A, B:

$$x(x-1)(x+1) \equiv Ax^2(x-1) + B(x^2 - 1).$$

By multiplying out and comparing the coefficients of powers of x, or otherwise, prove that the student is mistaken.

In an identity, the coefficients of each power of x must be the same on both sides of the equation: so one identity of this type generates multiple equations. In this case, consider the x term.

Multiplying out, we get

$$x^3 - x \equiv Ax^3 - Ax^2 + Bx^2 - B.$$

The RHS has no term in x, whereas the LHS does. Hence, no values A, B will make this an identity.

215. The numbers 1 to 5 are randomly assigned to the vertices of a pentagon. Find the probability that 1 and 2 are adjacent.

Place the 1 somewhere (the pentagon is symmetrical, so it makes no difference where), then consider the probability that the 2 ends up next door.

Place the 1 somewhere (the pentagon is symmetrical, so it makes no difference where). There remain four possible positions for the 2, of which two are adjacent to the 1. So the probability is $\frac{1}{2}$.

216. A functional instruction is given by

$$f(x) = \frac{2}{x-4}.$$

(a) The largest real domain over which this may be defined is $\mathbb{R} \setminus \{a\}$. Write down the value of a.

(b) By setting the function equal to y and rearranging to make x the subject, determine the functional instruction $f^{-1}(x)$.

In (a), consider division by zero.

(a) The set $\{2\}$ must be removed from the domain, to avoid division by zero.

(b) Set $y = \dfrac{2}{x-4}$

$$\implies xy - 4y = 2$$
$$\implies xy = 2 + 4y$$
$$\implies x = \frac{2}{y} + 4.$$

Redefining x, $f^{-1}(x) = \dfrac{2}{x} + 4$.

217. Solve the following simultaneous equations:

$$x^2 + y^2 = 1,$$
$$y = \tfrac{1}{2}x + 1.$$

Substitute the linear equation into the circle.

Substituting gives $x^2 + (\frac{1}{2}x + 1)^2 = 1$. So

$$\implies \tfrac{5}{4}x^2 + x = 0$$
$$\implies x(\tfrac{5}{4}x + 1) = 0$$

(x, y) is $(0, 1)$ or $(-\frac{4}{5}, \frac{3}{5})$.

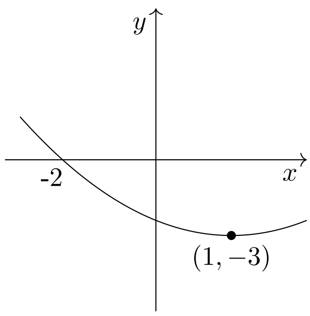
218. In a square $ABCD$ of side length 1 cm, 50% of the area is within a cm of vertex A. Determine the exact value of a.

Draw a diagram containing a shaded quarter circle.

The area of the square is 1. The area within a cm of vertex A is a quarter circle of radius a. Its area, therefore, is $\frac{1}{4}\pi a^2$. So we require $\frac{1}{4}\pi a^2 = \frac{1}{2}$. Taking the positive value of a, this rearranges to

$$a = \sqrt{\frac{2}{\pi}}.$$

219. Find the equation of the parabola shown below, on which a root and the vertex have been marked, in the form $y = ax^2 + bx + c$.



Begin with the form $y = a(x - p)^2 + q$.

With a vertex at $(1, -3)$, the parabola must have the form $y = a(x - 1)^2 - 3$. Substituting $(-2, 0)$ gives $a = \frac{1}{3}$. Multiplying out, we get $y = \frac{1}{3}x^2 - \frac{2}{3}x - \frac{8}{3}$.

220. Three forces, magnitudes $10, 20, 40$ N, act on an object of mass 5 kg, which accelerates at 2 ms^{-2}. Show that all three forces must have the same line of action.

Calculate the resultant force, and compare it to the magnitudes of the individual forces.

The resultant force is $F = ma = 10$ Newtons. Since the 40 N is 10 Newtons greater than the sum of the other two, the only way in which the resultant force can be 10 N is if the 20 N and 10 N forces act in the opposite direction to the 40 N force. Hence, all three lines of action must be the same.

221. Show that the points of intersection of the graphs $y = 2x + 5$ and $xy = 3$ are $\frac{7\sqrt{5}}{2}$ apart.

Solve simultaneously, then use Pythagoras.

Solving simultaneously gives intersections at $(-3, -1)$ and $(\frac{1}{2}, 6)$. The distance between these two is given by $\sqrt{3.5^2 + 7^2} = \frac{7\sqrt{5}}{2}$.

222. A die is rolled repeatedly until a six is attained. Show that the probability of requiring a total of r rolls to attain a six is

$$P(r) = \frac{5^{r-1}}{6^r}.$$

Consider the $r - 1$ non-sixes required.

We require $r - 1$ non-sixes, followed by a six. Hence, the probability is $p = \left(\frac{5}{6}\right)^{r-1} \times \frac{1}{6} = \frac{5^{r-1}}{6^r}$.

223. A curve is defined by $y = 2^x - 4^x$.

(a) Show that $y = 0$ is an asymptote.

(b) Find the axis intercepts of the curve.

(c) Hence, sketch the curve.

(a) An asymptote is a line which the curve approaches infinitely closely.

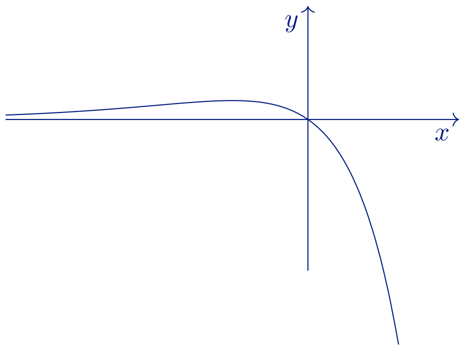
(b) ...

(c) Consider the behaviour as $x \to \pm\infty$.

(a) As $x \to -\infty$, both exponential terms tend to zero, hence $y \to 0$ and the curve is asymptotic to the x axis.

(b) For x axis intercepts, set $2^x - 4^x = 0$. Factorising gives $2^x(1 - 2^x) = 0$, which is only satisfied by $x = 0$. This must also be the y axis intercept.

(c) As $x \to \infty$, the curve diverges to negative infinity, so the behaviour is



224. A regular octahedron has eight faces, each of which is an equilateral triangle. Prove that the surface area of an octahedron is $2\sqrt{3}a^2$, where a is edge length.

Find the area of an equilateral triangle first, in terms of its edge length.

An equilateral triangle of side length a has perpendicular height $\frac{\sqrt{3}}{2}a$. Hence, it has area $\frac{\sqrt{3}}{4}a^2$. An octahedron contains eight such triangles, so the total surface area is $8 \cdot \frac{\sqrt{3}}{4}a^2 = 2\sqrt{3}a^2$.

225. By constructing a six-by-six grid representing the possibility space, find the probability that, when two dice are rolled, both show prime numbers.

Use $p = \frac{\text{successful}}{\text{total}}$.

Marking successful outcomes in the possibility space:

	1	2	3	4	5	6
1						
2		✓	✓		✓	
3						
4		✓	✓		✓	
5		✓	✓		✓	
6						

The probability is $\frac{9}{36} = \frac{1}{4}$.

226. A sample $\{x_i\}$ is given as follows:

x	0	1	2	3
f	6	14	9	2

Use the statistical functions on your calculator to find the mean $\bar{x}$ and standard deviation s.

If you have no frequency column, there is usually a setting to toggle the frequency on and off.

Mean $\bar{x} = 1.23$, $s = 0.831$ (3sf). (If you got $s = 0.845$ you were looking at the calculation of standard deviation using a divisor of $n-1$ rather than n. This is used as an unbiased estimator of population standard deviation.)

227. The terms of an AP have mean 10 and sum 420. Find the number of terms in the progression.

Consider the relationship between the mean, sum and number of terms of a sequence. There is no need to find (or, indeed, possibility of finding) individual terms or common differences.

The fact that the sequence is an AP is not relevant. The sum of the terms of a sequence is n times its mean. Hence, $n = \frac{420}{10} = 42$.

228. Disprove the following statements by providing a counterexample to each:

(a) "No kite is also a parallelogram."

(b) "A prime number cannot be a multiple of 11."

(a) Give the name of a quadrilateral which is both a kite and a parallelogram.

(b) Give the name of a prime number which is a multiple of 11.

(a) A rhombus or a square.

(b) 11 is technically a multiple of 11.

229. The locus L is defined as the boundary of the set of points which are closer to $(4, 0)$ than they are to $(0, 0)$. Find the equation of L.

Find the perpendicular bisector of the points.

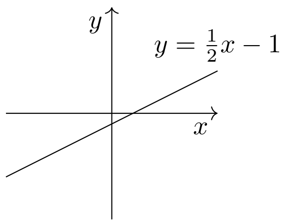
The boundary L is the perpendicular bisector of the points. The midpoint is $(2, 0)$, so the equation of L is $x = 2$.

230. By factorising, solve the equation $x^3 - 4x = 0$.

There are three factors.

$x(x-2)(x+2) = 0$, so $x = -2, 0, 2$.

231. The graph below is of $y = f'(x)$, for some function f defined over the real numbers.



Find all possible functions f.

Use integration.

Integrating $f'(x) = \frac{1}{2}x - 1$, we get a family of parabolae

$$f(x) = \frac{1}{4}x^2 - x + c.$$

232. Show that $f(x) = 10 + 24x^2 - 16x^4$ has maximum value 19.

Complete the square or differentiate.

This is a quadratic in x^2, so we can complete the square, treating x^2 as the variable. This gives

$$f(x) = 19 - 16(x^2 - \tfrac{3}{4})^2.$$

Since a square is always positive, this has maximum value 19.

233. A sample of size 24 has a datum added to it, after calculation of the mean $\bar{x} = 4$ and standard deviation $s = 1.2$. Write down the value of this datum, if its addition causes no alteration to the mean.

The standard deviation is not relevant.

Not to affect the mean, the new datum must have exactly the value of the old mean, so 4.

234. Evaluate $\displaystyle\sum_{k=1}^{4} \cos(90k - 15)^\circ$.

Use a symmetry argument on a unit circle or a graph.

In this sum, the four input values of the cosine function are symmetrically placed around the unit circle $\{90-15, 180-15, 270-15, 360-15\}$. Hence, their cosines must sum to zero.

235. An extreme runner is running across a glacier at 6 ms^{-1} when he spots a small ravine ahead. The ravine is 3 metres wide. He decides to jump it, without stopping. Assuming his horizontal speed remains constant, determine the vertical speed with which he must leap to make it safely across.

Use the horizontal information to find the length of time it takes for him to cross the ravine. Then look at the vertical motion.

From the horizontal information, he will spend half a second crossing the ravine. Assuming the minimal jump, he will land after half a second. So, using $s = ut + \frac{1}{2}at^2$, we require $0 = \frac{1}{2}u - \frac{1}{2}g \cdot \frac{1}{4}$. Solving, we get $u = \frac{1}{4}g = 2.45$ ms^{-1}.

236. Write down the roots of $(x^2 - a^2)(x^2 + b^2)$.

The roots are the values that make the expression zero. A fraction is only zero when its numerator is zero.

This expression is zero for $x = \pm a$. (It is also zero at $x = 0$ in the case that $b = 0$.)

237. In an attempted solution, a student writes

$$\begin{aligned} & 2x^2 - 3x > x^2 + x \\ \Longrightarrow\ & x^2 - 4x > 0 \\ \Longrightarrow\ & x - 4 > 0 \\ \Longrightarrow\ & x > 4. \end{aligned}$$

Explain the error, giving a counterexample to the relevant implication, and correct the solution.

It's the second line to the third line.

Division by x doesn't have a consistent effect on an inequality. If $x = -1$, then $x^2 - 4x > 0$, but $x - 4 \not> 0$.

In a correct solution, we find the roots of the boundary equation, which are $x = 0, 4$, and give the region outside those: $x \in (-\infty, 0) \cup (4, \infty)$.

238. If $f'(x) = \dfrac{1+x}{\sqrt{x}}$, find and simplify $f''(x)$.

$f''(x)$ is the derivative of $f'(x)$. Split the fraction up and differentiate.

Splitting the fraction up, we have

$$f'(x) = x^{-\frac{1}{2}} + x^{\frac{1}{2}}.$$

Differentiating the first derivative to get the second derivative gives $f''(x) = -\frac{1}{2}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$. Putting this over a common denominator of $2x^{\frac{3}{2}}$, we get

$$f''(x) = \frac{x-1}{2x^{\frac{3}{2}}}.$$

239. By writing each as fraction of integers, prove that a product pq of rational numbers p and q is rational.

Set $p = \frac{a}{b}, q = \frac{c}{d}$, where $a, b, c, d \in \mathbb{Z}$.

Since $p, q \in \mathbb{Q}$, we know that $p = \frac{a}{b}, q = \frac{c}{d}$, where $a, b, c, d \in \mathbb{Z}$. So $pq = \frac{ac}{bd}$, which is a quotient (fraction) of integers. Hence, $pq \in \mathbb{Q}$.

240. State, with a reason, whether $y = 2x + k$ intersects the following lines:

(a) $y = 1 - 2x + k$,

(b) $y = 1 + 2x + k$.

Consider the gradients of the lines. If two straight lines have different gradients, they must intersect.

(a) Yes, because their gradients are different.

(b) No, because they are parallel and distinct.

241. Solve $(x^2+1)^2 - (x^2-1)^2 = 0$

Expand and simplify the LHS, or rearrange and take the square root.

The LHS simplifies to $4x^2$, so the only root is $x = 0$.

242. The definite integral below gives the displacement, over a particular time period, for an object moving with constant speed:

$$s = \int_2^5 8\, dt.$$

Write down the speed and duration of the motion, and calculate the displacement.

The area under a velocity-time graph gives the displacement.

The speed is the integrand 8 ms^{-1}; the duration is $5-2=3$ seconds. The displacement is $s = 24$ m.

243. Prove that, for all positive real numbers x, y,

$$x + y > 5 \implies 2x + 3y > 10.$$

Double the first inequality.

If $x+y > 5$, then $2x+2y > 10$. Adding another y to the LHS can only increase this, since we are told that y is a positive number. Hence, $2x + 3y > 10$.

244. The point $(\sqrt{20}, 0)$ is rotated anticlockwise around the origin by an angle $\theta = \arctan \frac{1}{2}$. Find the new coordinates.

Sketch the scenario.

The point remains a distance $\sqrt{20}$ from the origin during the rotation. Since the rotation is by $\arctan \frac{1}{2}$, it must end up at a point of the form $(2k, k)$, for positive k. Solving $\sqrt{4k^2+k^2} = \sqrt{20}$ gives $k = 2$, so $(4, 2)$.

245. £1500 is invested at an annual interest rate of 4.3%, compounded every quarter. Show that the total interest received after 6 years is £438.84.

4.3% "compounded every quarter" means that the scale factor every quarter-year is 1.01075.

4.3% compounded every quarter means a scale factor of 1.01075 each quarter. So, after 6 years, the total amount is $1500 \times 1.01075^{24} = 1938.84$. Total interest is $1938.84 - 1500 = £438.84$.

246. A function g is such that $g'(x) = \frac{1}{3}$ for all real numbers x. Find $g(6) - g(0)$.

Calculate a definite integral.

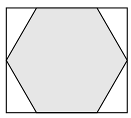
$$\begin{aligned} g(6) - g(0) &= \int_0^6 g'(x)\, dx \\ &= \int_0^6 \tfrac{1}{3}\, dx \\ &= \left[\tfrac{1}{3}x\right]_0^6 \\ &= 2. \end{aligned}$$

247. If $\log_3 y = x$, write 9^x in terms of y.

Write x in terms of y, and substitute.

Rewriting the logarithmic statement as an index statement, we have $y = 3^x$. Then we can express $9^x = (3^2)^x = 3^{2x} = (3^x)^2 = y^2$.

248. A rectangle is drawn around a regular hexagon.



Show that the ratio of the lengths of the sides of the rectangle is $2 : \sqrt{3}$.

Consider the hexagon as six equilateral triangles.

Let the hexagon have unit side length. Then the unshaded triangles have angles $(30°, 60°, 90°)$, and thus sides $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1)$. The rectangle measures $\frac{1}{2} + 1 + \frac{1}{2}$ wide by $2 \cdot \frac{\sqrt{3}}{2}$ high, which is a ratio of $2 : \sqrt{3}$.

249. Show that the line through $(b, 2b)$ and $(-b, 4b)$, for $b \neq 0$, is at 45° to the x axis.

Find the gradient of such a line.

The gradient is $\frac{\Delta y}{\Delta x} = \frac{2b}{-2b} = -1$. By symmetry, such a line is at 45° to the x axis.

250. Simplify $\dfrac{x+a}{b-a} \times \dfrac{a-x}{x+b} \times \dfrac{x^2-b^2}{x^2-a^2}$.

Remember that $x - a = -(a - x)$.

The right-hand fraction contains differences of two squares. Each of the factors cancels elsewhere, leaving $\frac{-1}{-1} = 1$.

251. Find the probability that, if two letters are chosen at random from the word DEUCE, neither is E.

Imagine picking the letters one at a time, without replacement: both need to be "not E".

Pick the letters one at a time. Both must be "not E", so the probability is $\frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$.

252. Prove that, if two triangles have a finite number n points of intersection, then $n \leq 6$.

Just sketch a few triangles to see what's going on, then explain.

For $n \geq 7$, we would require a straight line intersecting all three sides of a triangle. This is clearly not possible, since all triangles are convex.

253. A parabola has equation $y = 2x^2 - 3x + 6$.

(a) Find the gradient formula $\dfrac{dy}{dx}$.

(b) Evaluate the gradient formula at $x = 2$.

(c) Find the y coordinate at $x = 2$.

(d) Hence, show that the equation of the tangent to the curve at $x = 2$ is $y = 5x - 2$.

Use the standard differentiation rule for polynomials:

$$y = x^n \implies \frac{dy}{dx} = nx^{n-1}.$$

(a) $\dfrac{dy}{dx} = 4x - 3$,

(b) $4x - 3\,|_{x=2} = 5$,

(c) $y = 8$.

(d) The tangent is $y = 5x + c$. Substituting $(2, 8)$, we get $c = -2$, giving $y = 5x - 2$.

254. Using position vectors, prove that, for any three points A, B, C, the line joining the midpoint of AB to the midpoint of AC is parallel to BC.

Use the fact that, for any points P and Q with position vectors $\mathbf{p}$ and $\mathbf{q}$, the vector from P to Q is given by $\overrightarrow{PQ} = \mathbf{c} - \mathbf{b}$.

First, we set up position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ relative to an arbitrary origin. The midpoint of AB is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ and the midpoint of AC is $\frac{1}{2}(\mathbf{a}+\mathbf{c})$. The two vectors in the question are then $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$, and

$$\begin{aligned} &\tfrac{1}{2}(\mathbf{a}+\mathbf{c}) - \tfrac{1}{2}(\mathbf{a}+\mathbf{b}) \\ =\ &\tfrac{1}{2}(\mathbf{c}-\mathbf{b}) \\ =\ &\tfrac{1}{2}\overrightarrow{BC}. \end{aligned}$$

Since they are scalar multiples of one another, the vectors in question are parallel.

255. The definition of ${}^nC_r$ is

$${}^nC_r = \frac{n!}{r!(n-r)!}.$$

From this definition, show that

(a) ${}^nC_1 \equiv n$,

(b) ${}^nC_{n-2} \equiv \frac{1}{2}n(n-1)$, for $n \geq 2$.

Dividing one factorial by another allows for cancellation of many factors.

(a) $\dfrac{n!}{1!(n-1)!}$. All of the factors of $n!$ except n cancel, and ${}^nC_1 \equiv n$.

(b) $\dfrac{n!}{2!(n-2)!}$. The same approach works, only this time the factors $n(n-1)$ remain, as does the 2 in the denominator. So ${}^nC_2 \equiv \frac{1}{2}n(n-1)$.

256. Prove that the product of 7 consecutive integers is divisible by 7.

How often do factors of 7 appear?

Since one in every seven integers is a multiple of seven, a run of seven consecutive integers contains a factor of seven. Hence, the product does too.

257. State, with a reason, which of the implications $\implies$, $\impliedby$, $\iff$ links the following statements concerning a real number x:

- $(x-a)(x-b)(x-c) = 0$,
- $x \in \{a, b, c\}$.

Consider whether the implication in the factor theorem goes both ways.

The implication in the factor theorem goes both ways, so the statements should be linked by $\iff$.

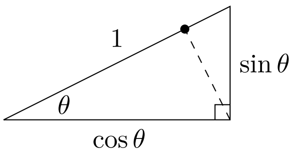
258. Express $x^2 + 2x + 5$ in terms of $(x-1)$.

Start with $(x-1)^2$, then deal with the x and constant terms.

Begin with $(x-1)^2$, which must be present to generate x^2. This gives $x^2 - 2x + 1$. Add $4(x-1)$ to sort the term in x, giving $x^2 + 2x - 3$. Lastly, add 8 to sort the constant term:

$$x^2 + 2x + 5 \equiv (x-1)^2 + 4(x-1) + 8.$$

259. The triangle below defines the functions sine and cosine over the domain $(0°, 90°)$. A perpendicular (dashed line) has been drawn to the hypotenuse.



(a) Show that the dashed line has length $\sin\theta\cos\theta$.

(b) The marked point splits the hypotenuse into two sections. Find these lengths, and write down the trigonometric identity which is verified by this.

Consider similar triangles.

All three triangles in the diagram are similar.

(a) The dashed line is the opposite side in a right-angled triangle with hypotenuse $\cos\theta$, so it has length $\sin\theta\cos\theta$.

(b) $\sin^2\theta + \cos^2\theta \equiv 1$.

260. Write down the roots of the following equation, in which the algebraic fraction is written in its lowest terms: $\dfrac{(x^2-p^2)(x^2-q^2)}{(x^2+r^2)(x^2+s^2)} = 0$.

For a fraction to be zero, it is necessary that...

Since the fraction is in its lowest terms, its roots are the roots of the numerator: $x = \pm p, \pm q$.

261. Two forces, with magnitudes 8 and 15 Newtons, act in perpendicular directions on an object of mass 34 kg. Find the acceleration of the object.

Combine the forces according to Pythagoras.

Since the forces are perpendicular, they combine by Pythagoras. This gives the resultant force as 17 Newtons, so the acceleration is 0.5 ms^{-2}.

262. Find the equation of the locus of points which are equidistant from $(4, 0)$ and $(0, 4)$.

Find the perpendicular bisector.

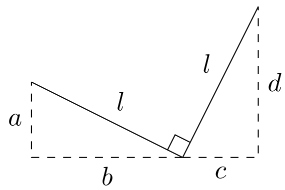
By symmetry, the perpendicular bisector of these points is the line $y = x$.

263. Explain what is wrong with the following argument: "In zero-gravity, there can be no reaction forces on an object, because reaction forces exist in reaction to gravity."

Here, the word "reaction" doesn't mean exactly what Newton originally meant in his third law.

In modern parlance, a "reaction force" is not simply a force that exists in response to something else, but has a more specific meaning: it means a contact force that is perpendicular to the surfaces in contact. (As opposed to friction, which is a contact force parallel to the surfaces in contact.) Two objects in contact in zero-gravity can exert reaction forces on each other, independently of gravity.

264. The diagram below shows gradient triangles drawn on two perpendicular lines of length l. Use it to show that the such gradients satisfy $m_1 m_2 = -1$.



Consider congruency.

The two triangles are congruent, as they share the same angles and a side length. Hence $a = c$ and $b = d$. So, $m_1 m_2 = \frac{-a}{b}\frac{d}{c} = \frac{-a}{b}\frac{b}{a} = -1$.

265. Find the value of the constant k, if the following curve has an asymptote at $x = 4$:

$$y = \frac{1}{x^2 + x + k}.$$

A fraction becomes undefined when its denominator is zero.

Since the curve has an asymptote at $x = 4$, its denominator must be zero at $x = 4$. Hence, $16 + 4 + k = 0$, and $k = -20$.

266. The small-angle approximation for $\cos\theta$, for θ in radians, is $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Find the percentage error in this approximation at

(a) $\theta = \dfrac{\pi}{24}$,

(b) $\theta = \dfrac{\pi}{6}$.

Use radian mode on your calculator.

(a) $\dfrac{\cos\frac{\pi}{24} - \left(1 + \frac{1}{2}\left(\frac{\pi}{24}\right)^2\right)}{\cos\frac{\pi}{24}} = 0.00123\%$.

(b) $\dfrac{\cos\frac{\pi}{6} - \left(1 + \frac{1}{2}\left(\frac{\pi}{6}\right)^2\right)}{\cos\frac{\pi}{6}} = 0.358\%$.

267. A sequence, with first term $u_0 = 1$, is defined by the iteration $u_{n+1} = 2u_n + n$. Find u_3.

Calculate u_1, then u_2, then u_3.

$u_1 = 2 \cdot 1 + 0 = 2$

$u_2 = 2 \cdot 2 + 1 = 5$

$u_3 = 2 \cdot 5 + 2 = 12.$

268. The resultant force on a particular object of mass m is given by $2m(t - t^2)$ N, where $t \in [0, \infty)$. Show that the maximum acceleration is given by $a_{\text{max}} = 0.5 \text{ ms}^{-2}$.

The acceleration is a quadratic in t, so either complete the square or use calculus to find its maximum value.

$a = F/m$, so the mass cancels, and $a = 2t - 2t^2$. This is a negative quadratic, so its maximum value is at the vertex. This is at $t = \frac{1}{2}$. Substituting in gives $a_{\text{max}} = 2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{4} = 0.5 \text{ ms}^{-2}$.

269. Find, in radians, the exterior angle of a regular decagon.

Consider a full journey around the perimeter, divided up into 10 turns at the vertices.

Quoting the formula, in radians, for the exterior angle of a regular 10-gon, we have $\frac{2\pi}{10} = \frac{1}{5}\pi$ radians.

270. Explain why the solution of the following equation is the same whatever the value of the constant a:

$$\frac{x^2 + 4x - 2}{x^2 + 1 + a^2} = 0.$$

Consider whether the fraction is in its lowest terms.

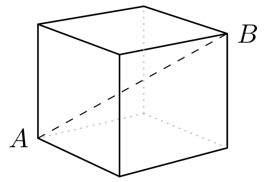
If a fraction is in its lowest terms, then its roots are the roots of its numerator. And the denominator here can have no factors, because $1 + a^2$ must always be positive. So the roots are the roots of the numerator, which doesn't depend on a.

271. Write down the area scale factor when $y = f(x)$ is transformed to $y = f(3x)$.

Consider whether this is an input or an output transformation.

This is an input transformation. Replacing x with $3x$ compresses the graph by factor 3 in the x direction. There is no change in y, so the area scale factor is $\frac{1}{3}$.

272. The diagram shows a cube of unit side length.



Determine the distance, through the centre of the cube, between the vertices marked A and B.

Use 3D Pythagoras.

3D Pythagoras is the same as 2D Pythagoras, just with more terms. So $d = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

273. Give, in radians, the sum of the interior angles of a heptagon.

A heptagon can be divided up into five triangles.

Since a heptagon can be divided up into five triangles, each of which has interior sum π radians, the total is 5π.

274. Show that $\dfrac{\sqrt{2} - \sqrt{8}}{\sqrt{8} - \sqrt{32}} = \dfrac{1}{2}$.

Simplify the individual surds first.

$$\begin{aligned} &\frac{\sqrt{2} - \sqrt{8}}{\sqrt{8} - \sqrt{32}} \\ &= \frac{\sqrt{2} - 2\sqrt{2}}{2\sqrt{2} - 4\sqrt{2}} \\ &= \frac{-\sqrt{2}}{-2\sqrt{2}} \\ &= \frac{1}{2}. \end{aligned}$$

275. Write down the range of the squaring function $x \mapsto x^2$, when it is defined over the given domains:

(a) $[0, 1]$,
(b) $[-1, 1]$,
(c) $\mathbb{R}$.

The range is the set of outputs attainable from a given domain. If in doubt, sketch a graph...

(a) $[0, 1] \mapsto [0, 1]$,
(b) $[-1, 1] \mapsto [0, 1]$,
(c) $\mathbb{R} \mapsto [0, \infty)$.

276. Find the equation of the perpendicular bisector of the points $(0, 4)$ and $(12, 0)$.

First, calculate the gradient and midpoint of the line joining the given points.

The perpendicular bisector passes through the average of the two points, which is $(6, 2)$, and its gradient is the negative reciprocal of $-\frac{1}{3}$, which is 3. So its equation is $y = 3x - 16$.

277. An arithmetic sequence has first term $u_1 = a$ and last term $u_n = l$. Show that the common difference is given by

$$\frac{l-a}{n-1}.$$

Consider the number of steps from 1 to n.

The difference between the first term and last term is $l-a$, and there are $n-1$ steps between the first and the n^{th} term. Each of these steps is a common difference, giving the required result.

278. Solve $\dfrac{(x-a)(x+b)}{(x-a)(x+b-1)} = 0$.

Consider common factors.

The numerator is zero at $x = a$ and $x = -b$. However, the denominator is also zero at $x = a$. Hence, the only root of the equation is $x = -b$.

279. A function has $g''(x) = 2$. By integrating twice, show that $y = g(x)$ is a monic parabola.

"Monic" means having a leading coefficient of 1.

If $g''(x) = 2$, then $g'(x) = 2x + c$, and $g(x) = x^2 + cx + d$, which is a monic parabola as it is a quadratic with leading coefficient 1.

280. A fair die is rolled twice. By drawing a diagram of the possibility space, find the probability that the second score is higher than the first score.

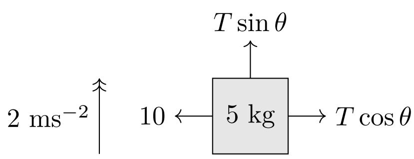
Use $p = \frac{\text{successful}}{\text{total}}$.

The possibility space is:

	1	2	3	4	5	6
1						
2	✓					
3	✓	✓				
4	✓	✓	✓			
5	✓	✓	✓	✓		
6	✓	✓	✓	✓	✓	

The probability is $\frac{15}{36} = \frac{5}{12}$.

281. Three forces, with magnitudes given in Newtons, cause a 5 kg mass to accelerate as depicted below:



Solve to find T and θ.

Set up vertical and horizontal $F = ma$'s, and solve simultaneously.

$F = ma$ gives us $T\cos\theta = 10$, and $T\sin\theta = 10$. Dividing the latter by the former tells us that $\tan\theta = 1$, so $\theta = 45°$, and $T = 10\sqrt{2}$.

282. Determine whether the point $(4, 5)$ lies inside, on, or outside the ellipse $2x^2 + 3y^2 = 100$.

An ellipse is a closed curve, so evaluating the LHS will determine whether a point is inside, on, or outside it.

An ellipse is a closed curve, so evaluating the LHS will determine whether a point is inside, on, or outside it. $2x^2 + 3y^2\big|_{(4,5)} = 107 > 100$, so the point lies outside the ellipse.

283. An irregular pentagon has sides whose lengths are in arithmetic progression. Its perimeter is 35. Explain carefully why the length l of its longest side must satisfy:

(a) $l > 7$,

(b) $l < 14$.

Consider the mean of the side lengths.

(a) The mean of the side lengths is $\frac{35}{5} = 7$. Since the pentagon is irregular, the longest must be longer than this.

(b) Since the sides are in AP, however much greater than the mean the largest is, the smallest must be that much smaller than the mean. The smallest must be greater than 0, and the mean is 7, so the largest must be less than 14.

284. The exam marks of class of twenty pupils are summarised with $\bar{x} = 69.3\%$. A new pupil then joins the class, whose mark is 86%. Calculate the new mean for the class.

Find $\sum x$, and add the new mark to that.

$\sum x$ for the twenty pupils is $69.3 \times 20 = 1386$. Adding 86 gives a new mean of $\frac{1386+86}{21} = 70.1\%$ (1dp).

285. For $p \neq 0$, determine the vertical asymptotes of

$$y = \frac{x^2 + p^2}{x^3 - px}.$$

Look for value for which the denominator is zero.

Because the numerator is never zero, this curve has vertical asymptotes wherever the denominator is zero. Setting $x^3 - px = 0$ gives $x = 0, \pm\sqrt{p}$, so these are the equations of the asymptotes.

286. True or false?

(a) $x^3 = y^3 \iff x = \pm y$,

(b) $x^4 = y^4 \iff x = \pm y$,

(c) $x^5 = y^5 \iff x = \pm y$.

Consider the implication right-to-left.

(a) False, because $x^3 = y^3 \;\not\Longleftarrow\; x = -y$.

(b) True.

(c) False, because $x^5 = y^5 \;\not\Longleftarrow\; x = -y$.

287. Give the exact area of the region of the (x, y) plane defined by the inequality $(x-1)^2 + (y-2)^2 \le 5$.

The region is circular.

The region is circular, with radius $\sqrt{5}$. The centre isn't relevant, and the area is 5π.

288. By differentiating, find all possible values of x for which the gradients of the graphs $y = x^3 + x^2$ and $y = (\sqrt{x}+1)(\sqrt{x}-1)$ are the same.

Multiply out before differentiating.

Multiplying out and differentiating, we require that

$$\begin{aligned} & 3x^2 + 2x = 1 \\ \Longrightarrow\ & 3x^2 + 2x - 1 = 0 \\ \Longrightarrow\ & (3x-1)(x+1) = 0 \\ \Longrightarrow\ & x = \tfrac{1}{3}, -1. \end{aligned}$$

289. Give the range of $f(x) = (\sin x + 1)^2 + 1$.

The range of $\sin x$ is $[-1, 1]$.

The range of $\sin x$ is $[-1, 1]$. So, the range of $(\sin x + 1)$ is $[0, 2]$. Hence, the range of $(\sin x + 1)^2$ is $[0, 4]$, and the range of $(\sin x + 1)^2 + 1$ is $[1, 5]$.

290. A quadrilateral $ABCD$ has vertices, in order, at $(0, 0)$, $(-1, 1)$, $(2, 4)$, $(3, 1)$.

(a) Find the perpendicular bisectors of

i. AC,

ii. BD.

(b) Find the intersection X of these bisectors.

(c) Show that $|AX| = |BX| = |CX| = |DX|$.

(d) Hence, show that $ABCD$ is cyclic.

A cyclic quadrilateral is one whose vertices are on the circumference of the same circle.

(a) Perpendicular bisectors:

i. $y = -\frac{1}{2}x + \frac{5}{2}$

ii. $x = 1$.

(b) Intersection is at $(1, 2)$.

(c) Each of A, B, C, D lies at distance of $\sqrt{2^2 + 1^1}$ from X.

(d) Hence, X is the centre of a circle of radius $\sqrt{5}$ which passes through A, B, C, D. so $ABCD$ is a cyclic quadrilateral.

291. Disprove the following statement:

If $f'(x) \equiv g'(x)$, then $f(x) \equiv g(x)$.

Provide a counterexample: two distinct functions with the same derivative.

Any two functions differing by a constant provide a counterexample, e.g. $f(x) = x^2$ and $g(x) = x^2 + 1$.

292. Show that the graphs $2x + 3y = 1$, $x - 5y = -6$ and $x^2 + y^2 = 2$ are concurrent.

"Concurrent" means having a mutual point of intersection.

The intersection of the lines is $(-1, 1)$. Substituting this into the LHS of the circle gives $(-1)^1 + 1^2 = 2$, so this intersection lies on the circle. Hence, the three graphs are concurrent.

293. Two of the constant acceleration formulae are true by definition: average velocity is rate of change of displacement, $\frac{1}{2}(u + v) = \frac{s}{t}$, and acceleration is rate of change of velocity, $a = \frac{v-u}{t}$. From these two, prove the formula $v^2 = u^2 + 2as$.

The result required contains no t, so rearrange the simpler of the two equation to make t the subject, substitute and simplify.

We rearrange the second equation to make t the subject, $t = \frac{v-u}{a}$. Multiplying the first equation by t and substituting this gives

$$\frac{1}{2}(u + v) \cdot \frac{v - u}{a} = s.$$

Multiplying by $2a$ and expanding the difference of two squares gives $v^2 - u^2 = 2as$, from which required result follows.

294. Using the discriminant $\Delta = b^2 - 4ac$, determine all possible values of k for which the equation $x^3 + kx^2 + 9x = 0$ has exactly two roots.

Remove the $x = 0$ root first.

The equation has one root at $x = 0$. We require the remaining quadratic factor to have exactly one root, so we set $\Delta = k^2 - 36 = 0$, giving $k = \pm 6$.

295. Find $\displaystyle\int f(t)\,dt$, for

(a) $f(t) = 0$,
(b) $f(t) = 1$,
(c) $f(t) = \sqrt{t}$.

If in doubt with integration, consider the process as anti-differentiation. Ask "What would differentiate to give this?"

(a) $\int 0\,dt = c$,
(b) $\int 1\,dt = t + c$,
(c) $\int \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}} + c$.

296. Find the equation of the line through $(a, a+1)$ and $(a+2, a+3)$.

The answer is independent of a.

A gradient triangle gives $m = 1$. Substituting, we get $y = x + 1$.

297. The parabola $y = ax^2 + 2x + 1$ passes through the point $(-2, 17)$. Find a.

Substitute $(-2, 17)$ to form an equation.

Substituting gives $4a - 3 = 17$, so $a = 5$.

298. True or false?

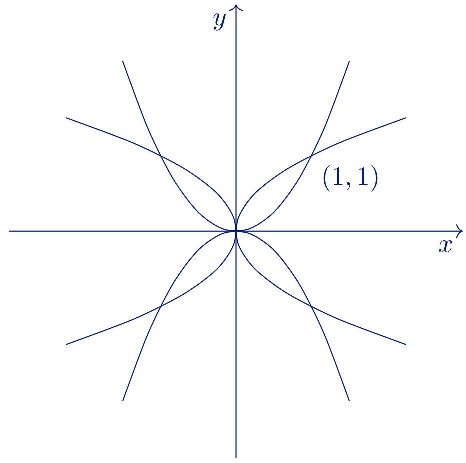
(a) Kites never have rotational symmetry,
(b) All parallelograms have rotational symmetry,
(c) Trapezia may have rotational symmetry.

Remember that, e.g. a square is a kite.

(a) False: a square is a kite.
(b) True.
(c) True: a square is a trapezium.

299. Sketch the four graphs $y = \pm x^2$, $x = \pm y^2$ on the same axes, labelling the point of intersection in the positive quadrant.

Draw $y = x^2$ first, then consider transformations.



300. Prove that, in a plane, precisely six unit circles may be placed around a seventh, such that each is tangent to the central circle and its neighbours.

Consider the angles involved when three unit circles are tangent to each other.

When three unit circles are placed tangent to each other, their centres form an equilateral triangle, so the angle subtended is $60°$. $360 = 6 \times 60$, so precisely six circles will fit around a seventh.

301. Solve $|3x - 1| = |2 - 5x|$.

Square both sides.

It is generally true that $|a| = |b| \implies a^2 = b^2$. So, we can solve $(3x-1)^2 = (2-5x)^2$. Multiplying out gives

$$\begin{aligned} & 9x^2 - 6x + 1 = 25x^2 - 20x + 4 \\ \implies & 16x^2 - 14x + 3 = 0 \\ \implies & (8x-3)(2x-1) = 0 \\ \implies & x = \tfrac{3}{8}, \tfrac{1}{2}. \end{aligned}$$

302. "The curves $y = x^2$ and $y + x^2 + 4x + 2 = 0$ are tangent to one another." True or false?

Consider Δ or equivalently double roots.

Substituting for y gives $(x+1)^2 = 0$. The double root means that the curves are tangent.

303. Two white counters and two black counters are placed in a bag. Two counters are then drawn out. Find the probability of drawing out

(a) two white counters,
(b) one of each colour.

For (b), consider both orders.

(a) $\frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$,

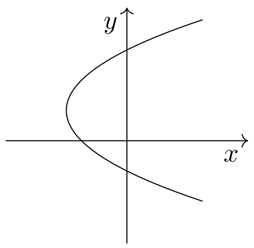
(b) There are two outcomes (BW and WB) which make up this event, so we have $2 \times \frac{2}{4} \times \frac{2}{3} = \frac{2}{3}$.

304. The parabola $y = ax^2 + bx + c$ passes through the origin. Explain whether any of the constants a, b, c can be determined from this information.

Substitute $(0, 0)$.

The constant term c must be 0, since $(0, 0)$ satisfies the equation. The other two coefficients cannot be determined.

305. State, with a reason, whether each of the equations could possibly be that of the sketched graph.



(a) $x = (y - 1)^2 - 2$,

(b) $x = (y + 1)^2 - 2$,

(c) $x = (y - 1)^2 + 2$.

Consider the coordinates of the vertex.

(a) $x = (y - 1)^2 - 2$ has a vertex at $(-2, 1)$: Yes.

(b) $x = (y + 1)^2 - 2$ has a vertex at $(-2, -1)$: No.

(c) $x = (y - 1)^2 + 2$ has a vertex at $(2, 1)$: No.

306. Triangle ABC has $|AB| = 2\sqrt{3}$, $|BC| = 2$, and $\angle CAB = 30°$. Find all possible values of $\angle BCA$.

This is an ambiguous case of the sine rule: there are two possible answers.

Using the sine rule, labelling $\angle BCA$ as θ, we have

$$\frac{\sin 30°}{2} = \frac{\sin \theta}{2\sqrt{3}}.$$

This gives $\sin \theta = 60°, 120°$. This is an ambiguous case of the sine rule: there are two triangles which fit the information given.

307. (a) Show that $x^2 + 1 = x$ has no solutions.

(b) Hence, show that the instruction $x \mapsto x^2 + 1$ maps all real numbers to numbers larger than themselves.

Sketch $y = x^2 + 1$ and $y = x$.

(a) $\Delta = 1 - 4 = -3 < 0$, so the quadratic has no real roots.

(b) The graph $y = x^2 + 1$ does not intersect $y = x$, and it is a positive parabola, so must be above the line $y = x$ everywhere, i.e. $y > x$. Hence, the instruction $x \mapsto x^2 + 1$ produces outputs that are greater than its inputs.

308. The equations $x + 2y = a$, $3x - 4y = 6$ are satisfied by $x = 4$, $y = b$. Find a and b.

Substitute values and solve.

Substituting values, we get $4 + 2b = a$ and $12 - 4b = 6$. The latter equation gives $b = \frac{3}{2}$. The former then gives $a = 7$.

309. For $f : x \mapsto 4(x - 2)^3 - 1$, find f^{-1}.

Set the function to equal a variable y, and rearrange to make x the subject.

Introducing a variable y, we have

$$\begin{aligned} & y = 4(x-2)^3 - 1 \\ \Longrightarrow\ & \frac{y+1}{4} = (x-2)^3 \\ \Longrightarrow\ & x = \sqrt[3]{\frac{y+1}{4}} + 2. \end{aligned}$$

Rewriting in terms of x, we have

$$f^{-1}(x) = \sqrt[3]{\frac{x+1}{4}} + 2.$$

310. Two of the following statements are true; the other is not. Prove the two and disprove the other.

(a) $(x - 2)(x^2 - 4x + 3) = 0 \implies x = 2$,

(b) $(x - 2)(x^2 - 4x + 4) = 0 \implies x = 2$,

(c) $(x - 2)(x^2 - 4x + 5) = 0 \implies x = 2$.

Use the factor theorem and the discriminant.

The discriminants of the quadratic factors are $\Delta = 4, 0, -4$ respectively. So, (a) is false, because its quadratic has real roots, and (c) is true, because its quadratic has no real roots. In (b), a further consideration is required: the quadratic factorises as $(x - 2)^2$, with one root at $x = 2$, hence, the implication also holds there.

311. By substituting the variable $z = x^3$, or otherwise, solve the equation $x^6 - 9x^3 + 8 = 0$.

Factorise. The equation factorises $(x^3 - 1)(x^3 - 8) = 0$. So $x = 1$ or $x = 2$.

312. Find and correct the error in the following:

$$\begin{aligned} &x^4 - x^2 = 0 \\ \implies &x^2 - 1 = 0 \\ \implies &x = \pm 1. \end{aligned}$$

Division by x^2 is impossible if...

Division by x^2 removes the possibility that $x^2 = 0$, deleting a root. The second line should be a factorisation: $x^2(x^2-1) = 0$, which maintains the possibility. The solution is $x = 0, \pm 1$.

313. Verify that the function $f(x) = 4x^{\frac{1}{3}}$ is a solution of the differential equation

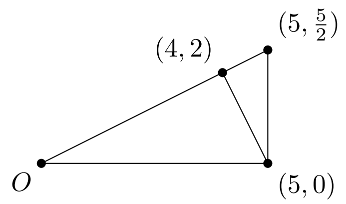
$$f'(x) = \frac{f(x)}{3x}.$$

Differentiate to find $f'(x)$ and substitute.

Differentiating gives $f'(x) = \frac{4}{3}x^{-\frac{2}{3}}$. And the RHS of the differential equation is

$$\begin{aligned} &\frac{f(x)}{3x} \\ = &\frac{4x^{\frac{1}{3}}}{3x} \\ = &\frac{4}{3}x^{\frac{1}{3}-1} \\ = &\frac{4}{3}x^{-\frac{2}{3}} \\ = &f'(x). \end{aligned}$$

314. Show that the triangles below are all similar.



Prove perpendicularity.

The large triangle is right-angled, so we need only show that the line segment inside it is perpendicular to the hypotenuse. The gradients in question are $\frac{1}{2}$ and -2, which are negative reciprocals. Hence, since all three triangles have the same interior angles, they must be similar.

315. Show that $2\int_{-1}^{k} t - 1 \, dt = (k-1)^2$.

Use the fact that $\int (t-1)\, dt = \frac{1}{2}t^2 - t + c$.

$$\begin{aligned} &2\int_{-1}^{k} t - 1 \, dt \\ = \;&2\Big[\tfrac{1}{2}t^2 - t\Big]_{-1}^{k} \\ = \;&2\Big(\big(\tfrac{1}{2}k^2 - k\big) - \big(\tfrac{1}{2}(-1)^2 - (-1)\big)\Big) \\ = \;&k^2 - 2k + 1 \\ = \;&(k-1)^2. \end{aligned}$$

316. A graph is given by $(x^2+y^2-1)(x^2+y^2-4) = 0$.

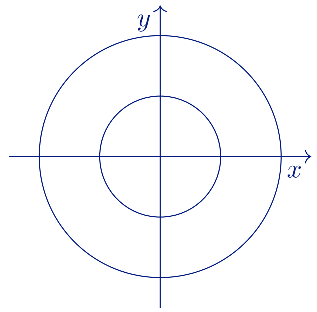
(a) Explain why the graph consists of a pair of distinct circles.

(b) Sketch the graph.

Use the factor theorem.

(a) By the factor theorem, one of the brackets must equal zero. Hence, $x^2+y^2 = 1$ or 4. These are the equations of two concentric circles.

(b) The circle are centred at the origin and have radii 1 and 2:



317. If $y = x - 2$, write $x^2 - 4x$ in simplified terms of y.

Write x in terms of y and substitute.

We rearrange to $x = y + 2$ and substitute, giving $(y+2)^2 - 4(y+2)$. This simplifies to $y^2 - 4$.

318. The dimensions of a sheet of A4 are defined such that, when the sheet is folded in half to make A5, the two shapes are similar. Show that the dimensions of such sheets of paper are in the ratio $1 : \sqrt{2}$.

Consider the area scale factor.

The area scale factor between A5 and A4 is 2, by definition. Hence, the length scale factor is $\sqrt{2}$. This means that the lengths of the sides of a sheet of A4 are in the ratio $1 : \sqrt{2}$.

319. Show that $(3x+1)^2(x+2) + 3x + 6$, where $x \in \mathbb{N}$, cannot be prime.

Take out a common factor.

Taking out a common factor of $(3x+1)$, we have

$$(3x+1)\big[(3x+1)(x+2)+2\big].$$

If $x \in \mathbb{N}$, then the two factors are both natural numbers greater than 1. Hence, their product cannot be prime.

320. (a) Evaluate $6x^3 - 23x^2 + 24x - 7\,\big|_{x=\frac{1}{2}}$.

(b) Hence, write down a factor of the expression $6x^3 - 23x^2 + 24x - 7$, giving the name of the theorem you use.

Factor theorem!

Evaluation gives 0, so, by the factor theorem, $(2x-1)$ is a factor of the cubic.

321. Write the following in simplified interval notation:

$$\{x \in \mathbb{R} : x > 0\} \cap \{x \in \mathbb{R} : x \le 3\}.$$

If in doubt, sketch a number line.

The set is everything which is simultaneously larger than zero and less than or equal to 3. In interval set notation, this is $(0, 3]$.

322. Two large, disgruntled elephants are shoving one another. Neither is giving an inch. Explain how Newton's laws tell you that the magnitudes of the following forces are equal:

(a) the force of elephant A on elephant B; the force of elephant B on elephant A,

(b) the frictional force of the ground on elephant A; the force of elephant B on elephant A.

Consider Newton II for equilibrium, and Newton III.

(a) These forces form a Newton III pair: they are two aspects of a single interaction.

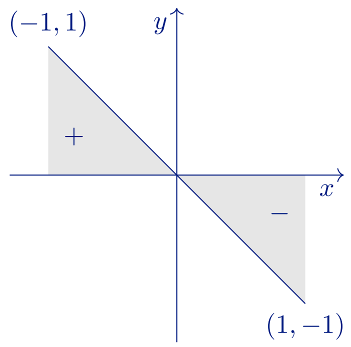
(b) The forces are equal in magnitude because of Newton II: there is no acceleration, so there is no resultant force.

323. Sketch a linear graph $y = f(x)$ for which

$$\int_{-1}^{1} f(x)\,dx = 0, \quad f(-1) = 1.$$

Remember that an integral calculates the signed area: negative-valued functions produce negative-valued signed areas.

Since the integral of f on $[-1, 1]$ is zero, the graph must be symmetrical about the origin. This is the only way in which the signed areas can cancel. Hence, we require



324. There are 4 routes from A to B. Counting a trip and its reverse as different, find the number of different return trips, if the routes out and back

(a) may be the same,

(b) must be different.

Multiply the numbers of options for the way out and the way back.

(a) $4 \times 4 = 16$,

(b) $4 \times 3 = 12$.

325. Solve $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}} = x$.

When fractions are inlaid like they are on the LHS, multiply top and bottom of the large fraction by the denominator of the little fraction.

Multiplying top and bottom of the large fraction by the denominator of the inlaid fraction, we have

$$\begin{aligned} & \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = x \\ \Longrightarrow\ & \frac{x+1}{x-1} = x \\ \Longrightarrow\ & x+1 = x^2 - x \\ \Longrightarrow\ & x^2 - 2x - 1 = 0 \\ \Longrightarrow\ & x = 1 \pm \sqrt{2}. \end{aligned}$$

326. The scores X and Y on two tests, each out of forty marks, are to be combined, with equal weighting, into a single mark M, which is out of a hundred. Determine a formula for M in terms of X and Y.

Consider the scale factor required.

The marks must be scaled by $\frac{5}{4}$ to produce 100 total marks. So $M = \frac{5}{4}(X+Y)$.

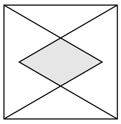
327. Show that $(1+\sqrt{3})^5 = 76 + 44\sqrt{3}$.

Use the binomial expansion on the LHS.

Using the binomial expansion, we have

$$\begin{aligned} &(1+\sqrt{3})^5 \\ =\ &1 + 5\cdot\sqrt{3} + 10\cdot 3 + 10\cdot 3\sqrt{3} + 5\cdot 9 + 9\sqrt{3} \\ =\ &76 + 44\sqrt{3}. \end{aligned}$$

328. Two equilateral triangles are drawn inside a square of side length 1.



Show that the area of the shaded region is $\dfrac{2\sqrt{3}}{3} - 1$.

Find the area of the two isosceles triangles.

The isosceles triangles at the top and bottom have base 1 and height $\frac{1}{2\sqrt{3}}$. So their combined area is $\frac{1}{2\sqrt{3}}$. Each equilateral triangle has area $\frac{\sqrt{3}}{4}$. So the shaded region has area "two equilateral triangles plus two isosceles triangles minus the square."

$$\begin{aligned} &2\cdot\frac{\sqrt{3}}{4} + \frac{1}{2\sqrt{3}} - 1 \\ =\ &\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{6} - 1 \\ =\ &\frac{2\sqrt{3}}{3} - 1. \end{aligned}$$

329. Two dogs pull on the opposite ends of a light toy. Explain, with reference to Newton's laws, why the difference between the magnitudes of the forces exerted by the two dogs must be negligible.

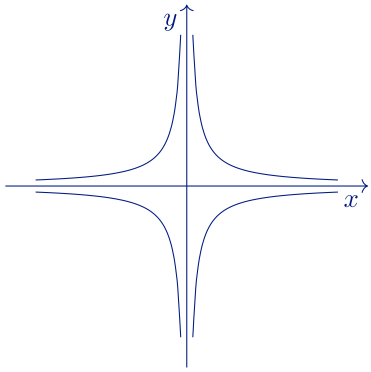
The key word is "light".

The toy is light, which means its mass is modelled as negligible, i.e. $m \approx 0$. So, it doesn't matter what acceleration the dogs are giving to the toy, $F = ma$ dictates that the resultant force on the toy must be negligible. This is equivalent to the required result.

330. Sketch the graph $\dfrac{x^2}{y^2} = 1$.

Take a square root, remembering the $\pm$.

This is equivalent to $xy = \pm 1$, which is two reciprocal graphs, one positive, one negative:



331. A particle has acceleration of magnitude 35 ms^{-2} in the direction of $-6\mathbf{i} + 8\mathbf{j}$. Find the component of its acceleration in the $\mathbf{j}$ direction.

Sketch a triangle representing the force.

$(6, 8, 10)$ is a Pythagorean triple, so the component in the $\mathbf{j}$ direction is $\frac{8}{10}\cdot 35 = 28$ ms^{-2}.

332. A function f is such that the indefinite integral of f is quadratic. Describe the function f.

Consider standard polynomial integration.

If the indefinite integral of f is quadratic, then f can be any linear function of the form $f(x) = ax + b$, where $a \neq 0$. (This non-zero condition is required, otherwise there would be no x^2 term in the integral of $f(x)$.)

333. A circle, in the positive quadrant, has radius 1 and is tangent to both axes.

(a) Write down the equation of the circle.

(b) The circle is stretched by a scale factor 2 in the x direction. Write down the equation of the ellipse formed.

(c) The ellipse is then reflected in the line $y = x$. Write down the equation of the new ellipse.

In (b), to stretch by a factor 2 in the x direction, replace x by $\frac{1}{2}x$. In (c), switch x and y.

(a) $(x-1)^2 + (y-1)^2 = 1$,

(b) $(\frac{1}{2}x - 1)^2 + (y-1)^2 = 1$,

(c) $(\frac{1}{2}y - 1)^2 + (x-1)^2 = 1$.

334. An arithmetic sequence has first term 1, last term 15 and sum 64. Find the common difference.

Consider the mean of the sequence.

The mean of an AP is the mean of the first and last terms. So $\frac{1}{2}(1+15)n = 64$. This gives $n = 8$, i.e. 7 steps. So the common difference is $\frac{15-1}{7} = 2$.

335. Polygon $OABC$ has vertices at $(5,1),(1,5),(6,6)$.

(a) Show that the polygon is a rhombus.

(b) Show that the area of the rhombus is 24.

The area of a rhombus is most easily calculated from the lengths of its diagonals.

(a) The squared distance between adjacent vertices is 26 in each case, so the polygon is a rhombus.

(b) The diagonals have length $\sqrt{72}$ and $\sqrt{32}$. So the area of the rhombus is $\frac{1}{2}\sqrt{72}\sqrt{32} = 24$.

336. You are given that the lines $2x + 3y = 5$ and $kx - 4y = 1$ are perpendicular to one another. Determine the value of k.

Use the negative reciprocal.

Since the gradients must be negative reciprocals, $k = 6$.

337. Explain why, if three events A, B, C are pairwise mutually exclusive, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

"Pairwise mutually exclusive" means that no two of the events overlap.

Since the events are pairwise mutually exclusive, they do not overlap, i.e. their intersections, both pairwise and as a three, must be empty. Hence, the probability of their union, which is the LHS, is given simply by the sum of their probabilities.

338. An object has exactly three forces acting on it, whose magnitudes are $2, 5, 8$ Newtons. Show that the object cannot be in equilibrium.

Consider the fact that $2 + 5 < 8$.

Since $2+5<8$, a triangle of forces cannot be constructed: there must always be a resultant force in the direction of the 8 N force. Hence, the object cannot remain in equilibrium.

339. Evaluate the following:

(a) $\left[2^x + 3^x\right]_0^1$,

(b) $\left[\log_2 x + \log_4 x\right]_1^2$.

$\left[F(x)\right]_a^b = F(b) - F(a)$.

(a) $\left[2^x + 3^x\right]_0^1 = 5 - 2 = 3$

(b) $\left[\log_2 x + \log_4 x\right]_1^2 = (1 + \dfrac{1}{2}) - (0) = \dfrac{3}{2}$.

340. A chord, which is tangent to $x^2+y^2 = 1$, is drawn to $x^2 + y^2 = 4$. Determine its exact length.

Draw a sketch, but don't use coordinate geometry.

Since tangent and radius are perpendicular, we know that half the chord has length $\sqrt{2^2 - 1^2}$. Hence, the chord has length $2\sqrt{3}$.

341. Prove that, if X has a binomial distribution with $2n$ trials, $X \sim B(2n, \frac{1}{2})$, then, for all k,

$$P(X = n + k) = P(X = n - k).$$

Consider symmetry.

Since the binomial distribution has probability $\frac{1}{2}$ and mean $2n \cdot \frac{1}{2} = n$, it is symmetrical about n. The two probabilities in the required result are symmetrically k away from n, so they must have equal probabilities.

342. A function f has $f(0) = 4$, $f'(0) = 2$, $f''(x) = 4$.

(a) Show that $f'(x) = 4x + 2$.

(b) Find $f(x)$.

(c) Sketch $y = f(x)$.

Integrate!

(a) Integrating f'', we get $f'(x) = 4x + c$. Then $f'(0)$ gives $c = 2$.

(b) Integrating again, $f(x) = 2x^2 + 2x + d$. Then $f(0)$ gives $d = 4$.

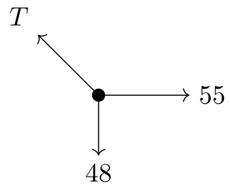
(c) The discriminant Δ is negative, so the parabola doesn't cross the x axis. Its gradient at $x = 0$ is positive, so its vertex is to the left of the y axis.

343. Show that $(x^2 - 1)$ is a factor of $x^3 - 4x^2 - x + 4$.

Show that $(x + 1)$ and $(x - 1)$ are both factors.

Since $x^2 - 1 = (x - 1)(x + 1)$, we need to test $x = \pm 1$. Both of these values, when substituted into the cubic expression, give 0, so $x^2 - 1$ is a factor.

344. Three forces, with magnitudes given in Newtons, hold a mass in equilibrium as depicted below. The known forces are perpendicular.



Find T.

Use a triangle of forces.

Setting up a right-angled triangle of forces, we have $T = \sqrt{48^2 + 55^2} = 73$ N.

345. The radian is defined so that arc length l is given by $l = r\theta$, where θ is the angle subtended at the centre of the circle. Prove that sector area A is given by $A = \frac{1}{2}r^2\theta$.

Consider a sector as a fraction of a circle.

From $l = r\theta$, we can scale to $\frac{l}{2\pi r} = \frac{\theta}{2\pi}$. The LHS is the arc length as a fraction of the circle. To find sector area, we multiply the same fraction by the area of the circle, giving $A = \frac{\theta}{2\pi}\pi r^2 = \frac{1}{2}r^2\theta$.

346. Find $\displaystyle\int \frac{(2x+1)(2x-1)}{x^2}\,dx$.

Multiply out and split up the fraction.

Multiplying out and splitting up the fraction, we get

$$\int 4 - x^{-2}\,dx = 4x + \frac{1}{x} + c.$$

347. Odd numbers can be thought of as an arithmetic progression with common difference 2. Use this fact to find the sum of the first 100 odd numbers.

Use $S_n = \frac{1}{2}n(2a + (n-1)d)$.

Using $S_n = \frac{1}{2}n(2a + (n-1)d)$, we have $S_{100} = \frac{1}{2}100(2 + 99 \cdot 2) = 10000$.

348. Consider a quadratic graph $y = x^2 + px + q$.

(a) Show that this can be written in the form

$$y = \left(x + \tfrac{1}{2}p\right)^2 + q - \tfrac{1}{4}p^2.$$

(b) Explain why the graph has a minimum at the point $\left(-\frac{1}{2}p, q - \frac{1}{4}p^2\right)$.

Complete the square.

(a) Completing the square gives the result.

(b) The square is minimised at $x = -\frac{1}{2}p$, when it has value zero. At this point, the quadratic has value $q - \frac{1}{4}p^2$.

349. A function h, for which $h(0) = 1$ and $h(1) = 4$, has constant derivative. Find $h(3)$.

Functions with constant derivatives are linear.

A function with a constant derivative is linear. So, counting explicitly, $h(2) = 7$ and $h(3) = 10$.

350. To solve the equation $x^3 = x$, a student writes the following: "Dividing by x gives $x^2 = 1$, so $x = \pm 1$." Explain what is wrong with this logic.

Consider $x = 0$.

Division by x ignores the possibility that x could be zero. Factorisation is better.

351. State whether the following functions have a sign change at $x = a$.

(a) $f : x \mapsto (x - a + 1)(x - a)$,

(b) $f : x \mapsto (x - a + 1)(x - a)^2$,

(c) $f : x \mapsto (x - a + 1)(x - a)^3$.

The factor $(x - a)$ changes sign at $x = a$. What about powers of the same factor?

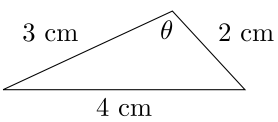
The factor $(x - a)$ changes sign at $x = a$. But an even power of it does not. The other factor is not zero at $x = a$, so it isn't relevant. Hence (a) and (c) change sign, but (b) doesn't.

352. Solve $x - (\sqrt{x} - 2)^2 = 1$.

Multiply out and rearrange to a quadratic in $\sqrt{x}$.

$$\begin{aligned}
& x - (\sqrt{x} - 2)^2 = 1 \\
\Longrightarrow\; & x - (x - 4\sqrt{x} + 4) = 1 \\
\Longrightarrow\; & 4\sqrt{x} - 4 = 1 \\
\Longrightarrow\; & x = \frac{25}{16}.
\end{aligned}$$

353. Triangle ABC has sides $2, 3, 4$ cm.



(a) Use the cosine rule to find $\cos\theta$.

(b) Hence, show that $\sin\theta = \frac{1}{4}\sqrt{15}$.

(c) Using $A = \frac{1}{2}ab\sin C$, determine the area of the triangle, giving your answer exactly.

In (b), use $\sin^2\theta + \cos^2\theta = 1$.

(a) $\cos\theta = \dfrac{2^2 + 3^2 - 4^2}{2 \cdot 2 \cdot 3} = -\dfrac{1}{4}.$

(b) Since $\sin^2\theta + \cos^2\theta = 1$, we have $\sin^2\theta = 1 - \frac{1}{16}$. For non-reflex angles, $\sin\theta$ is positive, so $\sin\theta = \frac{1}{4}\sqrt{15}$.

(c) $A = \frac{1}{2}\cdot 2\cdot 3\cdot \frac{1}{4}\sqrt{15} = \frac{3}{4}\sqrt{15}$.

354. Solve the equation $|y^2 - 2| = 5$.

Square both sides...

Squaring both sides get rid of the mod sign, giving $y^4 - 4y^2 + 4 = 25$, which factorises as $(y^2-7)(y^2+3) = 0$. The second factor has no real roots, so $y = \pm\sqrt{7}$.

355. Exactly three forces $\mathbf{F}_1 = 10\mathbf{i}$ N, $\mathbf{F}_2 = 20\mathbf{j}$ N, and $\mathbf{F}_3$ act on an object of mass 10 kg, which accelerates from rest in the $\mathbf{i}+\mathbf{j}$ direction. After t seconds, the object has covered a distance of $s = 2\sqrt{2}t^2$ metres. Find the force $\mathbf{F}_3$.

Find the magnitude of the resultant force using a one-dimensional suvat, then find $\mathbf{F}_3$.

From $s = ut + \frac{1}{2}at^2$, the magnitude of the acceleration must be $4\sqrt{2}$. Since this is in the direction of $\mathbf{i}+\mathbf{j}$, the acceleration is $4\mathbf{i}+4\mathbf{j}$. So the resultant force is $40\mathbf{i}+40\mathbf{j}$. This means $\mathbf{F}_3 = 30\mathbf{i}+20\mathbf{j}$

356. Write down the number of ways of rearranging:

(a) PQRS,

(b) PPQR.

Part (b) involves an overcounting argument: there are half as many arrangements in (b) as (a).

(a) $4! = 24$.

(b) Since two letters are identical, the number of orders is the total rearrangements divided by the arrangements among those two: $\frac{4!}{2!} = 12$.

357. A function f has $f''(x) = 6\sqrt{x}$. The graph $y = f(x)$ has gradient 5 as it crosses the y axis. Find $f'(x)$.

Integrate and find the $+c$.

Integrating $f''(x)$ gives $f'(x) = 4x^{\frac{3}{2}} + c$. Furthermore, we know that $f'(0) = 5$, so $c = 5$. Hence $f'(x) = 4x^{\frac{3}{2}} + 5$.

358. State whether the following hold:

(a) $x^2 = y^2 \implies x = y$,

(b) $x^2 = y^2 \impliedby x = y$,

(c) $x^3 = y^3 \implies x = y$,

(d) $x^3 = y^3 \impliedby x = y$.

Consider $x = -y$.

(a) No, $x = -y$ is a possibility.

(b) Yes.

(c) Yes.

(d) Yes.

359. Solve for x in the equation $ax^2 + bx = 0$, where $a, b \in \mathbb{R}$ and $a \neq 0$.

Despite the unknown constants, this is a standard quadratic and can be factorised as usual.

Factorising gives $x(ax+b) = 0$, so $x = 0$ or $x = -\frac{b}{a}$.

360. Two forces $\mathbf{F} = 4a\mathbf{i}+b\mathbf{j}$ and $\mathbf{G} = (2b+2)\mathbf{i}+(a+3)\mathbf{j}$ act on an object, which remains in equilibrium. Find a and b.

Consider the $\mathbf{i}$ and $\mathbf{j}$ directions separately.

The object must be in equilibrium in both $\mathbf{i}$ and $\mathbf{j}$ directions, so $4a = 2b+2$ and $b = a+3$. Solving simultaneously gives $a = 4$, $b = 7$.

361. Convert the following into units of radians, giving your answers in terms of π:

(a) $30°$,

(b) 4 revolutions.

One revolution is a full circle.

(a) $\frac{\pi}{6}$,

(b) 8π.

362. State, with a reason, whether the line $ax + by = c$ intersects the following lines:

(a) $ay - bx = c+1$,

(b) $ax + by = c+1$,

(c) $ax - by = c+1$.

Consider the gradients of the lines.

(a) Yes: perpendicular to original line.

(b) No: parallel to original line.

(c) Yes, except if $a = b$: reciprocal gradient.

363. Two hikers leave camp simultaneously. A walks on bearing $160°$ at 3 mph; B walks on bearing $250°$ at 4 mph. Determine the time it takes for them to separate by five miles.

Consider Pythagoras.

The bearings are at right-angles, so the speed at which the two are separating is 5 mph, via a $(3, 4, 5)$ triple. So it takes 1 hour.

364. Either prove or disprove the following implications:

(a) $x + y > 1 \implies x^2 + y^2 > 1$,

(b) $x^2 + y^2 > 1 \implies x + y > 1$.

Sketch regions on a graph.

Neither implication is true, as the line $x + y = 1$ and the circle $x^2 + y^2 = 1$ intersect twice. Anything above and to the right of the line, and inside the circle, is a counterexample to the first statement; anything outside the circle, and below and to the left of the line, is a counterexample to the second.

365. Find the length of the space diagonal of a cuboid measuring $4 \times 4 \times 7$.

The space diagonal is the longest diagonal. Use 3D Pythagoras: $d = \sqrt{x^2 + y^2 + z^2}$.

Using 3D Pythagoras, $d = \sqrt{4^2 + 4^2 + 7^2} = 9$.

366. The parabola $y = (x - a)^2 + b$ is rotated $180°$ around its vertex. Write down the equation of the new parabola.

Write down the coordinates of the vertex, and construct the new curve from there.

The vertex is (a, b), so $y = b - (x - a)^2$.

367. Find $\dfrac{dy}{dx}$ when

(a) $y = x(x - 3)(x + 4)$,

(b) $xy = x^2 + 2$.

(a) Multiply out.

(b) Make y the subject.

(a) $\dfrac{dy}{dx} = 3x^2 + 2x - 12$,

(b) Rearranging gives $y = x + \frac{2}{x}$.

So $\dfrac{dy}{dx} = 1 - 2x^{-2} = 1 - \dfrac{2}{x^2}$.

368. Determine the two integer values of x which satisfy $x^2 + 5x + 5 < 0$.

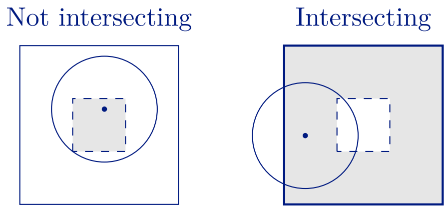
First, solve the inequality without reference to the "integer" fact.

Solving over the reals, the boundary equation gives $x = \frac{-5 \pm \sqrt{5}}{2} = -3.61, -1.38$. The quadratic is less than zero between these, so, over $\mathbb{Z}$, $x \in \{-3, -2\}$.

369. A circle of radius 1 is placed at random with its centre somewhere (equally likely to be anywhere) inside a square of side length 3. Find the probability that the shapes intersect.

Use a version of $p = \frac{\text{successful}}{\text{total}}$, with the possibility space measured by area.

Treating the square as the possibility space of locations of the centre of the circle, we can see that the total area is 9 and the area of the successful region is 8.



So the probability is $\frac{8}{9}$.

370. The following identity is proposed:

$$x^2 \equiv A(x + 1)^2 + B(x - 1)^2.$$

By multiplying out both sides and comparing the coefficients of powers of x, or otherwise, show that no such constants A, B can be found.

One polynomial identity generates multiple individual equations. These cannot necessarily be solved.

Multiplying out and gathering like terms, we get

$$x^2 \equiv (A + B)x^2 + (A + B).$$

The coefficient of x^2 requires $A + B = 1$, but this forces the constant term to be 1, and there is no constant term on the RHS. So this can never be an identity.

371. Prove that, if two functions f and g have the same derivative $f'(x) = g'(x)$, then $y = f(x)$ and $y = g(x)$ are a constant vertical distance apart.

Integrate.

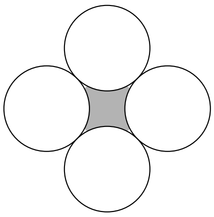
Integrating, we get $f(x) + c = g(x) + d$. These constants can be combined into one, so $f(x) = g(x) + k$. This is equivalent to the graphs $y = f(x)$ and $y = g(x)$ being a constant vertical distance apart.

372. Find all possible values of k such that $(x-1)$ is a factor of $k^2x^3 - 12x^2 + 2k^2$.

Use the factor theorem.

Substituting $x = 1$, we require $k^2 - 12 + 2k^2 = 0$, which implies $k^2 = 4$. So $k = \pm 2$.

373. Four circles of radius 1 are arranged symmetrically, such that each circle is tangent to two of the others and their centres lie at the vertices of a square.



Find the exact area of the shaded region.

Draw in the square and find its area.

The square has area 4, since each side is two radii. Subtracted from this are four quarter-circles, of total area π. So the area shaded is $4 - \pi$.

374. Prove that the difference of two odd numbers is even.

Call the odd numbers $2a+1$ and $2b+1$, for $a, b \in \mathbb{Z}$.

Call the odd numbers $2a + 1$ and $2b + 1$, for $a, b \in \mathbb{Z}$. Then their difference is $2a+1-(2b+1)$, which is equal to $2(a-b)$. Since $a-b$ is an integer, this is an even number. Q.E.D.

375. A hand of three cards is dealt from a standard deck. Find the probability that there are no face cards (Jack, Queen, King) in the hand.

Consider the cards one by one. There are 40 non-face cards in a pack.

We require each card to be one of the 40 non-face cards. This gives

$$p = \frac{40}{52} \times \frac{39}{51} \times \frac{38}{50} = \frac{38}{85}.$$

376. Show that $f(x) = 7x^2 - 9x + 12$ produces outputs that are always greater than 9.

Find the vertex.

Differentiating to find the vertex, we get $14x - 9 = 0$, so $x = \frac{9}{14}$. Substituting gives $y = \frac{255}{28} > 9$. Since the quadratic is positive, all outputs must be greater than 9.

377. True or false?

(a) $\sqrt{x} = \sqrt{y} \iff x = y$,

(b) $\sqrt[3]{x} = \sqrt[3]{y} \iff x = y$,

(c) $\sqrt[4]{x} = \sqrt[4]{y} \iff x = y$.

Consider negative numbers.

Implications (a) and (c) fail to hold for $x = y < 0$, because the roots are undefined. Implication (b) holds.

378. Shade the regions of the plane for which $xy \geq 0$.

Consider the boundary equation $xy = 0$.

The boundary equation $xy = 0$ is solved by $x = 0$ or $y = 0$. These are the x and y axes. So the region in question is two quadrants: $x, y > 0$ and $x, y < 0$.

379. The velocity of a moving particle, at time t, is given by $v = 4t + 6$, and the average velocity over the first k seconds of motion is 20 ms^{-1}.

(a) Show that $\dfrac{1}{k}\Big[2t^2 + 6t\Big]_0^k = 20$.

(b) Solve for k.

Calculate displacement with a definite integral.

(a) The displacement is given by

$$\int_0^k 4t + 6\,dt = \Big[2t^2 + 6t\Big]_0^k.$$

Dividing this by the duration k over which the displacement takes place gives the average velocity 20.

(b) We get $2k + 6 = 20$, so $k = 7$.

380. By factorising, or otherwise, prove that, for $n \in \mathbb{N}$, $n^2 + n$ is always even.

Consider the parity (evenness or oddness) of n and $n + 1$.

Factorising gives $n^2 + n = n(n+1)$. Since n and $n+1$ are two consecutive integers, one of them must be even. Hence, their product must be even. Q.E.D.

381. If $f(x) = 3x^2 - 2x + 1$, find $\int f'(x)\,dx$.

No calculation is needed.

Differentiating then integrating returns the original function, except with an unknown constant. So we get $3x^2 - 2x + c$.

382. Beginning with the first Pythagorean trig identity $\sin^2\theta + \cos^2\theta \equiv 1$, prove, by enacting a suitable division, that $\tan^2\theta + 1 \equiv \sec^2\theta$.

Use $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$.

Division by $\cos^2\theta$ and use of $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ gives the required result.

383. A student is looking for an algebraic slip in his work. Explain the error and correct it.

$$\begin{aligned} \frac{3x-2}{5x-1} - \frac{x-1}{5x-1} &= 1 \\ \Longrightarrow 3x - 2 - x - 1 &= 5x - 1. \end{aligned}$$

It's a minus sign.

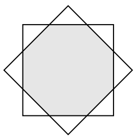
This minus sign $3x - 2 - x - 1 = 5x - 1$ should be a $+$.

384. A sequence is defined by $u_n = 100 - 85n + 3n^2$. Find the minimum value of the sequence.

Use calculus, then return to integers afterwards.

Since this is a quadratic, we can differentiate to find the minimum. We need $-85 + 6n = 0$, so $n = 14\frac{1}{6}$. Since a quadratic is symmetrical, $n = 14$ must give the minimum value, which is -502.

385. Two squares of side length 1 are overlaid to form the symmetrical pattern below.



Show that the area of the shaded octagon is $\sqrt{8} - 2$.

Find the area of the small triangles around the outside.

The small triangles around the outside are right-angled and isosceles. Their perimeters are 1, so their short sides have length l satisfying $2l + \sqrt{2}l = 1$. This gives $l = \frac{1}{2+\sqrt{2}}$. So, four of them together have area $2l^2$, which is $3 - 2\sqrt{2}$. The area of the octagon is then

$$1 - (3 - 2\sqrt{2}) = \sqrt{8} - 2.$$

386. Prove algebraically that, if three variables a, b, c are linked by $a \propto b$ and $b \propto c$, then $a \propto c$.

$a \propto b$ means $a = kb$ for some constant k.

Since $a \propto b$ and $b \propto c$, we know that $a = k_1 b$ and $b = k_2 c$. Substituting gives $a = k_1 k_2 c$. Hence, $a \propto c$.

387. A function g has first derivative $g'(x) = 2x + 3$. Evaluate $g(2) - g(1)$.

This is a definite integral.

$$\int_1^2 2x + 3\,dx = \left[x^2 + 3x\right]_1^2 = (4 + 6) - (1 + 3) = 6.$$

388. Two cards are dealt from a standard deck. State which, if either, of the following events has the greater probability:

- two jacks,
- a king and a queen.

Consider the number of successful outcomes in each case.

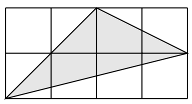
A king and a queen is likelier, as there are $4 \times 4 = 16$ such outcomes, compared to ${}^4C_2 = 6$ for the two jacks.

389. Describe the transformation which takes the graph $y = f(x)$ onto the graph $y = 3f(x) + 2$.

Both transformations are in the y direction.

A stretch factor 3 in the y direction, followed by a translation by vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

390. The grid shown below consists of unit squares.



Find the area of the shaded region.

Subtract the triangles from the rectangle.

The grid has area 8. The three triangles have area $2, 2, 1$. So the shaded region has area 3.

391. Solve the simultaneous equations

$$\begin{aligned} 0.1x + 0.3y &= 0.13, \\ x^2 - 0.6y &= 0.37. \end{aligned}$$

Rearrange the first to make y the subject, and substitute it into the second equation.

We rearrange the first equation:

$$\begin{aligned} &0.1x + 0.3y = 0.13 \\ \implies &30y = 13 - 10x \\ \implies &y = \tfrac{13}{30} - \tfrac{1}{3}x. \end{aligned}$$

Substituting into the second equation gives

$$\begin{aligned} &x^2 - 0.6(\tfrac{13}{30} - \tfrac{1}{3}x) = 0.37 \\ \implies &x^2 + \tfrac{1}{5}x - \tfrac{63}{100} = 0 \\ \implies &x = \tfrac{7}{10}, y = \tfrac{11}{15} \text{ or } x = \tfrac{7}{10}, y = \tfrac{1}{5}. \end{aligned}$$

392. A convex quadrilateral has vertices at $(0,0)$, $(3,8)$, $(9,4)$, and $(10,0)$. Show that its area is 50.

Since two of the vertices of the quadrilateral are on the x axis, it can be split up into three parts: two triangles and a trapezium.

Since two of the vertices of the quadrilateral are on the x axis, it can be split up into three parts: two triangles and a trapezium. So the total area is

$$A = \tfrac{1}{2} \cdot 3 \cdot 8 + \tfrac{1}{2} \cdot 6 \cdot (8 + 4) + \tfrac{1}{2} \cdot 1 \cdot 4 = 50.$$

393. Vectors are given as $\mathbf{p} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{q} = 2\mathbf{i} + \mathbf{j}$. The vector $11\mathbf{i} + 13\mathbf{j}$ is to be written as a linear combination $a\mathbf{p} + b\mathbf{q}$. Find a and b.

Set up and solve simultaneous equations by comparing coefficients of $\mathbf{i}$ and $\mathbf{j}$.

Comparing coefficients of $\mathbf{i}$ and $\mathbf{j}$, we have $a + 2b = 11$ and $2a + b = 13$. Solving simultaneously yields $a = 5$, $b = 3$.

394. Position x takes the following values at time t:

t	0	5	10	15
x	4	5	8	13

Show that this data is consistent with the assumption of constant acceleration.

Consider the second differences.

Since t increases linearly, we can treat the x values as a sequence. We require that the second difference (corresponding to the second derivative) is constant. The first differences are $1, 3, 5$ and the second differences are $2, 2$. So, the data is consistent with the assumption of constant acceleration.

395. Consider the function $g(x) = 224x^2 - 649x + 255$.

(a) Show that g has two rational roots.

(b) Explain why, if $x = \frac{b}{a}$ is a root of g, then $(ax - b)$ must be a factor of $g(x)$.

(c) Hence, factorise $g(x)$.

In (a), just solve using the formula.

(a) Using the quadratic formula, if $g(x) = 0$, then

$$\begin{aligned} x &= \frac{649 \pm 649^2 - 4 \cdot 224 \cdot 255}{2 \cdot 224} \\ &= \frac{649 \pm 439}{448} \\ &= \frac{17}{7}, \frac{15}{16}. \end{aligned}$$

(b) The factor theorem says that if $x = \frac{b}{a}$ is a root, then $(x - \frac{b}{a})$ is a factor. A constant multiple of a linear factor is also a factor, so, multiplying by the constant a, we know that $(ax - b)$ is a factor.

(c) From the above, we know that $(7x - 17)$ and $(16x - 15)$ are factors. Since e.g. $7 \times 16 = 112$, we also need a constant factor of two. Hence, $g(x)$ factorises as

$$224x^2 - 649x + 255 \equiv 2(7x - 17)(16x - 15).$$

396. By listing them explicitly, show that there are ten possible rearrangements of $AAABB$.

List alphabetically.

Listing alphabetically:

$AAABB$	$ABBAA$
$AABAB$	$BAAAB$
$AABBA$	$BAABA$
$ABAAB$	$BABAA$
$ABABA$	$BBAAA$

397. The interior angles of a triangle are in arithmetic progression. Give, in radians, the set of possible values for the smallest angle.

Consider the average value of the angles.

The average value of the angles must be $\frac{\pi}{3}$. The smallest angle can be anything from zero to this average (since $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$ is trivially an AP). Hence, the set of possible values for the smallest angle is $(0, \frac{\pi}{3}]$.

398. You are given that the equations $3x - py = 5$ and $6x = 2y + q$ have no simultaneous solutions (x, y).

(a) Explain why the two equations must describe a pair of parallel lines.

(b) Write down the value of p.

(c) Determine all possible values of q.

...

(a) Any pair of non-parallel lines intersects at precisely one point, generating a simultaneous solution.

(b) For the lines to be parallel $p = 1$.

(c) For the lines to be distinct (otherwise they would have infinitely many solutions), $q \neq 10$.

399. Explain why no quadratic function has range $\mathbb{R}$.

Consider the shape of a parabola.

The range is the set of values attainable by the function. Every quadratic function produces a parabolic graph, which by definition has either a minimum or a maximum. Values one side of this are not attainable by the function, hence there must always be real numbers which are not in the range of any given quadratic function.

400. State, with a reason, whether the following claims are true in the Newtonian system:

(a) "An object with mass cannot be weightless."

(b) "An object with weight cannot be massless."

Consider an object floating in space.

(a) This is false. An asteroid in deep space has mass but is weightless.

(b) This is true. In the Newtonian system, only massive objects experience gravitational force (weight).

401. Assuming that none of a, b, c, d have common factors greater than 1, write down the roots of

$$\frac{(x-a)^2(x+b)^3}{(x-c)^2(x+d)^3} = 0.$$

A fraction is only zero if...

A fraction in its lowest terms is zero if and only if its numerator is zero. Since none of a, b, c, d have common factors greater than 1, that is the case here. The powers of the brackets make no difference, so the roots are $x = a, -b$.

402. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $x^4 = 9$,
- $x^2 = 3$.

...

The implication goes both ways, since neither quantity x^4 or x^2 can be negative: $\Longleftrightarrow$.

403. Find $\dfrac{dy}{dx}$, when $y = \dfrac{2(x+1)^2(x+2)}{\sqrt{x}}$.

Multiply out and split up the fraction.

Multiplying out and splitting up the fraction:

$$\begin{aligned} y &= \frac{2x^3 + 8x^2 + 10x + 4}{\sqrt{x}} \\ &= 2x^{\frac{5}{2}} + 8x^{\frac{3}{2}} + 10x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}. \end{aligned}$$

Then differentiating, we get

$$\frac{dy}{dx} = 5x^{\frac{3}{2}} + 12x^{\frac{1}{2}} + 5x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}}.$$

404. By considering the equation as a quadratic in $\sqrt{p}$, rearrange $ap + b\sqrt{p} + c = 0$ to make p the subject.

Use the quadratic formula.

Since this is a standard quadratic, with coefficients a, b, c, in the variable $\sqrt{p}$, it is solved by $\sqrt{p} =$ quadratic formula. Hence

$$p = \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)^2.$$

405. Simplify the following, where $n \in \mathbb{Z}$, giving your answers in standard form:

(a) $3 \times 10^n - 2.8 \times 10^n$,

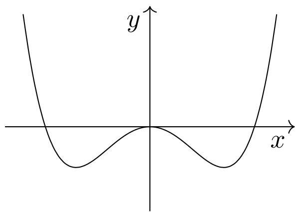
(b) $3 \times 10^n \div (4 \times 10^{n+5})$.

After performing the calculation, trade factors of 10 between the two parts of the standard form.

(a) Trivially, this is 0.2×10^n, but this is not in standard form, so move a factor of ten from the right to the left. This gives $2 \times 10^{n-1}$.

(b) We get 0.75×10^{-5}, which, in standard form, is 7.5×10^{-6}.

406. State, with a reason, whether the equations below could possibly be that of the sketched graph.



(a) $y = x^4 - x$,

(b) $y = x^4 - x^2$,

(c) $y = x^4 - x^3$.

Factorise the equations given.

Factorising the equations given, we see that only $y = x^4 - x^2 = x^2(x+1)(x-1)$ has a double root at zero. Hence, only (b) could be the equation of the graph.

407. A student writes: "$x(x-1) = 0$, which implies that $x = 0$ and $x = 1$." Explain the logical error, and correct it.

The word "and" is wrong.

The words "implies" and "and" do not fit here. It would be reasonable to say either 1) "The roots of the equation are $x = 0$ and $x = 1$" or 2) "The equation implies $x = 0$ or $x = 1$."

408. It is possible to measure the depth of a well by dropping a stone in and timing the splash. Show that the formula for depth d in metres is given by $d \approx 5t^2$, where t is measured in seconds.

Use suvat.

Using suvat, we have an object dropped from rest, so, neglecting air resistance, $s = \frac{1}{2}gt^2$. In this case, displacement until the splash is depth d, and $g \approx 10$, so $d \approx 5t^2$.

409. A cuboid has dimensions $x \times 2x \times 3x$ units. The total surface area is 88 square units. Find x.

Find the surface area in terms of x.

Forming an equation for surface area, we have $2(x \cdot 2x + x \cdot 3x + 2x \cdot 3x) = 88$. This gives $x^2 = 4$. Since x must be positive, $x = 2$.

410. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $\sin x = \frac{1}{2} \implies x = \arcsin \frac{1}{2}$,

(b) $\sin x = \frac{1}{2} \impliedby x = \arcsin \frac{1}{2}$.

Consider the unit circle or a graph.

The first statement is incorrect. There are infinitely many values x which gives $\sin x = \frac{1}{2}$, for instance $x = 150°$.

411. Find the area of triangle ABC, with vertices at $A : (0, 2)$, $B : (3, 4)$ and $C : (1, 6)$.

Enclosed the triangle in a rectangular box, whose sides are parallel to the x and y axes.

Enclose the triangle in a rectangular box, with vertices at $(0, 2)$ and $(3, 6)$. This has area 12, from which we must subtract triangles with areas $2, 2, 3$. This leaves 5.

412. Events X, Y, Z are such that $P(X \cap Y \cap Z) = \frac{1}{2}$ and $P(X \cap Y) = P(Y \cap Z) = P(Z \cap X) = \frac{2}{3}$. By drawing a Venn diagram, or otherwise, show that at least two of the events must occur.

Find $P(X \cap Y \cap Z')$ and similar.

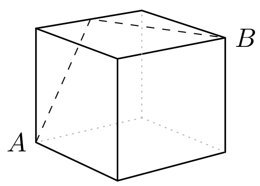
Removing the three-way intersection, we can see that $P(X \cap Y \cap Z') = \frac{2}{3} = \frac{1}{2} = \frac{1}{6}$. The same is true for rearrangements of X, Y, Z. This gives a total probability, within the two-way intersections, of $\frac{1}{2} + 3 \cdot \frac{1}{6} = 1$, which is all of it. Hence, it is guaranteed that at least two of the events happen.

413. By completing the square, show that the parabola $y = ax^2 + bx + c$ has a line of symmetry at $x = -\frac{b}{2a}$.

...

Completing the square, we get $y = a(x - \frac{b}{2a})^2 - \frac{b^2}{4a} + c$. Since the algebra is symmetrical around $x = -\frac{b}{2a}$, so must the graph be.

414. The diagram shows a solid cube of unit side length, with the shortest path between vertices A and B marked.



Determine the length of marked path.

Unfold the two faces to form a flat rectangle.

Unfolding the two faces to a flat rectangle, the length of the path is given by the hypotenuse of a right-angled triangle with sides of length 1 and 2. Hence, the path has length $\sqrt{5}$.

415. Simplify $\left(\dfrac{x-a}{x-b}\right)^{-\frac{1}{3}} \div \left(\dfrac{a-x}{b-x}\right)^{\frac{2}{3}}$.

Remember that $(x - a) = -(a - x)$.

The fractions inside the the brackets are the same, in fact, because $(x-a) = -(a-x)$. Hence, we can simply subtract the powers by the usual index law. This gives a power of -1, which is a reciprocal. So the full simplification is

$$\left(\frac{x-a}{x-b}\right)^{-\frac{1}{3}} \div \left(\frac{a-x}{b-x}\right)^{\frac{2}{3}} \equiv \frac{x-b}{x-a}.$$

416. You are given that, for some constants $p, q, r > 0$, the following is an identity x and y:

$$(px-qy)^2 \equiv 4x^2 - 12xy + ry^2.$$

Find the constants p, q, r.

Multiply out and compare coefficients.

Multiplying out and comparing coefficients, we get $p^2 = 4$, $-2pq = -12$, and $q^2 = r$. Since $p, q, r > 0$, we must have $p = 2, q = 3$ and $r = 9$.

417. Prove, from first principles, that the gradient of the locus of the equation $y = 4x + 7$ is 4.

Set up the gradient triangle $\dfrac{4(x+h)+7-(4x+7)}{h}$.

We set up and simplify a gradient triangle, in which the width of the triangle h is not equal to zero, as follows:

$$\begin{aligned} &\frac{4(x+h)+7-(4x+7)}{h} \\ = &\frac{4x+4h+7-4x-7}{h} \\ = &\frac{4h}{h} \\ = &\ 4 \text{ since } h \neq 0. \end{aligned}$$

Hence, the gradient is 4.

418. The circle $(x-2)^2 + (y-3)^2 = 1$ is translated by $4\mathbf{i}$. Write down the equation of the new graph.

Replace x by $x-4$.

Replacing x by $x-4$, we get $(x-6)^2 + (y-3)^2 = 1$.

419. Write the following speeds of rotation in units of revolutions per minute:

(a) $60°$ per second.

(b) 240π radians per hour.

One revolution is 360°is 2π radians.

(a) $60°$ per second is $\frac{1}{6}$ of a revolution per second, which is 10 rpm.

(b) 240π radians per hour is 120 revolutions per hour, which is 2 rpm.

420. Describe the transformation that takes the graph $y = (x-a)^2$ onto the graph $y = k(x-b)^2$.

Consider the input and the output transformations separately.

The output transformation is a stretch factor k in the y direction. The input transformation is a replacement of $(x-a)$ by $(x-b)$, which is equivalent to a replacement of x by $(x-(b-a))$. This is a shift by vector $(b-a)\mathbf{i}$.

421. (a) Evaluate $6x^3 - 23x^2 + 24x - 7\,\big|_{x=\frac{7}{3}}$.

(b) Hence, write the expression $6x^3-23x^2+24x-7$ as the product of three linear factors.

Use the factor theorem.

Evaluation gives zero. So $(3x-7)$ is a factor. Taking out this factor, we get

$$6x^3 - 23x^2 + 24x - 7 \equiv (3x-7)(2x-1)(x-1).$$

422. A die is rolled. Given that the result is at least 3, find the the probability that it is at least 5.

Consider the restriction of the possibility space by the given information.

Since the result is at least three, the possibility space is restricted to $\{3, 4, 5, 6\}$. So, $p = \frac{1}{2}$.

423. A programmer sets up a function to answer the following question: "How many pairs of parallel sides does quadrilateral Q have?" Write down

(a) the largest possible domain,

(b) a suitable codomain,

(c) the range, with the domain given.

The domain is the set of all possible inputs; the codomain is a coverall term for the type of output; the range is the specific set of outputs attainable.

(a) The set of quadrilaterals.

(b) $\mathbb{N}$ (or anything containing $\{0, 1, 2\}$).

(c) $\{0, 1, 2\}$.

424. Evaluate $\left.\dfrac{x! + \sin x}{x! + \cos x}\right|_{x=0}$

0! is defined to be 1.

Since 0! is defined to be 1, this is $\frac{1+0}{1+1} = \frac{1}{2}$.

425. A value x is chosen randomly from $[-2, 2]$. By solving the inequality $x^2 < x$, find $P(x^2 < x)$.

Consider the possibility space as $[-2, 2]$, and find the successful subset.

The inequality $x^2 < x$ has solution $(0, 1)$. As a fraction of the possibility space $[-2, 2]$, this is $\frac{1}{4}$. (The inclusion or exclusion of the endpoints is not relevant to this calculation, because, the reals being continuous, the probability of exactly 1 is zero.)

426. Determine the area of the region of the (x, y) plane whose points simultaneously satisfy the following inequalities:

$$\begin{aligned} -1 &\le x \le 4, \\ 2 &\le y \le 6. \end{aligned}$$

The region is a rectangle.

The region is a rectangle with width 5 and height 4, so its area is 20.

427. Determine the coefficient of x^4 in $(3x - 1)^5$.

Use the binomial expansion.

The coefficient is ${}^5C_4(3)^4(-1) = -405$.

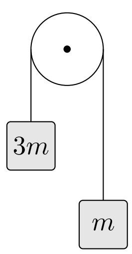
428. By factorising, solve the following quadratic in 2^x:

$$8 \cdot (2^x)^2 - 33 \cdot (2^x) + 4 = 0.$$

Write as $(8 \cdot 2^x - ...)(2^x - ...) = 0$.

Factorising gives $(8 \cdot 2^x - 1)(2^x - 4) = 0$. So the solution is $x = -3, 2$.

429. Two masses are connected by a light, inextensible string, which is passed over a smooth, light, fixed pulley as shown in the diagram.



(a) State the assumptions which allow modelling
   i. the tension in the string as constant,
   ii. the accelerations of the masses as equal in magnitude.

(b) Draw separate force diagrams for the masses.

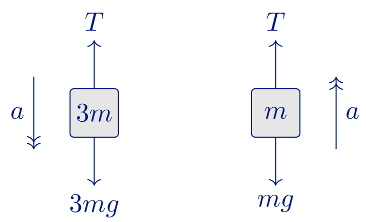
(c) Show that the acceleration is $\frac{1}{2}g$, and find the tension in the string.

(d) State, with a reason, what would happen to those values if the pulley were rough.

...

(a) To model
   i. the tension in the string as constant, we require that the string is light and that the pulley is smooth,
   ii. the accelerations of the masses as equal in magnitude, we require that the string is inextensible (and that all motion is vertical).

(b) Force diagrams:



(c) Vertical $F = ma$ gives $3mg - T = 3ma$ and $T - mg = ma$. Adding these gives $2mg = 4ma$, so $a = \frac{1}{2}g$. Then $T = \frac{3}{2}mg$.

(d) The acceleration would be reduced for obvious reasons. The tension would no longer be the same on either side of the pulley: it would be larger on the left and smaller on the right.

430. If $z = a^x$, write $(a^3)^x$ in terms of z.

Use an index law.

$(a^3)^x = a^{3x} = (a^x)^3 = z^3$.

431. Prove by contradiction that every quadrilateral must have an interior angle that is not acute.

Begin with the line "Assume, for a contradiction, that quadrilateral Q has four acute interior angles."

Assume, for a contradiction, that quadrilateral Q has four acute interior angles. An acute angle is $\theta < 90°$, so the four angles $\theta_1 + \theta_2 + \theta_3 + \theta_4 < 360°$. But this is impossible, since the interior angles of a quadrilateral add to $360°$. Hence, every quadrilateral must have an interior angle that is not acute. Q.E.D.

432. Two variables x and y depend on t as follows: x is proportional to the cube root of t, and y^2 depends linearly on t. Show that $y^2 = ax^3 + b$, for constants $a, b \in \mathbb{R}$.

Turn the worded statements into algebra, and eliminate t.

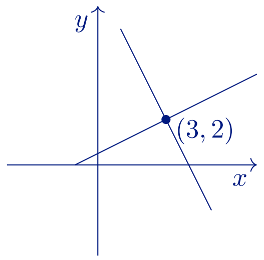
The worded statements translate algebraically as $x = k\sqrt[3]{t}$ and $y^2 = mt + b$. Rearranging the first to make t the subject, we get $t = k'x^3$, using k' to represent a new version of k. Substituting, this gives $y^2 = mk'x^3 + b$. Renaming the first constant a gives the required result.

433. On the same axes, for some constant $m \in (0, 1)$, sketch the graphs

(a) $\dfrac{y-2}{x-3} = m$, (b) $\dfrac{y-2}{x-3} = -\dfrac{1}{m}$.

These are both straight lines through $(3, 2)$.

These are perpendicular lines, intersecting at the point $(3, 2)$:



434. True or false?

(a) $\displaystyle\sum_{i=1}^{n} ax_i = a\sum_{i=1}^{n} x_i$,

(b) $\displaystyle\sum_{i=1}^{n}(b + x_i) = b + \sum_{i=1}^{n} x_i$.

Write the sums out longhand.

(a) This is true. The factor a is common to all terms and can be taken out.

(b) This is false. The term b is added to all terms, so, if taken out, it should be $n \times b$.

435. Find the area of the annular region satisfying

$$1 \geq x^2 + y^2 \geq 4.$$

"Annular" means shaped like a ring.

The annular region is a ring: a disc of radius 2 minus a disc of radius 1. (Disc is the technical name for a circle together with the area within it, i.e. a solid circle rather than a circumference). So the area is $4\pi - 1\pi = 3\pi$.

436. "The coordinate axes are normal to the curve $x^2 + y^2 = 1$." True or false?

In geometry, "normal" means "perpendicular to".

The coordinate axes are perpendicular to the unit circle, so this is true.

437. You are given that, for some constant $p \in \mathbb{R}$, the quadratic equation $2px^2 + 8x + 2p + 15 = 0$ has exactly one root.

(a) Show that the discriminant $\Delta = b^2 - 4ac$ of this quadratic equation is $64 - 120p - 16p^2$.

(b) Solve a suitable equation to determine the two possible values of p.

...

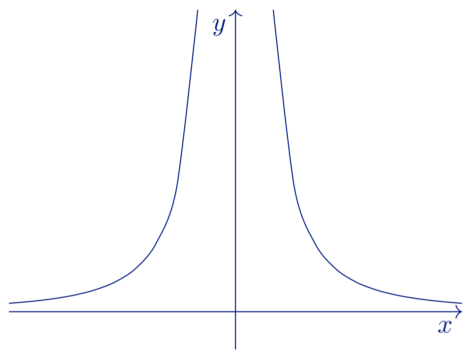
(a) $\Delta = 8^2 - 4 \cdot 2p \cdot (2p + 15)$, which simplifies to $64 - 120p - 16p^2$.

(b) The original quadratic has exactly one root, so $\Delta = 0$. Factorising, we have $(8+p)(8-16p) = 0$, so $p = -8, \frac{1}{2}$.

438. Sketch the curve $x^2y = 1$.

Compare to $xy = 1$, for negative x.

This is $y = \frac{1}{x^2}$, which is similar to $y = \frac{1}{x}$, except that negative y values are squared to give positive y values. So the graph is



439. Simplify the following sets:

(a) $[0, 2) \cap [1, 3)$,

(b) $(0, 2) \cap [1, 3]$,

(c) $[0, 2] \cap (1, 3)$.

Square brackets are inclusive (they include the "corners"), while round brackets are not inclusive (they don't include the "corners").

(a) $[0, 2) \cap [1, 3) = [1, 2)$,

(b) $(0, 2) \cap [1, 3] = [1, 2)$,

(c) $[0, 2] \cap (1, 3) = (1, 2]$.

440. Find the range of the function $f(x) = x^2 - 4$, on the domain $(-1, 3)$.

Sketch the curve.

The domain includes the vertex of the parabola, which is the minimum -4. The maximum is given, then, by the higher of the values at the extremes of the domain, which must, by symmetry, be $f(3) = 5$. Hence the range is $[-4, 5)$.

441. Two forces of magnitude 3 N and 4 N act on an object of mass 5 kg. Determine the magnitude of the acceleration of the object if the forces are

(a) parallel,

(b) perpendicular,

(c) antiparallel.

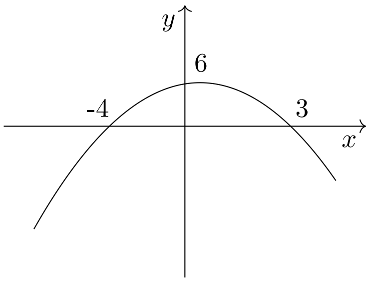
Use Pythagoras for (b).

(a) parallel: $a = \frac{7}{5}$ ms^{-2},

(b) perpendicular: $a = 1$ ms^{-2},

(c) antiparallel: $a = \frac{1}{5}$ ms^{-2}.

442. Find the equation of the parabola shown below, on which the axes intercepts have been marked, giving your answer in expanded polynomial form.



Use the factor theorem.

By the factor theorem, we require $y = k(x-3)(x+4)$. Substituting $x = 0, y = 6$, $k = -\frac{1}{2}$. Multiplying out gives $y = -\frac{1}{2}x^2 - \frac{1}{2}x + 6$.

443. Determine the coordinates of the intersections of $x^2 + (y+2)^2 = 1$ and $(x+3)^2 + (y+2)^2 = 10$.

Solve simultaneously by eliminating $(y+2)^2$.

Subtracting the equations, we get $(x+3)^2 - x^2 = 9$, so $6x + 9 = 9$, so $x = 0$. This gives $y = \pm 1 - 2$, so the coordinates are $(0, -1)$ and $(0, -3)$.

444. Find the surface area of an equilateral triangular prism, all of whose edges have length 1.

Such an equilateral triangular prism has three square faces and two equilateral faces.

Such an equilateral triangular prism has three square faces and two equilateral triangular faces. The squares have total area 3; the equilateral triangles have total area $\frac{\sqrt{3}}{2}$. So the total surface area is $A = 3 + \frac{\sqrt{3}}{2}$.

445. $(3, 0)$ and $(-2, 0)$ lie on a parabola $y = f(x)$.

(a) Explain why the parabola must have the form $y = k(x^2 - x - 6)$.

(b) Determine the value of k if the parabola also passes through the point $(2, -16)$.

Use the factor theorem.

(a) By the factor theorem, since $(x-3)(x+2) = x^2 - x - 6$.

(b) Substituting $(2, -16)$ gives $k = 4$.

446. Find the probability that two consecutive dice rolls give different scores.

Consider the rolls one by one.

It makes no difference what the first roll is. The probability that the second is different is then $\frac{5}{6}$.

447. The definite integral below gives the displacement, over a particular time period, for an object moving with constant acceleration:

$$s = \int_0^3 2t + 5\, dt.$$

(a) Write down the initial velocity, acceleration and duration of the motion.

(b) Calculate the displacement.

Displacement is the integral of velocity.

(a) The integrand $2t + 5$ is the velocity, so the initial velocity is 5, the acceleration is 2, and the duration is $3 - 0 = 3$.

(b) Integrating gives a displacement of 24. (There are no units given in the question, so this quantity is unitless.)

448. Write down the area scale factor when $y = f(x)$ is transformed to $y = f(kx)$.

This is an input transformation: consider the effect on $y = f(x)$ in the x direction.

Replacing x by kx is a stretch factor $\frac{1}{k}$ in the x direction. Since there is no change in the y direction, this is also the area scale factor.

449. Find p in terms of q, if the quadratic $x^2 + x + p$ has a factor of $x - q$.

Use the factor theorem.

Since the quadratic has a factor of $x - q$, we know that $x = q$ makes the quadratic zero. Hence, $q^2 + q + p = 0$. Therefore $p = -q^2 - q$.

450. State, with a reason, whether the line $y = x + k$ intersects the following lines:

(a) $x = k + 1$,

(b) $y = k + 1$.

Sketch the behaviour for a particular value of k.

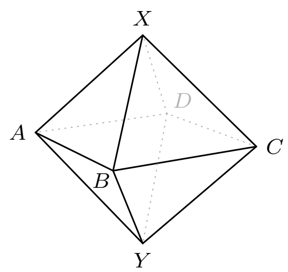
Since $y = x + k$ is a line, k must be a constant. Thus $x = k + 1$ and $y = k + 1$ are horizontal and vertical lines. So $y = x + k$, which has gradient 1, must intersect them both.

451. A cat has velocity $\mathbf{i} - \sqrt{3}\mathbf{k}$ ms^{-1}. Find its speed.

Use Pythagoras.

By Pythagoras, $v = \sqrt{1 + 3} = 2$ ms^{-1}.

452. Depicted is a regular octahedron of edge length l.



Determine the distance XY.

By symmetry, $|XY| = |AC|$.

XY is the diagonal of a square of length l, so $|XY| = \sqrt{2}l$.

453. Solve for a in the mechanical equations

$$\begin{aligned} T_1 - 4mg &= 4ma, \\ T_2 - T_1 &= 7ma, \\ 9mg - T_2 &= 9ma. \end{aligned}$$

Add all three equations together.

Adding all three equations together eliminates T_1 and T_2, and gives $5mg = 20ma$. So $a = \frac{1}{4}g$.

454. Find the linear function g for which

$$\int_0^3 g(x)\,dx = 6, \quad g(0) = 2.$$

Substitute $g(x) = ax + 2$, then solve for a.

Since $g(0) = 2$, we know that $g(x) = ax + 2$. Integrating gives $\frac{1}{2}a(3^2) + 2(3) = 6$. Hence, $a = 0$ and the linear function is constant: $g(x) = 2$.

455. Graph G is given by $y = \sin 6x + \tan 4x$, with the trigonometric functions defined in degrees.

(a) Write down the periods of

i. $y = \sin x$ and $y = \tan x$,

ii. $y = \sin 6x$ and $y = \tan 4x$.

(b) Hence, show that G has period 180°.

The periods of sin and tan are 360° and 180°. Transform these according to the relevant input transformations.

(a) The periods are

i. 360° and 180°,

ii. 60° and 45°

(b) The period of the sum is given by the lowest common multiple of the individual periods, so G has period 180°.

456. Find the distance between $(a - 1, b + 2, c - 4)$ and $(a + 1, b - 1, c + 2)$.

Use 3D Pythagoras.

3D Pythagoras gives $d = \sqrt{2^2 + 3^2 + 6^2} = 7$.

457. By integrating, determine all functions f which satisfy the following differential equation, in which m is a constant: $f'(x) = m$.

...

Integrating gives $f(x) = mx + c$. So the family of solutions is the set of straight lines of gradient m.

458. You are given that 2 is a root of f, where

$$f(x) = 8x^3 - 22x^2 + 13x - 2.$$

Factorise $f(x)$.

Use the factor theorem.

We take out $(x - 2)$, leaving a quadratic. This factorises exactly, giving $(x - 2)(4x - 1)(2x - 1)$.

459. Show that the normal to $y = x^3$ at $x = 1.2$ crosses the y axis close to $y = 2$.

Differentiate $y = x^3$ and find the equation of the normal.

At $x = 1.2$, $\frac{dy}{dx} = \frac{108}{25}$. So the gradient of the normal is $-\frac{25}{108}$. The equation of the normal is then $y = -\frac{25}{108}x + c$. Substituting $x = 1.2, y = 1.2^3$, we get $c = 2.0057 \approx 2$.

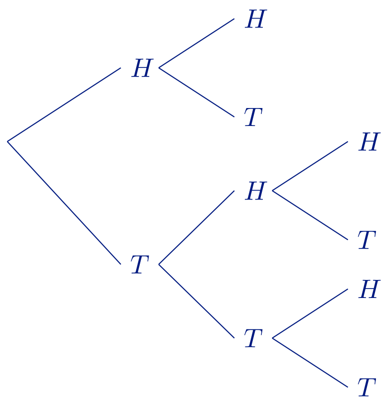
460. A coin is tossed. If it shows heads, then it is tossed again once. If it shows tails, then it is tossed again twice.

(a) Draw a tree diagram to represent this.

(b) Show that the probability of exactly two heads showing overall is $\frac{3}{8}$.

The two main branches of the tree diagram should have different lengths.

(a) Tree diagram, in which all branches have probability $\frac{1}{2}$.

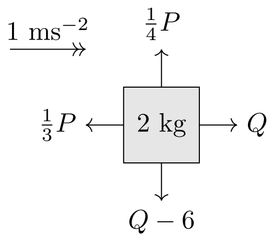


(b) The probability of exactly two heads showing is that of HH and THH: $\frac{1}{2}^2 + \frac{1}{2}^3 = \frac{3}{8}$.

461. The parabola $y = ax^2 + bx + c$ is reflected in $y = x$. Write down the equation of the new parabola.

Consider the effect on the point (p, q) of reflecting in $y = x$. Reflecting in $y = x$ switches the x and y coordinates, giving $x = ay^2 + by + c$.

462. Four forces, with magnitudes given in Newtons, cause a 2 kg mass to accelerate as depicted below:



Solve to find P and Q.

Set up and solve two simultaneous equations, horizontal $F = ma$ and vertical $F = ma$.

Horizontal and vertical $F = ma$ give $Q - \frac{1}{3}P = 2$ and $\frac{1}{4}P - (Q - 6) = 0$. Adding the equations gives $P = 48$ N, then $Q = 18$ N.

463. Give the formula for the interior angle θ, defined in radians, of a regular n-gon.

A regular n-gon can be split up into $n-2$ triangles.

Since a regular n-gon can be split up into $n-2$ triangles, the sum of the interior angles is $(n-2)\pi$. Dividing by n gives $\frac{n-2}{n}\pi$.

464. A piece is placed on a random square of an eight by eight chessboard. Find the probability that it is away from the edge of the board.

Treat the board as the possibility space.

Counting outcomes in the possibility space, 36 squares of 64 are away from the edge. Hence, the probability is $\frac{36}{64} = \frac{9}{16}$.

465. Show that the line through the points (a, b) and (b, a), where $a \neq b$, has gradient -1.

Calculate $\frac{\Delta y}{\Delta x}$ and simplify.

The gradient is $\frac{\Delta y}{\Delta x} = \frac{b-a}{a-b} = -1$.

466. Disprove the following statement:

If $f(x) \not\equiv g(x)$, then $f'(x) \not\equiv g'(x)$.

A counterexample to this is two functions f and g which are not identical, but do have identical derivatives.

Any two functions which differ by a constant are a counterexample, such as $f(x) = e^x$ and $g(x) = e^x + 1$.

467. An isosceles triangle, area 60, has sides $(13, 13, n)$.

(a) Using the area formula, find the two possible angles between the sides of length 13.

(b) Hence, use the cosine rule to determine the two possible values of n.

Use $\frac{1}{2}ab \sin C$.

(a) Setting up $\frac{1}{2} \cdot 13^2 \cdot \sin\theta = 60$, we have $\theta = \arcsin\frac{120}{169}$ or $\theta = 180° - \arcsin\frac{120}{169}$.

(b) The cosine rule gives $n^2 = 2 \cdot 13^2 - 2 \cdot 13 \cdot \cos\theta$. Plugging in the two possible angles and taking the positive square root gives $n = 10, 24$.

468. Simplify $(a-1, a] \cup [a, a+1)$.

Sketch a number line.

The two intervals overlap at a, so the union is just $(a-1, a+1)$.

469. Two polynomial functions f and g are such that $f(x) = g'(x)$. Prove that, if the minimum value of $g(x)$ occurs at $x = \alpha$, then f has a root at α.

Consider stationary points.

If the minimum value of $g(x)$ occurs at $x = \alpha$, then $g(x)$ must be stationary there, so $g(\alpha) = 0$. This means $f(\alpha) = 0$, which in turn means that f has a root at α.

470. A ballistic missile is fired at 60° above horizontal, with speed $196\sqrt{3}$ ms^{-1}.

(a) Find the initial vertical speed.

(b) Assuming projectile motion, show that the greatest height attained is 4.41 km.

In (b), the greatest height is attained when the vertical velocity is instantaneously zero.

(a) $u_y = 196\sqrt{3}\sin 60 = 294$ ms^{-1}.

(b) Assuming projectile motion, acceleration is g downwards. For the greatest height attained, $0^2 = 294^2 - 2gh$. With $g = 9.8$, we get $h = 4410$ m, which is 4.41 km.

471. A polynomial f has $f''(k) = 0$ for some $k \in \mathbb{R}$. Show that f cannot be a quadratic.

Prove this by contradiction.

Assume, for a contradiction, that f is quadratic, so $f(x) = ax^2 + bx + c$. Differentiating twice gives $f''(x) = 2a$, which is constant. So, if $f''(k) = 0$ for some $k \in \mathbb{R}$, then a must be zero. But if $a = 0$, f is not a quadratic function, which is a contradiction. So, f cannot be a quadratic function. $\square$

472. Two cards are picked together from a standard deck of 52. Find the probability that

(a) both cards are aces,

(b) neither card is red.

Consider the cards one after another.

(a) $\frac{4}{52} \times \frac{3}{51} = 0.00452$ (3sf).

(b) $\frac{26}{52} \times \frac{25}{51} = 0.245$ (3sf).

473. State whether the following hold:

(a) $x \in [a, b] \implies x \in (a, b)$,

(b) $x \in [a, b] \impliedby x \in (a, b)$,

(c) $x \notin [a, b] \implies x \notin (a, b)$,

(d) $x \notin [a, b] \impliedby x \notin (a, b)$.

Consider the boundary values $x = a$ and $x = b$ as possible counterexamples to the implications.

(a) No. (b) Yes. (c) Yes. (d) No.

Note that (a) and (d) are equivalent statements, as are (b) and (c).

474. The graph $y = px^2 - x - 2$ crosses the x axis at $x = -\frac{2}{3}$ and $x = q$. Find p and q.

Substitute values and solve.

Substituting $x = -\frac{2}{3}$, we get $0 = \frac{4}{9}p + \frac{2}{3} - 2$, so $p = 3$. Then substituting $x = q$, we get $3q^2 - q - 2 = 0$. This factorises as $(3q+2)(q-1) = 0$. We are looking for the root that is not $-\frac{2}{3}$, so $q = 1$.

475. For a function $f(x)$ and constants a, b, k,

$$\int_a^b f(x)\,dx = k.$$

Evaluate the following definite integrals, leaving your answers in terms of a, b and k.

(a) $\displaystyle\int_a^b 3f(x)\,dx$,

(b) $\displaystyle\int_a^b 2x - f(x)\,dx$.

Constant factors can be taken out of integrals, and sums can be split up.

(a) $\displaystyle 3\int_a^b f(x)\,dx = 3k$

(b) $\displaystyle\int_a^b 2x\,dx - \int_a^b f(x)\,dx = b^2 - a^2 - k$.

476. Complete the square in $6x^4 + 24x^2 + 13$.

This is a quadratic in x^2.

Completing the square gives $6(x^2+2)^2 - 24 + 13$, which simplifies to $6(x^2+2)^2 - 11$.

477. Consider the formula $\displaystyle\sum_1^n r = \frac{1}{2}n(n+1)$.

(a) Verify the formula explicitly, for $n = 5$.

(b) Prove the formula, by adding the following two equations:

$$S_n = 1 + 2 + ... + (n-1) + n,$$
$$S_n = n + (n-1) + ... + 2 + 1,$$

In (b), adding the sums gives n copies of the same quantity.

(a) $1 + 2 + 3 + 4 + 5 = 15$, and $\frac{1}{2} \cdot 5 \cdot 6 = 15$.

(b) Adding the equations gives $2S_n = n(n+1)$, since each pair of terms on the RHS adds to $n + 1$. Dividing by 2 gives the required result.

478. Find $\displaystyle\int \frac{5}{16}\left(2\sqrt{t}\right)^3\, dt$.

Multiply out first.

This is $\displaystyle\int \frac{5}{2} t^{\frac{3}{2}}\, dt = t^{\frac{5}{2}} + c$.

479. Prove that $(2m+1)(2n+1) - 1$ cannot be prime for any for any $m, n \in \mathbb{N}$.

Multiply out and factorise.

Multiplying out gives $4mn + 2m + 2n$, which is $2(2mn + m + n)$. And, since $m, n \in \mathbb{N}$, $(2mn + m + n)$ is an integer greater than 1. Hence, the number cannot be prime.

480. The straight line segment $x = 4\lambda$, $y = 3 - 2\lambda$, for $\lambda \in [0, 4]$ is reflected in the line $y = x$ and then translated by $5\mathbf{j}$. Write down the equation of the new line, in the same form.

Switch x and y, then add 5 to y.

Reflection is enacted by switching the variables x and y. Then we add 2 to y. Overall, this gives

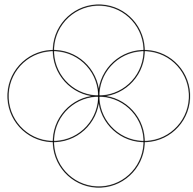
$$x = 3 - 2\lambda, y = 4\lambda + 5 \text{ for } \lambda \in [0, 4].$$

481. By factorising fully, solve $x^5 - 8x^3 + 16x = 0$.

Factorise to a biquadratic, which is a quadratic in x^2.

Take out the factor of x first, leaving a quartic: $x(x^4 - 8x^2 + 16) = 0$. But the quartic is a biquadratic, so can be factorised as $x(x^2 - 4)(x^2 - 4) = 0$. Hence, $x = 0, \pm 2$.

482. Four unit circles are arranged symmetrically such that all four circles are concurrent, as in the diagram.



(a) Show that each pair of circles overlaps by a distance of $2 - \sqrt{2}$.

(b) Hence, show that the smallest square which contains all four circles has area $6 + 4\sqrt{2}$.

The smallest square is arranged obliquely. So calculate the width of the group of circles, in a NW-SE direction.

Each circle has diameter 2. The length of overlap between two circles is given by $2 - \sqrt{2}$. So the edge length of the square is $4 - (2 - \sqrt{2})$, which is $2 + \sqrt{2}$. Squaring this gives $6 + 4\sqrt{2}$.

483. Show that, in the case of a right-angled triangle, the cosine rule reduces to Pythagoras's theorem.

Set $\theta = 90°$ in the cosine rule.

The cosine rule is $c^2 = a^2 + b^2 - 2ac\cos C$. In a right-angled triangle, $C = 90°$, and $\cos 90° = 0$. Hence, the cosine rule reduces to $c^2 = a^2 + b^2$, which is Pythagoras.

484. By dividing numerator and denominator of the fraction by x, show that

$$\lim_{x\to\infty} \frac{6x + 1}{2x - 11} = 3.$$

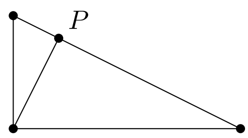
Having performed the division, consider the limit of the small fractions $\frac{1}{x}$ and $-\frac{11}{x}$.

Enacting the suggested division, we get

$$\lim_{x\to\infty} \frac{6 + \frac{1}{x}}{2 - \frac{11}{x}}.$$

As $x \to \infty$, the inlaid fractions tend to zero, leaving $\frac{6}{2} = 3$ as the limit.

485. In the triangle depicted, a perpendicular has been drawn. Find the coordinates of point P.



Use perpendicular gradients.

The equation of the hypotenuse is $y = 5 - \frac{1}{2}x$. The equation of OP, since it is perpendicular to this, is $y = 2x$. Solving simultaneously gives $2x = 5 - \frac{1}{2}x$, so P is $(2, 4)$.

486. Find the double root of $x^3 - a^2x + ax^2 - a^3 = 0$.

Factorise and group the factors.

Factorising fully, we get $(x+a)^2(x-a) = 0$, so $x = -a$ is the double root.

487. Prove that the perimeter and area of a hexagon are linked by the formula $24\,A = \sqrt{3}\,P^2$.

Write both quantities in terms of edge length l, then eliminate l.

The perimeter of the hexagon is $6l$, where l is edge length. An equilateral triangle of edge length l has area $\frac{\sqrt{3}}{4}l^2$, so the hexagon has area $A = \frac{3\sqrt{3}}{2}l^2$. This is $A = \frac{3\sqrt{3}}{2}(\frac{1}{6}l)^2$, which simplifies to the required result.

488. Write down the largest real domains over which the following functions may be defined:

(a) $x \mapsto \sqrt{x}$,

(b) $x \mapsto \sqrt[3]{x}$,

(c) $x \mapsto \sqrt[4]{x}$.

Consider the square, cube and fourth roots of negative numbers.

(a) $[0, \infty)$,

(b) $\mathbb{R}$,

(c) $[0, \infty)$.

489. Find constants a, b, c such that

$$ax + b + \frac{c}{x-4} \equiv \frac{2x^2 - x - 16}{x-4}.$$

Multiply the identity by $(x-4)$, then multiply out and equate coefficients.

Multiplying by $(x-4)$, we get

$$\begin{aligned} &(ax+b)(x-4) + c \equiv 2x^2 - x - 16, \\ \Longrightarrow\ &ax^2 + (b-4a)x + (c-4b) \equiv 2x^2 - x - 16. \end{aligned}$$

The coefficient of x^2 tells us that $a = 2$. Then the coefficient of x tells us that $b = 7$. Then the constant term tells us that $c = 12$.

490. State, with a reason, whether the following are valid identities:

(a) $|x|^2 \equiv x^2$,

(b) $x|x| \equiv -x^2$.

Test $x = \pm 1$.

(a) This holds. Both sides are the same magnitude and positive.

(b) This does not hold. For positive x, the LHS is positive, while the RHS is negative.

491. Determine which of the points $(0, 0)$ and $(6, 3)$ is closer to the line $y = 5 - x$.

The distance of a point from a line is defined as the shortest distance from the point to a line. This is the distance along a perpendicular dropped to the line.

Distance to the line $y = 5 - x$ is along a line of the form $y = x + k$. So we must compare $(0, 0)$ to $(\frac{5}{2}, \frac{5}{2})$ and $(6, 3)$ to $(4, 1)$. The relevant squared distances are $\frac{25}{2}$ and 8. So $(6, 3)$ is closer.

492. A random variable X is normally distributed $N(0, 1)$. Find the following probabilities:

(a) $P(X \in (0, 1)$,

(b) $P(X \in \{0, 1\}$,

(c) $P(X \in [0, 1]$.

Use the normal distribution faculty on your calculator, for (a), then write down the answers to (b) and (c).

(a) $P(X \in (0, 1) = 0.3413$,

(b) $P(X \in \{0, 1\} = 0$, because individual points have zero probability in a continuous distribution.

(c) $P(X \in [0, 1] = 0.3413$, for the reason of (b).

493. The integral $\displaystyle\int_0^k 6x^2 + 2\,dx$ has value 20. Find all possible values of k.

...

We get $2k^3 + 2k = 20$, which is $k^3 + k - 10 = 0$. Spotting the integer root, or finding it with any numerical method, we get $k = 2$. Taking out the factor $(x-2)$, we are left with $k^2 + 2k + 5 = 0$. Since $\Delta < 0$, $k = 2$ is the only possibility.

494. A rhombus is constructed, with interior angles of 45° and 135°, and side length l. Prove that the area of the rhombus is $\frac{\sqrt{2}}{2}l^2$.

Use the cosine rule to find the lengths of the diagonals of the rhombus.

Using the cosine rule, the diagonals of the rhombus are given by $d_1^2 = l^2(2 - 2\cos 45)$ and $d_2^2 = l^2(2-2\cos 135)$. These simplify to $d_1 = l\sqrt{2-\sqrt{2}}$ and $d_2 = l\sqrt{2+\sqrt{2}}$. The area of the rhombus is then given, counting right-angled triangles, by $\frac{1}{2}d_1 d_2$, which is

$$\begin{aligned} &\frac{1}{2}l^2\sqrt{2-\sqrt{2}}\sqrt{2+\sqrt{2}} \\ = &\ \tfrac{1}{2}l^2\sqrt{(2-\sqrt{2})(2+\sqrt{2})} \\ = &\ \tfrac{1}{2}l^2\sqrt{4-2} \\ = &\ \tfrac{\sqrt{2}}{2}l^2. \end{aligned}$$

495. Show that $A' \cap B'$ and $(A \cup B)'$ are the same set.

Draw a Venn diagram.

$A' \cap B'$ means everything that is simultaneously in not-A and not-B. This is the complement of everything that is in either one or the other of A and B. This latter set is $(A \cup B)'$.

496. Express $x^2 + 12x$ in simplified terms of $w = x - 3$.

Write x in terms of w, and substitute.

Substituting $x = w+3$, we get $(w+3)^2+12(w+3)$, which simplifies to $w^2 + 18w + 45$.

497. A particle of mass 10 grams is accelerating in the direction of $\mathbf{i}-2\mathbf{j}$. The component parallel to the y axis has magnitude 14 ms^{-2}. Find the magnitude of the force acting on the particle.

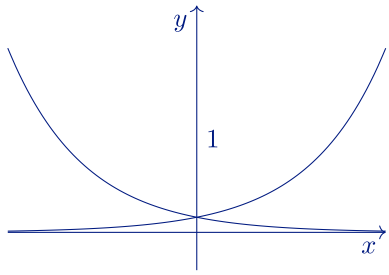
Find the magnitude of the acceleration as the size of the hypotenuse of a right-angled acceleration triangle. You have one of the other sides.

Since the acceleration is in the direction of $\mathbf{i} - 2\mathbf{j}$, and its component parallel to y has magnitude 14 ms^{-2}, the acceleration must be $7\mathbf{i} - 14\mathbf{j}$. So the magnitude of the acceleration is $a = \sqrt{7^2 + 14^2} = 7\sqrt{5}$ ms^{-2}. Then $F = ma$ gives $F = 0.01 \times 7\sqrt{5} = 0.157$ N.

498. Sketch the graphs $y = 2^x$ and $y = 2^{-x}$.

The first graph is a standard exponential. Consider the second graph as a reflection of the first.

The first graph is a standard exponential. The second graph is the first reflected in the x direction, i.e. reflected in the y axis.



499. You are given that the equation $ax^2 + bx + c = 0$ has exactly two real roots. State, with a reason, whether the following equations can be guaranteed to have exactly two real roots:

(a) $(a+1)x^2 + (b+1)x + (c+1) = 0$,

(b) $a(x+1)^2 + b(x+1) + c = 0$,

(c) $ax^4 + bx^2 + c = 0$.

...

(a) No. This equation is not related to the first in any obvious way.

(b) Yes. This equation has the same roots as the original one, shifted by 1.

(c) No. The biquadratic must give $x^2 = a$ and $x^2 = b$, but if a and b are both negative or both positive, then this has either zero or four roots.

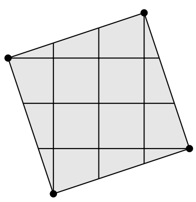
500. Find the length of the line segment

$$x = t, \quad y = 5 - t, \quad t \in [0, \sqrt{2}].$$

Substitute in the values of the parameter at the endpoints of the interval.

For a unit change in t, position along the line changes by $\sqrt{2}$, since this is the hypotenuse of a $(1, 1, \sqrt{2})$ right-angled triangle. So, over an interval of length $\sqrt{2}$, position changes by $\sqrt{2} \times \sqrt{2} = 2$.

501. The grid within the large square consists of squares of unit length. The marked vertices lie on the grid.



Find the area of the large square.

Find the edge lengths using Pythagoras.

Pythagoras gives the edge lengths of the large square as $\sqrt{10}$. Hence the area of the large square is 10 square units.

502. Variables a, b, c, d are related by $a \propto b^2$, $b \propto c^3$, $c \propto d^4$. Find the relationship between a and d.

Substitute in, leaving the statements as proportionality statements.

Substituting in, we get $a \propto b^2 \propto c^6 \propto d^{24}$.

503. It is given that the curves $x^2+y^2 = 1$ and $x+y = k$ intersect. Determine all possible values of k, giving your answer as a set.

Solve simultaneously, and use Δ.

Solving simultaneously, we get $x^2 + (k - x)^2 = 1$. Multiplying out gives $2x^2 - 2kx + k^2 - 1 = 0$. We require this to have at least one root, so $\Delta = 4k^2 - 8(k^2 - 1) > 0$. This inequality has boundary equation $-4k^2 + 8 = 0$, which gives $k^2 = 2$ and $k = \pm 2$. So, we need $k \in [-\sqrt{2}, \sqrt{2}]$.

504. Functions f and g have first derivatives which are proportional. Show that $f(x)$ is related linearly to $g(x)$.

Write the first sentence algebraically, then integrate.

Writing the first sentence algebraically, we have $f'(x) = kg'(x)$. Integrating this, we get $f(x) = kg(x) + c$. This is the same as saying that $f(x)$ is related linearly to $g(x)$.

505. Show that $(x+1)$ leaves a non-zero remainder when dividing $4x^3 - 12x^2 + 18$.

Use the factor theorem.

A zero remainder would imply that $(x + 1)$ is a factor. So, we test $x = -1$. But $4x^3 - 12x^2 + 18\,\big|_{x=-1} = 2$. This is a non-zero remainder.

506. Explain how Newton's first law may be thought of as a special case of Newton's second law.

Consider constant velocity as zero acceleration.

Newton II states that $F = ma$, where F is the resultant force acting on an object. If no unbalanced force acts on an object, then, according to Newton II, $a = 0$, so velocity is constant. This is the statement of Newton I.

507. Three cards are put into a hat. One is red on both sides, one is green on both sides, one is red on one side and green on the other. Find the probability that, if two cards are drawn out and laid on the table, they both show red.

Draw a tree diagram.

In order for RR to show, the two cards with red faces must be drawn. The probability of this is $\frac{1}{3}$. Then, in order for RR to show, the RG card must show R. So the probability is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

508. An odd integer can be expressed as $2k + 1$, where $k \in \mathbb{Z}$, i.e. where k is any integer.

(a) Show that $(2k+1)^2$ can be written as $2(2k^2 + 2) + 1$.

(b) Hence, prove that the square of an odd number is odd.

Multiply out and simplify.

(a) $$\begin{aligned}(2k + 1)^2 &= 4k^2 + 4k + 1\\ &= 2(2k^2 + 2k) + 1.\end{aligned}$$

(b) Since k is an integer, $2k^2 + 2k$ is an integer. Therefore $2(2k^2+2k)$ is even, so $2(2k^2+2k)+1$ is odd. Hence, the square of an odd number is odd.

509. Show that, at half past three, the angle between the hands of a clock is $\frac{5\pi}{12}$*radians*.

One clock hour/five clock minutes is $\frac{2\pi}{12} = \frac{\pi}{6}$ radians.

Since one clock hour/five clock minutes is $\frac{2\pi}{12} = \frac{\pi}{6}$ radians, the angle between the minute and hour hands at $3:30$ is the angle between 3.5 clock hours and 6 clock hours, which is $2.5 \times \frac{\pi}{6} = \frac{5\pi}{12}$.

510. This question concerns proving that a triangle of side lengths $(a, b, c) = (5, 6, 7)$ will not fit inside a rectangle with a side of length 4.

(a) Use the cosine rule to show that the largest angle in the triangle is $C = \arccos \frac{1}{5}$.

(b) Determine the area of the triangle.

(c) Hence, find the length of the perpendicular from point C to the side of length c and thus prove the result.

(a) The largest angle is opposite the largest length.

(b) Use $A_\triangle = \frac{1}{2}ab \sin C$.

(c) Use $A_\triangle = \frac{1}{2}\text{base} \times \text{height}$.

(a) The largest angle is opposite the largest length, so

$$\begin{aligned}\cos C &= \frac{5^2+6^2-7^2}{2\cdot 5\cdot 6}\\ &= \frac{12}{60}\\ &= \frac{1}{5}.\end{aligned}$$

(b) $A_\triangle = \frac{1}{2}\cdot 5\cdot 6\sin(\arccos\frac{1}{5}) = 6\sqrt{6}$.

(c) Using $A_\triangle = \frac{1}{2}\text{base}\times\text{height}$, we have

$$\begin{aligned}6\sqrt{6} &= \frac{1}{2}\cdot 7\cdot h\\ \Longrightarrow h &= \frac{12\sqrt{6}}{7} = 4.20 \text{ (3sf)}.\end{aligned}$$

Since the shortest perpendicular is longer than 4, the triangle cannot be placed inside a rectangle of side length 4.

511. Give the meaning of the following adjectives used in mechanical modelling:

(a) "smooth",

(b) "rigid",

(c) "light".

Each definition should have the word "negligible" in it.

Each modelling term means "negligible something", where negligible means that, while the quantity in question is not zero, it can be assumed to be zero, i.e. it can be neglected.

(a) Smooth means negligible friction,

(b) Rigid means negligible deformation,

(c) Light means negligible mass.

512. For functions f and constants a, b in the domain of f, state whether the following is true or false:

$$f(x)\Big|_{x=b} - f(x)\Big|_{x=a} \equiv \Big[-f(x)\Big]_b^a$$

...

True. A minus sign reverses the direction of the limits.

513. Determine the four roots of $\sin 2\theta = \frac{\sqrt{3}}{2}$ in $[0, 2\pi)$.

Find all values of 2θ in $[0, 4\pi)$.

$$\begin{aligned}2\theta &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}\\ \Longrightarrow \theta &= \frac{\pi}{6}, \frac{2\pi}{6}, \frac{7\pi}{6}, \frac{8\pi}{6}\end{aligned}$$

514. A triangle has two sides of lengths 20 and 21, and area 126. Find all possible lengths of the third side.

Use $A_\triangle = \frac{1}{2}ab\sin C$, then the cosine rule.

$A_\triangle = \frac{1}{2}ab\sin C$ gives $\sin\theta = \frac{3}{5}$. Using the Pythagorean identity, we get $\cos\theta = \pm\frac{4}{5}$. Hence, using the cosine rule,

$$c^2 = 20^2 + 21^2 - 2\cdot 20\cdot 21\cdot \pm\frac{4}{5}.$$

This tells us that $c = 13$ or $c = 38.9$ (3sf).

515. True or false?

(a) $\dfrac{d}{dx}(x+1) = \dfrac{d}{dx}(x+2)$,

(b) $\displaystyle\int (x+1)\,dx = \int (x+2)\,dx$.

...

(a) True; the constants differentiate to nothing.

(b) False; the constants integrate to x and $2x$ respectively.

516. State, with a reason, whether getting cards of the same suit is more probable if the cards are picked

(a) with replacement,

(b) without replacement.

... Getting cards of the same suit is more probable with replacement, as, once one card of particular suit has been picked, fewer cards of that suit remain.

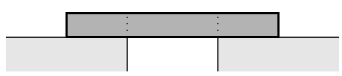
517. The unit circle is $x^2 + y^2 = 1$. Using geometric methods, or otherwise, prove that

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Use the fact that tangent and radius are perpendicular.

This can be solved geometrically, using the fact that tangent and radius are perpendicular. Alternatively, differentiate the equation implicitly, using the chain rule: $2x + 2y\frac{dy}{dx} = 0$. Rearranging gives the required result.

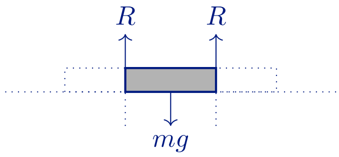
518. A bridge across a stream consists of a uniform wooden beam of mass 20 kg, laid symmetrically with 60% of its length resting on flat ground.



Modelling the beam in three sections, as denoted by the dotted lines, determine the upward force exerted by each of the outer sections on the middle section.

Draw a force diagram for the central section.

Considering the central section as an object in its own right, we have



Since the beam is uniform, the middle section contains 40% of the mass, i.e. 8 kg. Hence $R = 4g$.

519. To differentiate $y = x^3$ from first principles, the following limit is constructed:

$$\lim_{\delta x \to 0} \frac{x^3 - (x - \delta x)^3}{\delta x}.$$

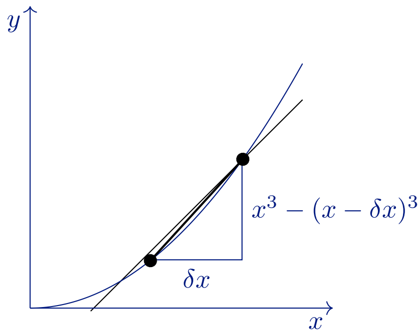
(a) Sketch the chord whose gradient is given by the fraction inside the limit.

(b) Explain the meaning and role of δx.

(c) By expanding and simplifying the numerator, prove that the derivative of x^3 is $3x^2$.

In this differentiation from first principles, the gradient triangle is set up to the left of the point (x, x^3).

(a) In this differentiation from first principles, the chord is set up to the left of the point at which the tangent is drawn.



(b) The quantity δx is a small change in x, often called h.

(c)
$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{x^3 - (x - \delta x)^3}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{3x^2\delta x + \text{ terms in } \delta x^2 \text{ and } \delta x^3}{\delta x} \\ &= \lim_{\delta x \to 0} 3x^2 + \text{ terms in } \delta x \text{ and } \delta x^2 \\ &= 3x^2.\end{aligned}$$

520. By treating it as a quadratic, solve the equation

$$x^5 - x^2\sqrt{x} - 992 = 0.$$

Simplify the middle term first.

Rewriting the middle term, we have

$$\begin{aligned}&x^5 - x^2\sqrt{x} - 992 = 0 \\ \Longrightarrow\ &x^5 - x^{\frac{5}{2}} - 992 = 0 \\ \Longrightarrow\ &x^{\frac{5}{2}} = \frac{1 \pm \sqrt{1 + 4 \cdot 992}}{2} \\ \Longrightarrow\ &x^{\frac{5}{2}} = 32, -31.\end{aligned}$$

There are no real roots of $x^{\frac{5}{2}} = -31$, so $x = 32$.

521. Give the interior angles of the following regular polygons in radians, as fractions of π:

(a) a square,

(b) a hexagon,

(c) a dodecagon.

Use the formula $\theta = \pi - \frac{2\pi}{n}$.

Using $\theta = \pi - \frac{2\pi}{n}$, which comes from considering the exterior angles first, we get

(a) a square: $\frac{\pi}{4}$ radians,

(b) a hexagon: $\frac{\pi}{3}$ radians,

(c) a dodecagon: $\frac{5\pi}{6}$ radians.

522. Brahmagupta studied triangles with side lengths a, b, c generated by

$$\begin{aligned}a &= n(m^2 + k^2), \\ b &= m(n^2 + k^2), \\ c &= (m + n)(mn - k^2).\end{aligned}$$

Prove that the perimeter of such a triangle is given by $P = 2mn(m + n)$.

Add and simplify.

Summing the side lengths, we get

$$\begin{aligned}&n(m^2 + k^2) + m(n^2 + k^2) + (m + n)(mn - k^2) \\ \equiv\ &m^2n + k^2n + mn^2 + k^2m + m^2n + mn^2 - k^2m - k^2n \\ \equiv\ &m^2n + mn^2 + m^2n + mn^2 \\ \equiv\ &2m^2n + 2mn^2 \\ \equiv\ &2mn(m + n).\end{aligned}$$

523. Prove that, if p_1 and p_2 are primes greater than 2, then $p_1p_2 + 1$ cannot be prime.

Consider parity, i.e. evenness/oddness.

If p_1 and p_2 are primes greater than 2, then they must both be odd. Since the product of two odd numbers is odd, p_1p_2+1 is even. It must be greater than 2, which means it cannot be prime.

524. Three interior angles of a quadrilateral are given as $\frac{\pi}{5}$, $\frac{2\pi}{5}$ and $\frac{3\pi}{5}$ radians. Find the fourth angle.

Interior angles of a quadrilateral add to 2π radians.

Interior angles of a quadrilateral add to 2π radians. So $\theta = 2\pi - \frac{\pi}{5} - \frac{2\pi}{5} - \frac{3\pi}{5}$. This gives $\theta = \frac{4\pi}{5}$.

525. The parabolae $y = x^2 + 4x + 6$ and $-x^2 - 12x - 30$ may be transformed onto each other by a rotation of 180° around point P. By completing the square, or otherwise, determine the coordinates of point P.

Find the coordinates of the vertices of the parabolae.

Since the parabola rotate onto one another, their vertices must rotate onto one another. Hence, since the rotation is by 180°, the centre of rotation C must be the midpoint of V_1 and V_2. By differentiating or completing the square, we have $V_1 : (-2, 2)$ and $V_2 : (-6, 6)$. So C is $(-4, 4)$.

526. Find $\displaystyle\int_{-k}^{k} x^3 - x\, dx$ and interpret your result.

Consider the fact that integrals calculate <u>signed</u> area, not simply area.

The value of the integral is zero. Interpreting this in terms of the cubic graph $y = x^3 - x$, the integral is zero because $y = x^3 - x$ has rotational symmetry around the origin. Hence, any signed-area contribution to the integral for positive x is cancelled out by the equivalent signed-area contribution for negative x.

527. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $x \in \{1\}$,
- $x \in \{1, 2\}$.

Remember that an implies arrow carries an implicit "if ... then".

$x \in \{1\} \implies x \in \{1, 2\}$. If x is an element of $\{1\}$, then it must be an element of $\{1, 2\}$.

528. A horizontal, uniform beam is supported at two points, which divide the length of the beam in the ratio 2 : 3 : 4. Show that the ratio of reaction forces at the supports is 1 : 5.

Consider moments around the centre of mass.

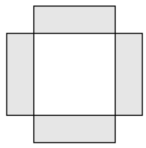
The two reaction forces are at distances 2.5 parts and 0.5 parts from the centre of mass. Taking moments around the centre of mass, the ratio of force magnitudes is $0.5 : \frac{2}{5}$, which is $1 : 5$.

529. The equation $x^3 + y^3 = 100$ forms a closed loop. Determine whether the point $(3, 4)$ lies inside, on, or outside this loop.

The quantity $x^3 + y^3$ is zero at the origin, and increases with distance from the origin.

Hence, we need only evaluate $x^3 + y^3 \big|_{(3,4)} = 91 < 100$ to show that $(3, 4)$ is inside the curve.

530. Two identical rectangles are drawn at right angles, as shown below. Each has perimeter 32, and the shaded area is 48.



Assuming that the diagram is to scale, find the area of the central square.

Label the sides of the rectangles x, y, and set up and solve simultaneous equations.

Labelling the sides of the rectangles $x > y$, we get: from the perimeter, $x + y = 16$, and from the area $4 \cdot \frac{1}{2}(x - y)y = 48$. The latter is $xy - y^2 = 24$. Substituting for x, this yields

$$\begin{aligned} &(16 - y)y - y^2 = 24 \\ \Longrightarrow\ &y^2 - 8y + 12 = 0 \\ \Longrightarrow\ &y = 2, 6. \end{aligned}$$

The root $y = 2$ gives $x = 14$, which does not correspond to the diagram. So $y = 6$, giving the area of the central square as 36.

531. Prove that one of the diagonals of a cyclic kite must be a diameter of the circumscribing circle.

Consider the symmetry of the kite.

A kite has a line of reflective symmetry. Since this kite is cyclic, its opposite angles add to 180°, which means two of them must be right angles. Hence, by the angle-in-a-semicircle theorem, the kite's line of symmetry must be a diameter.

532. A fly has velocity $\mathbf{i} + a\mathbf{j} + 2\mathbf{k}$ ms^{-1} and speed 3 ms^{-1}. Find a.

Use 3D Pythagoras.

Using 3D Pythagoras, $3 = \sqrt{1 + a^2 + 4}$. Hence, $a = \pm 2$.

533. At the point with x coordinate p, the tangent line to $y = x^2$ has equation $y = 2px + c$.

(a) Explain the presence of "$2p$".

(b) By substituting the point (p, p^2), show that a general tangent line to the curve $y = x^2$ has equation $y = 2px - p^2$.

Differentiate.

(a) $\left.\dfrac{dy}{dx}\right|_{x=p} = 2p$.

(b) Substituting (p, p^2) into $y = 2px + c$ gives $p^2 = 2p^2 + c$, so $c = -p^2$. Hence, the equation of the tangent to $y = x^2$ at (p, p^2) is $y = 2px - p^2$.

534. In a game, three coins are tossed, then two, then one. Find the probability that a total of four tails show during the game.

This is a binomial distribution.

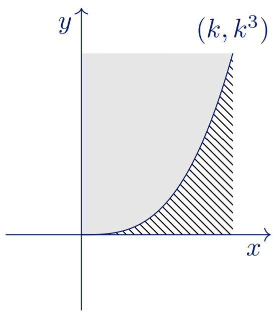
There is no need to split up the solution into $3, 2, 1$. Rather, the distribution is $X \sim B(6, \frac{1}{2})$. So $p = {}^6C_4 \cdot \frac{1}{2}^6 = \frac{15}{64}$.

535. Given the graph $y = x^3$, sketch, on a single set of axes, the regions whose areas are calculated by the following integrals:

(a) $\displaystyle\int_0^k y\,dx$, (b) $\displaystyle\int_0^{k^3} y\,dy$.

Consider the point (k, k^3).

These integrals represent the two areas defined by the point (k, k^3):



536. Prove that a product of six consecutive integers ends in zero.

Consider factors of 2 and 5.

To finish in zero, a number must have a factor of 10, i.e. factors of 2 and 5. But, in any six consecutive integers, there must be three multiples of 2 and either one or two multiples of 5. Hence, the number will end in zero.

537. A student is trying to prove that $f(x) \equiv g(x)$, for some two algebraic functions f and g. By valid algebra, he ends up with the line $f(x) = g(x) \implies 0 = 0$, and writes Q.E.D. Explain the error.

Consider the direction of the implication.

This implication show that, <u>if</u> $f(x) \equiv g(x)$, <u>then</u> $0 = 0$. But, the student is trying to prove that $f(x) \equiv g(x)$, which requires an implication in the opposite direction.

538. Show that the equation of the normal to the curve $y = x^{-\frac{1}{2}}$ at $x = 4$ is $2y = 32x - 127$.

Differentiate and use the negative reciprocal: the normal is perpendicular to the tangent.

Having differentiated, evaluate $-\frac{1}{2}x^{-\frac{3}{2}}$ at $x = 4$, giving $-\frac{1}{16}$. Taking the negative reciprocal, the normal has equation $y = 16x + c$. Substituting $(4, \frac{1}{2})$ gives $\frac{1}{2} = 16 \cdot 4 + c$, so $c = -\frac{127}{2}$. Hence, the equation of the normal is $2y = 32x - 127$.

539. A biquadratic in x is a quadratic in x^2. Solve the biquadratic $5x^4 - 6x^2 + 1 = 0$.

Factorise in terms of x^2.

Factorising, we get

$$\begin{aligned} & 5x^4 - 6x^2 + 1 = 0 \\ \implies & (5x^2 - 1)(x^2 - 1) = 0 \\ \implies & x^2 = 1, \frac{1}{5} \\ \implies & x = \pm 1, \pm\frac{1}{\sqrt{5}}. \end{aligned}$$

540. A set of data, whose mean is 4, is given as follows:

x	1	2	3	4	5
f	1	4	5	17	n

Find n.

Form an equation in n, and solve.

Forming an equation in n, we get $1 \cdot 1 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 17 + 5 \cdot n = 4(1 + 4 + 5 + 17 + n)$. This gives $5n - 4n = 108 - 92$, so $n = 16$.

541. A triangle T has vertices $(0,0)$, $(a,0)$, and (b,c), where $a,b,c>0$. Show that the area of T does not depend on b.

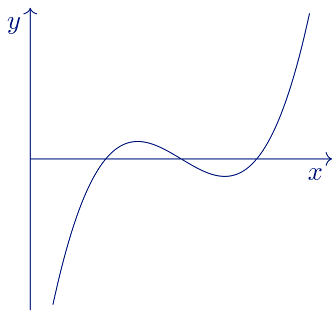
Sketch the triangle, and consider perpendicular heights.

Since the point $(a,0)$ is on the x axis, the base of the triangle is on the x axis. Hence, its perpendicular height is in the y direction, and is given by c. The area is then $\frac{1}{2}ac$, which does not depend on b.

542. By sketching or considering signs of factors, solve the inequality $(x-1)(x-2)(x-3) \leq 0$, giving your answer in set notation.

This is a positive cubic with three distinct real roots.

This is a negative cubic with three distinct real roots:



We are looking for the set of x values for which the curve is at or below the x axis, which is the set $(-\infty,1] \cup [2,3]$.

543. Find the constant term in $\left(x-\dfrac{1}{x}\right)^4$.

Use the binomial expansion.

Using the binomial expansion, the relevant term is ${}^4C_2 x^2 \frac{1}{x^2} = 6$.

544. State, with a reason, which of the following shapes has the larger area:

- a regular n-gon of side length l,
- a regular $(n+1)$-gon of side length l.

...

If an n-gon and an $(n+1)$-gon have the same perimeter, then the $(n+1)$-gon has larger area. So, since the perimeter of this $(n+1)$-gon is, in fact, larger, so must its area be.

545. Explain how you know that the following equation has no real roots:

$$\frac{(x^2+a^2+1)(x^2+b^2+1)}{(x^2+c^2+1)(x^2+d^2+1)} = 0.$$

Consider the roots of the numerator.

We need only consider the roots of the numerator. Because a^2+1 and b^2+1 are both strictly positive, the numerator has no roots. Hence, the equation has no roots.

546. Show, with detailed reasoning, that the area enclosed by the curve $y=x^2$ and the line $y=2x+1$ is $\frac{8\sqrt{2}}{3}$.

Solve to find intersections, then calculate a definite integral of the difference $2x+1-x^2$.

The intersections are at $x^2=2x+1$, so $x=1\pm\sqrt{2}$. Hence, since the line is above the parabola in between these intersections, the area between the curves is given by the definite integral

$$\begin{aligned} A &= \int_{1-\sqrt{2}}^{1+\sqrt{2}} 2x+1-x^2\,dx \\ &= \left[x^2+x-\tfrac{1}{3}x^3\right]_{1-\sqrt{2}}^{1+\sqrt{2}} \\ &= (4+3\sqrt{2}-\tfrac{1}{3}(1+\sqrt{2})^3) - (4-3\sqrt{2}-\tfrac{1}{3}(1-\sqrt{2})^3) \\ &= 6\sqrt{2}-\tfrac{2}{3}(3\sqrt{2}+2\sqrt{2}) \\ &= \frac{8\sqrt{3}}{2} \end{aligned}$$

547. State, with a reason, whether $y=x^2$ intersects the following curves:

(a) $y=x+1$,

(b) $y=x^2+1$,

(c) $y=x^3+1$.

Sketch the curves.

(a) Yes, as $y=x+1$ crosses the y axis at $(0,1)$.

(b) No. The curves are vertical translations of one another.

(c) Yes. Intersections are given by roots of a cubic $x^3+1=x^2$, and every cubic has at least one root.

548. A geometric sequence has n^{th} term u_n. Show that $w_n = u_{n-1}+u_n$ is also geometric.

Find an simplify an expression for the common ratio $\frac{w_{n+1}}{w_n}$ of the new sequence.

We need to show that the new sequence has a common ratio, i.e. that $\frac{w_{n+1}}{w_n}$ is constant. Simplifying algebraically, we get

$$\begin{aligned}\frac{w_{n+1}}{w_n} &= \frac{u_n + u_{n+1}}{u_{n-1} + u_n} \\ &= \frac{ru_{n-1} + ru_n}{u_{n-1} + u_n} \\ &= \frac{r(u_{n-1} + u_n)}{u_{n-1} + u_n} \\ &= r.\end{aligned}$$

Hence, w_n is geometric.

549. Describe the locus of the following equation, for non-zero constants a, b:

$$x^2 + y^2 = (x - a)^2 + (y - b)^2 .$$

Multiply out and simplify.

Multiplying out and simplifying, the terms in x^2 and y^2 cancel, and we are left with $0 = -2ax + a^2 - 2by + b^2$, which we can rearrange to $2ax + 2by = a^2 + b^2$. For non-zero constants a, b, this is the equation of a straight line.

550. Two inequalities are given below:

$$\begin{aligned}&x^4 + y^4 < 1 \\ &x + y > 2.\end{aligned}$$

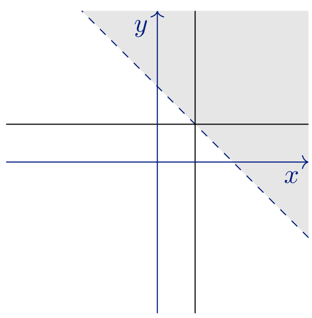
(a) Sketch the region $x + y > 2$ on a set of axes.

(b) Show that, for all points in the region, at least one of x or y is greater or equal to 1.

(c) Hence, show that no (x, y) points satisfy both inequalities simultaneously.

For (c), consider the fact that x and y are raised to an even power.

(a) The region $x + y > 2$, with red lines added for part (b), is



(b) Every shaded point is either to the right of $x = 1$ or above $y = 1$.

(c) Since x^4 and y^4 are even powers, they are always positive. Hence, if one of x or y is at least 1, then $x^4 + y^4$ is at least 1.

551. Let $X \sim N(0, 1)$. State, with a reason, whether the following variables are normally distributed:

(a) $-X$ (b) $|X|$ (c) $10 - X$ (d) X^2.

Every normal distribution is a transformation (via stretches and translations) of every other.

(a) Yes. The normal distribution is symmetrical.

(b) No. This distribution, on $[0, \infty)$, must be asymmetrical.

(c) Yes. This is a normal distribution $N(10, 1)$.

(d) No. This distribution, on $[0, \infty)$, must be asymmetrical.

552. The mean of a and c is b. Prove that a, b, c are in arithmetic progression.

Form and manipulate an equation in a, b, c.

We know that $\frac{1}{2}(a + c) = b$, hence $a + c = 2b$. This can be rearranged to $c - b = b - a$, which tells us that successive differences are the same. So, the sequence is an AP.

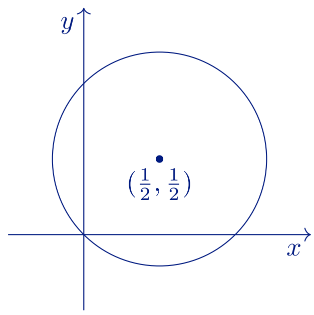
553. Sketch the graph $x^2 + y^2 = x + y$.

Put everything on one side and complete the square.

Grouping and completing the square, we have

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 = \frac{1}{2}.$$

This is a circle, centred on $(\frac{1}{2}, \frac{1}{2})$, with radius $\frac{\sqrt{2}}{2}$. This passes through the origin.



554. True or false?

(a) $\sin x = \sin y \implies x = y$,

(b) $\sin x = \sin y \impliedby x = y$,

(c) $\sin x = \sin y \iff x = y$.

Consider $x = 30°$ and $y = 150°$.

For any x, the equation $\sin x = \sin y$ is satisfied by infinitely many y values. Hence, the implication is only backwards, as in (b).

555. Simplify the following expression, in which $a \neq 0$:

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

The numerators form a difference of two squares.

The numerators form a difference of two squares, which gives

$$\begin{aligned} &\frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a} \\ \equiv\ &\frac{b^2-(b^2-4ac)}{4a^2} \\ \equiv\ &\frac{c}{a}. \end{aligned}$$

556. A curve is given by $y = \dfrac{(x-1)^2}{\sqrt{x}}$.

(a) Find $\dfrac{dy}{dx}$.

(b) Hence, find the coordinates of the stationary point of this curve.

Split up the fraction before differentiating, or else use the quotient rule.

(a) Using the quotient rule, we get

$$\frac{dy}{dx} = \frac{2(x-1)x^{\frac{1}{2}} - \frac{1}{2}(x-1)^2 x^{-\frac{1}{2}}}{x}.$$

(b) Solving to find stationary points, we require the numerator to be zero. Multiplying by $2x^{\frac{1}{2}}$, this is

$$\begin{aligned} &4(x-1)x + (x-1)^2 = 0 \\ \Longrightarrow\ &(x-1)(4x-(x-1)) = 0 \\ \Longrightarrow\ &x = 1, -\frac{1}{3}. \end{aligned}$$

The function is not defined at $x = -\frac{1}{3}$, so the only stationary point is at $(1, 0)$.

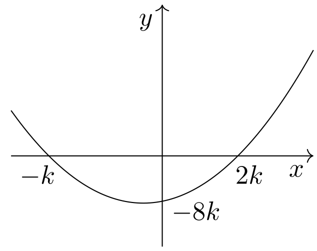
557. Evaluate $1 + 2 + 4 + 8 + ... + 1048576$.

The sum of a GP is $S_n = \dfrac{a(1-r^n)}{1-r}$.

The ordinal formula of this GP is $u_n = 2^{n-1}$. So, solving $u_n = 1048576$ gives $n = 21$. Then the sum is given by

$$S_{21} = \frac{1-2^{21}}{1-2} = 2097151.$$

558. Find the equation of the monic parabola shown, on which the axes intercepts have been marked, giving your answer in expanded polynomial form.



Use the factor theorem.

The parabola is monic, so its leading coefficient is 1. Hence, using the factor theorem, it is $y = (x+k)(x-2k)$. Substituting $(0, -8k)$, we get $-8k = \cdot - 2k^2$. This has roots $k = 0$ or $k = 4$. But $k = 0$ is not the parabola shown. So $k = 4$ and the equation is $y = x^2 - 4x - 32$.

559. Give, in radians, the average of the interior angles of a decagon.

Consider the sum of the interior angles.

The sum of the interior angles of a decagon is 8π, so the average of them is $\frac{8}{10}\pi = \frac{4}{5}\pi$.

560. *Euclid's formula*, in which $p > q > 0$ are natural numbers, gives triples (a, b, c) as follows:

$$a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2.$$

(a) Show that this generates Pythagorean triples.

(b) Hence, prove that there are infinitely many different right-angled triangles with edges of integer length.

...

(a) Pythagorean triples obey $a^2 + b^2 = c^2$. So, consider

$$\begin{aligned} &a^2 + b^2 \\ =\ &(p^2-q^2)^2 + (2pq)^2 \\ \equiv\ &p^4 - 2p^2q^2 + q^4 + 4p^2q^2 \\ \equiv\ &p^4 + 2p^2q^2 + q^4 \\ \equiv\ &(p^2+q^2)^2 \\ =\ &c^2. \end{aligned}$$

(b) Since there are infinitely many pairs $p > q > 0$, each of which generates a different right-angled triangle, there must be infinitely many different right-angled triangles.

561. Show that $\displaystyle\int_0^1 4x^3 - 5x^4\,dx = 0$.

...

Integrating, we get

$$\begin{aligned} &\int_0^1 4x^3 - 5x^4\,dx \\ =\ & \left[x^4 - x^6\right]_0^1 \\ =\ & (1-1) - (0-0) \\ =\ & 0. \end{aligned}$$

562. Find a, b, c such that the following is an identity:

$$20x^2 - 23x - 21 \equiv (5x + b)(cx + d).$$

If you can't spot the factorisation, use the quadratic formula to find roots, and then use the factor theorem.

$20x^2 - 23x - 21 = (5x + 3)(4x - 7)$. So $a = 3, b = 4, c = -7$.

563. The graph $y = x^2 + x$ is translated by the vector $2\mathbf{i} + 3\mathbf{j}$. Find, in the form $y = ax^2 + bx + c$, the equation of the new graph.

Consider two separate translations, one in the x direction and one in the y direction.

A translation by $2\mathbf{i}$ requires a replacement of x by $x-2$, and a translation by $3\mathbf{j}$ requires an addition of 3 (equivalent to a replacement of y by $y-3$). This gives $y = (x-2)^2 + (x-2) + 3$, which is $y = x^2 - 3x + 5$.

564. True or false?

(a) $f(a) = 0 \iff (f(a))^2 = 0$,

(b) $f(a) = 1 \iff (f(a))^2 = 1$.

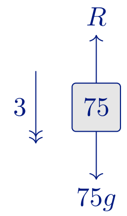
Remember that $f(a)$ is a value, so these statements are equivalent to (a) $x = 0 \iff x^2 = 0$ and ...

The first statement is true, as $x = 0 \iff x^2 = 0$. The second is not, since the backwards implication does not hold. The correct linkage would be $f(a) = 1 \implies (f(a))^2 = 1$.

565. A fridge of mass 75 kg is standing in a lift, which is accelerating downwards at 3 ms^{-2}. Find the force exerted by the fridge on the lift floor.

Draw a force diagram.

The question asks for the force exerted by the fridge on the lift floor, which is the Newton III pair of the force exerted by the lift on the fridge. So, we draw a force diagram for the fridge:



$75g - R = 75 \cdot 3$, so $R = 510N$.

566. Prove the area formula $A = \dfrac{1}{2}ab\sin C$.

Set up a triangle, and drop a perpendicular.

In triangle ABC, drop a perpendicular from vertex A to side BC. This has length $b\sin C$. Hence, the triangle has area $\frac{1}{2}ab\sin C$.

567. Write down the ranges of the following functions, when they are defined over the largest possible real domains:

(a) $x \mapsto \sin 2x$,

(b) $x \mapsto \cos 2x$,

(c) $x \mapsto \tan 2x$.

Ignore the input transformations, they do not affect the ranges here.

The input transformations $X \mapsto 2x$ do not affect the ranges, which are set of y outputs. So the ranges are

(a) $[-1, 1]$,

(b) $[-1, 1]$,

(c) $\mathbb{R}$.

568. Four points are $(0, 0)$, $(20, 0)$, $(0, 10)$, $(4, 8)$.

(a) Show that three of these points are collinear.

(b) Show that any other set of three forms a right-angled triangle.

Three points A, B, C are collinear iff $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R}$.

(a) Three points A, B, C are collinear iff $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R}$. Labelling the points O, A, B, C, we have

$$\overrightarrow{AB} = \begin{pmatrix} -20 \\ 10 \end{pmatrix} = \frac{5}{4}\begin{pmatrix} -16 \\ 8 \end{pmatrix} \frac{5}{4}\overrightarrow{BC}.$$

(b) Clearly triangle OAB is right-angled. Now, consider $\overrightarrow{OD}$ and $\overrightarrow{AB}$. These vectors are perpendicular. Hence triangles OAC and OBC are both right-angled.

569. Using the notation $\binom{n}{r} \equiv {}^nC_r$, *Dixon's identity* is

$$\sum_{k=-a}^{a} (-1)^k \binom{2a}{k+a}^3 \equiv \frac{(3a)!}{(a!)^3}.$$

Verify the identity for $a = 1$.

Evaluate the LHS and RHS separately.

With $a = 1$, the LHS is

$$\begin{aligned} & (-1)^{-1}\binom{2}{0}^3 + (-1)^0\binom{2}{1}^3 + (-1)^1\binom{2}{2}^3 \\ = & \; -1^3 + 2^3 - 1^3 \\ = & \; 6. \end{aligned}$$

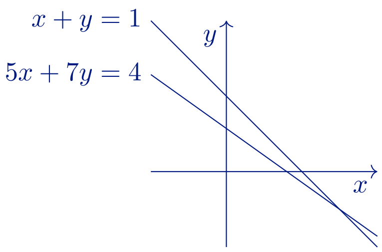
The RHS is $\dfrac{3!}{(1!)^3 = 6}$.

570. Prove that, for all positive real numbers x, y,

$$x + y > 1 \implies 5x + 7y > 4.$$

Sketch the regions on (x, y) axes.

Sketching the relevant regions on (x, y) axes, we see that, in the positive quadrant, the boundary line $5x + 7y = 4$ is below and to the left of the boundary line $x + y = 1$. Hence, if $x + y > 1$, (x, y) is above and to the right of both lines.



571. The equations below give two chords of one circle.

$$\begin{aligned} x &= s, \quad y = 2 - s, \quad s \in [0, 1] \\ x &= -1 + t, \quad y = 1 + t, \quad t \in [0, 1] \end{aligned}$$

Find the equation of the circle.

Since the line segments are chords, their endpoints, which have $s \in \{0, 1\}$ and $t \in \{0, 1\}$, must lie on the circle.

Since the line segments are chords, their endpoints, which have $s \in \{0, 1\}$ and $t \in \{0, 1\}$, must lie on the circle. These points are $(0, 2)$, $(1, 1)$, $(-1, 1)$, $(0, 2)$. Since the two chords are at right-angles to each other, $(1, 1)$ and $(-1, 1)$ must be endpoints of a diameter. Hence, the equation of the circle is $x^2 + (y - 1)^2 = 1$.

572. A sequence is defined iteratively by the following rule: A_{n+1} is three more than twice A_n. Show that, unless the sequence is constant, it is neither arithmetic nor geometric.

Set up the iteration algebraically, calculate the term-to-term difference and ratio, and show that neither is constant.

The iteration is $A_{n+1} = 3 + 2A_n$. Subtracting A_n gives the term-to-term difference as

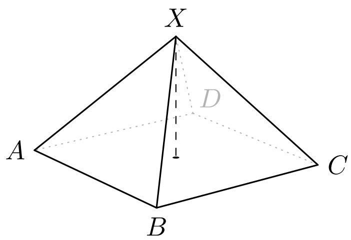
$$A_{n+1} - A_n = 3 + A_n.$$

Alternatively, dividing by A_n gives the term-to-term ratio as

$$\frac{A_{n+1}}{A_n} = \frac{3}{A_n} + 1.$$

Neither of these is constant unless A_n is constant.

573. The square-based pyramid shown below is formed of eight edges of unit length.



Determine the length of the dashed perpendicular shown dropped from X to $ABCD$.

Use triangle AXC.

Triangle AXC is congruent to triangle ABC. So the length of the dashed perpendicular is $\frac{1}{2}|AC| = \frac{\sqrt{2}}{2}$.

574. "The curves $y = x^2 + 1$ and $y = -4x - 3x^2$ are tangent to one another." True or false?

Solve to find intersections, and show that the resulting equation has a double root.

Solving simultaneously for intersections, we have $x^2 + 1 = -4x - 3x^2$, which rearranges to $4x^2 + 4x + 1 = 0$. This factorises as $(2x + 1)^2 = 0$, which has a double root at $x = -\frac{1}{2}$. Hence, the two curves are tangent at this point.

575. Functions F and G are periodic, with periods 3 and 5 respectively. State the periods of the following functions:

(a) $F(\frac{1}{2}x + 1)$,

(b) $3G(x) + 2$,

(c) $F(x) + G(x)$,

Consider input and output transformations for (a) and (b), and lowest common multiples for (c).

(a) Since the input transformation stretches the curve by a factor 2 in the x direction, the period of the new function is 6.

(b) The output transformations have no effect on the period, which remains at 5.

(c) The sum has period $\text{lcm}(3,5) = 15$.

576. Find the area enclosed by the curve $y = x^2$ and the line $y = 4$.

Find intersections, and set up a single definite integral.

The curves intersect at $x = \pm 2$. Hence, the area is given by the integral of the difference, which is

$$\begin{aligned} A &= \int_{-2}^{2} 4 - x^2 \, dx \\ &= \left[4x - \tfrac{1}{3}x^3 \right]_{-2}^{2} \\ &= (8 - \tfrac{8}{3}) - (-8 + \tfrac{8}{3}) \\ &= \tfrac{32}{3}. \end{aligned}$$

577. Three coins are tossed simultaneously. Event S is defined as all three coins showing the same result, and S' is the complement of this event. Show that $P(S') = 3P(S)$.

List the eight outcomes of the possibility space, and group them as S and S'.

There are eight outcomes in the possibility space, of which HHH and TTT have all three coins showing the same. Hence, $P(S) = \frac{2}{8}$ and $P(S') = \frac{6}{8}$, which gives $P(S') = 3P(S)$ as required.

578. Sketch the graph $x^3y^3 = 1$.

Simplify the algebra first.

Since there is only one real cube root of a number, this curve is identical to $xy = 1$, which is the standard reciprocal graph $y = \frac{1}{x}$.

579. A quadrilateral Q has vertices at $(0, 0)$ and $(6, 0)$, and its diagonals intersect at $(3, 4)$. Find the set of possible values of the perimeter of Q.

Sketch the scenario.

Let us name the current vertices A, B and the intersection X. The remaining two vertices must be on the lines AX and BX, continued beyond X. Since they could be arbitrarily far away, there is no upper bound on P. The lower bound, which cannot be attained, is the perimeter of the triangle ABX, since the remaining two vertices could be arbitrarily close to X. This perimeter is 16, so $P \in (16, \infty)$.

580. Most of the time, a human being has a mass on which weight and a reaction force act. State, with a reason, whether there is any physical scenario, as described by the Newtonian model, in which each of these quantities can drop to zero for a human being:

(a) mass,

(b) weight,

(c) reaction force.

Consider zero gravity.

(a) Mass is a conserved, permanent quantity in the Newtonian model, so cannot drop to zero.

(b) Weight drops to zero (or, more precisely) is negligible, in deep space.

(c) Reaction force drops to zero in full gravity when, for example, a human being jumps in the air.

581. Show that $\mathbf{r} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ is a unit vector.

Use 3D Pythagoras.

Using 3D Pythagoras, we have

$$|\mathbf{r}| = \sqrt{\frac{1}{3}^2 + \frac{2}{3}^2 + \frac{2}{3}^2} = 1.$$

582. The equation $x = 4y^2 + 12y + 11$ defines a parabola.

(a) Complete the square for y.

(b) Hence, state the coordinates of the vertex.

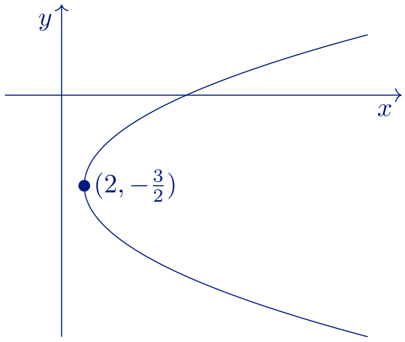
(c) Sketch the parabola.

...

(a) $x = 4(y + \frac{3}{2})^2 + 2$.

(b) The vertex is at $(2, -\frac{3}{2})$.

(c) The parabola is positive, so looks like:



583. Without multiplying out, solve the equation

$$(x-1)(x-2)+(x-1)(x-3)=0.$$

Take out a factor of $(x-1)$.

Taking out a factor of $(x-1)$, we have

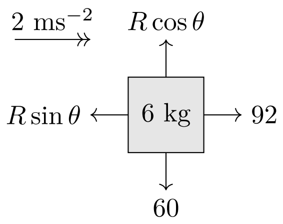
$$\begin{aligned}&(x-1)\left((x-2)+(x-3)\right)=0\\ \Longrightarrow\ &(x-1)(2x-5)=0\\ \Longrightarrow\ &x=1,\tfrac{5}{2}.\end{aligned}$$

584. A straight line is given parametrically as $x=t+4$, $y=3t$, for $t\in\mathbb{R}$. This line is then translated by the vector $5\mathbf{i}$. Write down the equation of the new line, in the same form.

Consider the fact that, for $x=t+4$, this is an output transformation.

To translate by $5\mathbf{i}$, we replace x by $x-5$. This gives $x-5=t+4$, and doesn't affect y, so the equation of the transformed line is $x=t+9$, $y=3t$, for $t\in\mathbb{R}$.

585. Four forces, with magnitudes given in Newtons, cause a 5 kg mass to accelerate as depicted below:



Solve to find R and θ.

Solve simultaneously, using the fact that $\tan\theta\equiv\frac{\sin\theta}{\cos\theta}$.

$F=ma$ in horizontal and vertical directions produces the simultaneous equations

$$\begin{aligned}R\sin\theta&=80\\ R\cos\theta&=60.\end{aligned}$$

Squaring both equations and adding them gives $R^2(\sin^2\theta+\cos^2\theta)=80^2+60^2$. According to the Pythagorean identity, $R^2=100$. In the diagram, R may be assumed to be a magnitude, so $R=10$. Dividing the equations gives $\theta=\arctan\frac{4}{3}$.

586. A circular ripple is spreading across a pond. Its radius in increasing linearly with time, at a rate of 5 centimetres per second. Find the rate of change of the circumference.

Differentiate both sides of $c=2\pi r$.

We know that $c=2\pi r$. Differentiating both sides gives

$$\frac{dc}{dt}=2\pi\frac{dr}{dt}.$$

Hence, the rate of change of the circumference is 2π times the rate of change of the radius, i.e. 10π cm/s.

587. Consider the quartic $y=(x-p)^2(x-q)^2$, where p and q are constants satisfying $0<p<q$.

(a) Without doing any calculations, explain how you know that the quartic has two stationary points on the x axis.

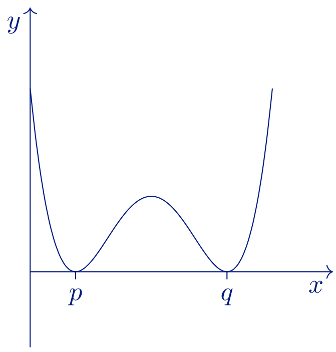
(b) Sketch the curve.

(c) Without using any calculus, write down, in terms of p and q, the x coordinate of the third stationary point on the curve.

Consider the squared factors.

(a) Each factor $(x-p)$ and $(x-q)$ is squared. These double factors correspond to double roots, which are points at which the curve is parabolically tangent to the x axis. Hence $x=p$ and $x=q$ must both be stationary points.

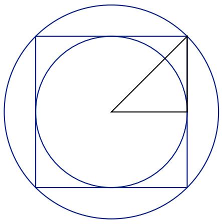
(b) Sketch of $y=(x-p)^2(x-q)^2$:



(c) Since the graph is symmetrical, the third stationary point must be at the x-midpoint of p and q, so $x=\frac{1}{2}(p+q)$.

588. A circle is inscribed in a square, which is inscribed in a circle. Show that the ratio of the areas of the circles is 1 : 2.

Sketch and use Pythagoras/trigonometry.



The triangle drawn in red has side lengths in the ratio $1 : \sqrt{2}$, so the ratio of areas is $1 : 2$.

589. Simplify $\dfrac{p^{\frac{5}{2}} - p^{\frac{3}{2}}}{p^{\frac{3}{2}} - p^{\frac{1}{2}}}$.

Factorise numerator and denominator.

$$\begin{aligned} &\frac{p^{\frac{5}{2}} - p^{\frac{3}{2}}}{p^{\frac{3}{2}} - p^{\frac{1}{2}}} \\ \equiv\ &\frac{p^{\frac{3}{2}}(p-1)}{p^{\frac{1}{2}}(p-1)} \\ \equiv\ &p, \text{ for } p \neq 0, 1. \end{aligned}$$

590. A student suggests that the following is an identity for some suitable choice of constants P, Q:

$$\frac{1}{x} \equiv \frac{P}{1-x} + \frac{Q}{1+x}.$$

By multiplying up and comparing the coefficients of powers of x, or otherwise, prove that the student is incorrect.

Multiply both sides by $x(1-x)(1+x)$.

Multiplying both sides by $x(1-x)(1+x)$, we get

$$1 - x^2 \equiv Px(1+x) + Qx(1-x).$$

Comparing coefficients gives

$$\begin{aligned} x^2 &: -1 = P - Q \\ x^1 &: 0 = P + Q \\ x^0 &: 1 = 0. \end{aligned}$$

Since the constant term cannot match, such an identity can never hold.

591. Find simplified expressions for the sets

(a) $\mathbb{Z} \cup \mathbb{R}$,

(b) $\mathbb{Z} \cap \mathbb{N}$,

(c) $\mathbb{Z} \cup \mathbb{Q}$.

Use $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Using the fact that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$:

(a) $\mathbb{Z} \cup \mathbb{R} = \mathbb{R}$,

(b) $\mathbb{Z} \cap \mathbb{N} = \mathbb{N}$,

(c) $\mathbb{Z} \cup \mathbb{Q} = \mathbb{Q}$.

592. The derivative of the function $f(x) = x^2(1+x^n)$ is $f'(x) = x(a+bx^3)$, where $n, a, b \in \mathbb{N}$. Find n, a, b.

Multiply out, differentiate, then factorise.

We have $f(x) = x^2 + x^{n+2}$. This differentiates to

$$\begin{aligned} f'(x) &= 2x + (n+2)x^{n+1} \\ &= x(2 + (n+2)x^n). \end{aligned}$$

Hence $a = 2, n = 3$ and $b = 5$.

593. Find the exact diameter, from vertex to vertex, of a regular octagon, side length 1, using the fact that

$$\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Consider a triangular sector of the octagon.

A single triangular sector (pizza slice) of the octagon subtends an angle $\frac{\pi}{4}$. So, it can be split into two right-angled triangles, each subtending $\frac{\pi}{8}$. The diameter is given by twice the hypotenuse:

$$\begin{aligned} &2 \times \frac{1}{2\sin\frac{\pi}{8}} \\ =\ &\frac{2}{\sqrt{2-\sqrt{2}}}. \end{aligned}$$

594. A *harmonic progression* is defined as a sequence of the reciprocals of an arithmetic progression. Show that every harmonic progression is either constant or tends to zero.

Consider the long-term behaviour of an AP.

Every AP is either constant, or diverges to $\pm\infty$. Hence, a harmonic progression will either be constant, or tend to zero, since the reciprocals of all large numbers, whether positive or negative, do so.

595. Prove that the product of two square numbers is a square number.

Express the number algebraically.

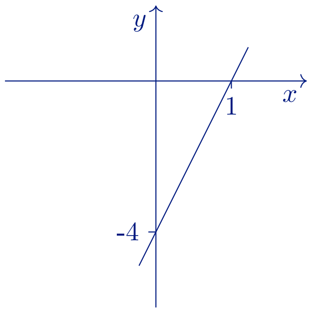
Two square numbers are given by a^2 and b^2, where $a, b \in \mathbb{Z}$. Their product is $a^2b^2 \equiv (ab)^2$, which, since $ab \in \mathbb{Z}$, is a square number. Q.E.D.

596. Sketch a linear graph $y = f(x)$ for which

$$\int_0^1 f(x)\,dx = -2, \quad f(1) = 0.$$

Either work algebraically with $f(x) = ax + b$, or sketch directly.

Since the integral between 0 and 1 is -2, the average value of the linear function, on the domain $[0, 1]$, must be -2. Hence, since $f(1) = 0$, we must have $f(0) = -4$.



597. Solve $2^x \times 4^{x-1} \times 8^{x-2} = 1$.

Express the powers with base 2.

Writing the powers with base 2, we get

$$\begin{aligned} & 2^x \times 4^{x-1} \times 8^{x-2} = 1 \\ \Longrightarrow\ & 2^x \times 2^{2(x-1)} \times 2^{3(x-2)} = 1 \\ \Longrightarrow\ & 2^{6x-8} = 1 \\ \Longrightarrow\ & 6x - 8 = 0 \\ \Longrightarrow\ & x = \tfrac{4}{3}. \end{aligned}$$

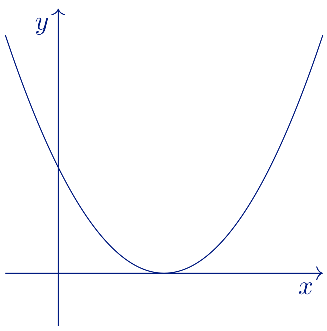
598. You are given that g is a monic quadratic function, and that $g(2) = g'(2) = 0$. Sketch these graphs:

(a) $y = g(x)$,

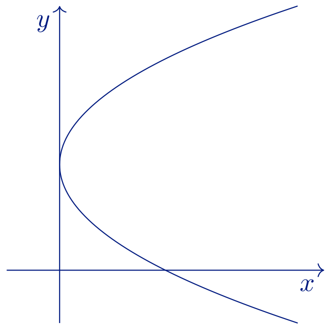
(b) $x = g(y)$.

Find the coordinates of the vertex of each parabola.

(a) The information $g(2) = g'(2) = 0$ tells us that the vertex of the first parabola is at $(2, 0)$.



(b) The second parabola is then a reflection of the first in $y = x$:



599. Explain why, if an object is in equilibrium under the action of three forces, then the vectors representing those forces, if connected tip-to-tail, form the closed triangle known as a “triangle of forces”.

Consider the graphical interpretation of the sum of three vectors.

When vectors are added, they are connected tip-to-tail in the manner described in the question. And the sum of three vectors, if they maintain equilibrium, must be zero. Hence, if the three vectors are connected in this manner, the distance from the overall tail to the overall tip must be zero. This is the same as saying that the vectors must form a closed triangle.

600. Express the following angular speeds in radians per second, in standard form to 3sf:

(a) $45°$ per hour,

(b) 1.2×10^{-4} revolutions per minute.

...

(a) $45°$ per hour is $\frac{\pi}{4}$ radians per hour, which is $\frac{\pi}{4} \div 3600 = 2.18 \times 10^{-4}$ rad/s (3sf).

(b) 1.2×10^{-4} rpm is $2.4\pi \times 10^{-4}$ radians per minute, which is $2.4\pi \times 10^{-4} \div 60 = 1.26 \times 10^{-5}$ rad/s (3sf).

601. Show that $(x+2)^3 - (x-2)^3 = 0$ has no real roots.

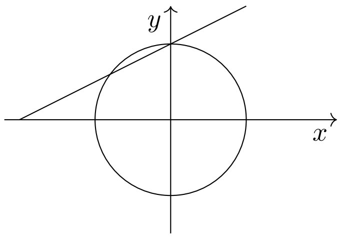
Multiply out to form a quadratic, then use Δ.

Multiplying out, using the binomial expansion:

$$x^3 + 6x^2 + 12x + 8 - (x^3 - 6x^2 + 12x - 8) = 0.$$

The odd-powered terms cancel, and we are left with a quadratic $x^2 = -\frac{4}{3}$. This has no real roots.

602. The unit circle $x^2 + y^2 = 1$ and the line segment $\mathbf{r} = t\mathbf{i} + (1 + \frac{1}{2}t)\mathbf{j}$, for $t \in [-2, 1]$ are drawn.



Show that $\frac{4}{15}$ of the length of the line segment lies within the circle.

Find the t intersections of the line and the circle. (You don't need to work out the (x, y) coordinates).

Substituting the parametric equation into the equation of the circle, we get $t^2 + (1 + \frac{1}{2}t)^2 = 1$, which simplifies to $\frac{5}{4}t^2 + t = 0$. So the line is inside the circle for $t \in (0, -\frac{4}{5})$. Since the t-domain of the line is $[-2, 1]$, this means that $\frac{4}{5} \div 3 = \frac{4}{15}$ of the length of the line lies within the circle.

603. Two dice are rolled, and the values are added. By drawing the possibility space, show that 7 is the most likely score.

...

The possibility space is:

	1	2	3	4	5	6
1						✓
2					✓	
3				✓		
4			✓			
5		✓				
6	✓					

In the possibility space, outcomes with the same total form diagonal lines. The longest of these is the diagonal corresponding to a total of 7.

604. Solve the equation $x^{10} + 8x^6 = 0$.

Factorise.

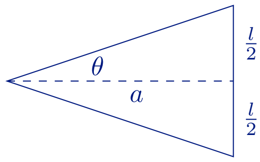
Factorising, we get $x^6(x^4 + 8) = 0$. The latter factor has no real roots, so the solution is $x = 0$.

605. The *apothem* of a regular polygon is the distance from the centre to the midpoint of one of the sides. Prove that, in an n-sided polygon, the apothem a is related to the side length l by the formula

$$a = \tfrac{1}{2} l \cot \tfrac{180°}{n}.$$

Split a sector of an n-gon up into two right-angled triangles.

A sector of a regular n-gon is isosceles by definition. Splitting this into two right-angled triangles, we get



The angle subtended by the sector is $\frac{360°}{n}$, so $\theta = \frac{180°}{n}$. Trigonometry then gives

$$\begin{aligned} \tan \frac{180°}{n} &= \frac{l}{2a} \\ \implies a &= \tfrac{1}{2} l \cot \tfrac{180°}{n}. \end{aligned}$$

606. Write 16^t in terms of $y = 8^t$.

...

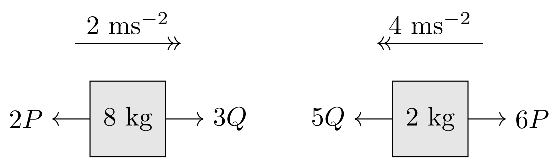
$16^t = (8^{\frac{4}{3}})^t = (8^t)^{\frac{4}{3}} = y^{\frac{4}{3}}$.

607. Show that there are no simultaneous solutions to the equations $x^2 + y^2 = 1$, $x^2 - y^2 = 2$.

Treat this as a pair of linear simultaneous equations in the variables x^2 and y^2.

We can eliminate y^2 by adding the equations. This gives $2x^2 = 3$. Substituting back for x^2, we get $y^2 = -\frac{1}{2}$. This has no real roots.

608. Two objects are modelled with the following force diagrams:



Determine the values of P and Q.

Solve simultaneous equations.

$F = ma$ gives simultaneous equations

$$\begin{aligned} 3Q - 2P &= 16 \\ 5Q - 6P &= 8. \end{aligned}$$

Solving gives $P = 7$, $Q = 10$ Newtons.

609. The sum of the first k natural numbers is equal to $3k$. Find k.

Use the formula $S_n = \frac{1}{2}n(n + 1)$.

In algebra, this is $\frac{1}{2}k(k + 1) = 3k$, which is a quadratic. Solving it gives $k = 0$ or $k = 5$. We may assume the question is looking for $k = 5$.

610. Write down the largest real domains over which the following functions may be defined:

(a) $x \mapsto \sqrt{1-x}$,
(b) $x \mapsto \sqrt{1-x^2}$,
(c) $x \mapsto \sqrt{1-x^3}$.

Set up inequalities $1 - x \geq 0$ etc.

We require the inputs of the square root function to be non-negative. This gives the largest domains as

(a) $\{x : x \leq 1\}$,
(b) $\{x : -1 \leq x \leq 1\}$,
(c) $\{x : x \leq 1\}$.

611. Explain why the factor theorem cannot be used, over the real numbers, to establish whether the expression (x^2+1) is a factor of $4x^5 + x + 1$.

Consider $x^2 + 1 = 0$.

To establish whether (x^2+1) is a factor, we would have to find the roots of $x^2 + 1 = 0$. This has no real roots, however. So, unless we use the complex numbers $\mathbb{C}$, the factor theorem gives us no help.

612. Two ships leave port simultaneously. Ship A travels on bearing 020° at 14 mph; ship B travels on bearing 280° at 16 mph. To the nearest minute, determine the time taken for the ships to separate by 10 miles.

Use the cosine rule.

Using the cosine rule, the speed at which the two ships are separating is given by

$$v^2 = 14^2 + 16^2 - 2 \cdot 14 \cdot 16 \cos 100°.$$

Taking the positive square root, $v = 23.01726...$ mph. So, it takes $\frac{10}{23.01726...} \approx 26$ minutes.

613. Write the set $\{x \in \mathbb{R} : x^2 - x \leq 0\}$ in interval notation.

Solve the inequality.

The boundary equation is $x^2 - x = 0$, which has solution set $\{0, 1\}$. Hence, the set given consists of all values between these, inclusively. In interval set notation, this is $[0, 1]$.

614. Provide a counterexample to the following claim: "If a_n and b_n are APs, then so is $a_n b_n$."

Almost any pair of APs will do!

Consider the sequences of the odd and even numbers, which are both APs. The products of these are $1 \cdot 2, 3 \cdot 4, 5 \cdot 6$. Since $12 - 2 \neq 30 - 12$, the sequence of products in not an AP.

615. Find, in terms of the positive constants a, b, the area of the region of the (x, y) plane whose points simultaneously satisfy the following inequalities:

$$\begin{aligned} -a &\leq x \leq a, \\ -b &\leq y \leq b. \end{aligned}$$

Sketch the boundary lines on a graph.

The region described is a rectangle, bounded by the lines $x = \pm a$, $y = \pm b$. Hence, the area is $4ab$.

616. Explain the two ways in which the equation below could have less than two real roots, giving explicit conditions on the constants a, b:

$$\frac{(x+a)(x-a)}{(x+b)(x-b)} = 0.$$

A fraction in its lowest terms is zero iff its numerator is zero.

A fraction in its lowest terms is zero iff its numerator is zero. So, if this is to have less than two real roots, either the fraction must not be in its lowest terms, or the numerator must have fewer than two real roots. The first condition is $a = b$, and the second is $a = 0$.

617. Determine all possible values of $P(A \cap B)$, given that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$.

Draw a Venn diagram, and consider the amount of overlap between the events A and B.

$A \cap B$ is the overlap of A and B. This cannot be zero, since $P(A) + P(B) > 1$. It is minimised with $P(A \cap B) = \frac{1}{6}$. The overlap is maximised if $A \subset B$, giving $P(A \cap B) = \frac{1}{2}$. So, the set of possible values is $P(A \cap B) \in [\frac{1}{6}, \frac{1}{2}]$.

618. Prove that $\tan\theta + \cot\theta \equiv \operatorname{cosec}\theta \sec\theta$.

Begin with the LHS, and use $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$.

Starting with the LHS:

$$\begin{aligned} & \tan\theta + \cot\theta \\ \equiv\ & \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\ \equiv\ & \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} \\ \equiv\ & \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\ \equiv\ & \frac{1}{\sin\theta\cos\theta} \\ \equiv\ & \operatorname{cosec}\theta\sec\theta. \end{aligned}$$

619. It is given that $\frac{d}{dx}F(x) = f(x)$, and that $F(0) = 2$, $F(4) = 6$. Evaluate the following:

(a) $\displaystyle\int_0^4 f(x)\,dx$,

(b) $\displaystyle\int_0^4 5f(x) - 1\,dx$.

$F(x)$ is the integral of $f(x)$.

(a) $$\begin{aligned} & \int_0^4 f(x)\,dx \\ =\ & \big[F(x)\big]_0^4 \\ =\ & F(4) - F(0) \\ =\ & 4. \end{aligned}$$

(b) $$\begin{aligned} & \int_0^4 5f(x) - 1\,dx \\ =\ & \big[5F(x) - x\big]_0^4 \\ =\ & (5F(4) - 4) - (5F(0) - 0) \\ =\ & 16. \end{aligned}$$

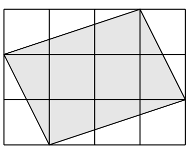
620. Factorise $x^3 - 2x^2 - 5x + 6$.

Look for a root first, then use the factor theorem.

By inspection $x = 1$ is a root. So $(x-1)$ is a factor. This gives

$$\begin{aligned} & x^3 - 2x^2 - 5x + 6 \\ \equiv\ & (x-1)(x^2 - x - 6) \\ \equiv\ & (x-1)(x+2)(x-3). \end{aligned}$$

621. The grid shown below consists of unit squares.



Find the area of the shaded region.

Subtract the area of the four unshaded triangles from the area of the rectangular grid.

The grid has area 12. Subtracted from this are four triangles of total area 5. So the shaded parallelogram has area 7.

622. Show that, if the iteration $x_{n+1} = x_n^2 - 3(x_n+1)^{-1}$ has a fixed point α, then $(x - \alpha)$ is a factor of the cubic expression $x^3 - x - 3$.

A fixed point of a function f is a value such that $f(x) = x$.

If α is a fixed point of this iteration, then it is a root of the equation

$$x = x^2 - \frac{3}{x+1}.$$

Multiplying by $x + 1$ gives $x^2 + x = x^3 + x^2 - 3$, which simplifies to $x^3 - x - 3 = 0$. Hence, by the factor theorem, $(x - \alpha)$ must be a factor of $x^3 - x - 3$.

623. Make x the subject of $y = \dfrac{1 + \frac{1}{x+1}}{1 - \frac{1}{x+1}}$.

Multiply top and bottom of the large fraction by the denominator of the small fractions.

Such fractions are best simplified by multiplying top and bottom of the large fraction by the denominator of the small fractions. This gives

$$\begin{aligned} & y = \frac{x+2}{x} \\ \Longrightarrow\ & xy = x + 2 \\ \Longrightarrow\ & x(y-1) = 2 \\ \Longrightarrow\ & x = \frac{2}{y-1}, \text{ for } y \neq 1. \end{aligned}$$

624. Show that $\displaystyle\int_1^2 \frac{16x + 48}{x^3}\,dx = 26$.

Split the fraction up before integrating.

Splitting the fraction up, we have

$$\begin{aligned} & \int_1^2 16x^{-2} + 48x^{-3}\,dx \\ =\ & \big[-16x^{-1} - 24x^{-2}\big]_1^2 \\ =\ & (-8 - 6) - (-16 - 24) = 26. \end{aligned}$$

625. A function has instruction $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

(a) Find the largest real domain over which f may be defined.

(b) Without using any calculus, write down the least possible value of $f(x)$ on this domain.

(c) Hence, state the range of f, giving your answer in set notation.

...

(a) For f to be well defined, we require $1-x^2>0$, with strict inequality to avoid division by zero. So the largest real domain is $(-1,1)$.

(b) The maximum of $1-x^2$ so 1, so the minimum of f is $\frac{1}{1}=1$.

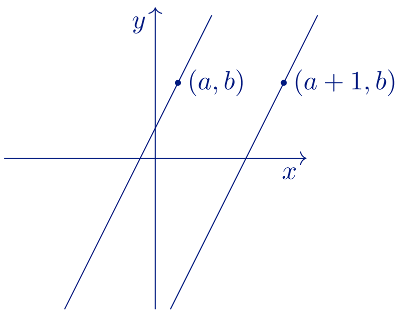
(c) The range is $[1,\infty)$.

626. On the same axes, for positive constants a, b, m, sketch the graphs

(a) $\dfrac{y-b}{x-a}=m$, (b) $\dfrac{y-b}{x-a-1}=m$.

These are parallel straight lines.

These are parallel straight lines, with gradient m, through the points (a,b) and $(a+1,b)$:



627. By taking out a common factor, solve the equation

$$(x+1)^3-4x^2-4x=0.$$

The factor is $(x+1)$.

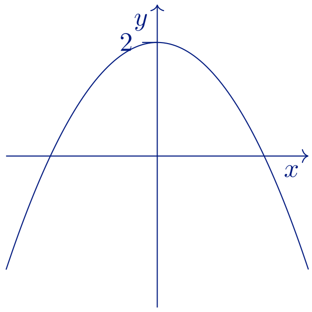
Taking out a common factor of $(x+1)$ directly, without multiplying out:

$$\begin{aligned}
&(x+1)^3-4x^2-4x=0\\
\Longrightarrow\ &(x+1)\big((x+1)^2-4x\big)=0\\
\Longrightarrow\ &(x+1)(x^2-2x+1)=0\\
\Longrightarrow\ &(x+1)(x-1)^2=0\\
\Longrightarrow\ &x=\pm 1.
\end{aligned}$$

628. A particular quadratic function g has $g(0)=2$, $g'(0)=0$, $g''(0)=-2$. Sketch the graph.

The second derivative is constant in a quadratic, so you can integrate it.

You can spot immediately, from the information at $x=0$, that the graph must look as follows:



Alternatively, since the second derivative of any quadratic is constant, $g''(x)=-2$. Integrating, we get $g'(x)=-2x+c$. Substituting $g'(0)=0$ gives $c=0$, so the first derivative is $g'(x)=-2x$. Integrating again, $g(x)=-x^2+d$. Substituting $g(0)=2$ yields $g(x)=-x^2+2$, as above.

629. Using a Venn Diagram, or otherwise, simplify

(a) $(A'\cap B')'$,

(b) $(A\cup B)\setminus(A'\cap B)$.

P' is the complement of P, i.e. not-P; $P\setminus Q$ is P minus Q, i.e. P with the elements of Q removed.

(a) $(A'\cap B')'=A\cup B$.

(b) $(A\cup B)\setminus(A'\cap B)=A$.

630. Find the area of the circle which passes through $(0,0)$, $(4,0)$ and $(-2,6)$.

The centre of the circle must lie on $x=2$.

By symmetry, the centre is at $(2,y)$. Equating the squared distances to points $(0,0)$ and $(-2,6)$, we get $4+y^2=16+(6-y)^2$. This simplifies to $4=16+36-12y$, so $y=4$. The squared radius is therefore 20 and the area is 20π.

631. Simplify $[-1,0]\cap\{x : x^2<\frac{1}{4}\}$.

Consider the intervals on a number line.

The right-hand set is $(-\frac{1}{2},\frac{1}{2})$, so the intersection is $(-\frac{1}{2},0]$.

632. The definite integral below gives the displacement, over a particular time period, for an object moving with constant acceleration:

$$s=\int_0^t u+ak\,dk.$$

Use this to prove the relevant constant acceleration formula.

You are looking for $s=ut+\frac{1}{2}at^2$.

$$\begin{aligned} s &= \int_0^t u + ak\, dk \\ &= \left[uk + \tfrac{1}{2}ak^2\right]_0^t \\ &= u(t-0) + \tfrac{1}{2}a(t-0)^2 \\ &= ut + \tfrac{1}{2}at^2. \end{aligned}$$

633. Solve $2^x - 2^{1-x} = 1$.

Because the second term contains 2^{-x}, multiply the whole equation by 2^x.

Multiplying by 2^x, we get $(2^x)^2 - 2 = 2^x$. This is a quadratic in 2^x. Rearranging and factorising, we have $(2^x-2)(2^x+1) = 0$. The latter factor has no roots, so $x = 1$.

634. The derivative of a function f is constant. Show that $f(x+1) - f(x-1)$ is constant.

Write the first sentence algebraically, integrate, and substitute into the expression given.

We know that $f'(x) = m$, for some constant m. So $f(x) = mx + c$. Substituting, this gives

$$\begin{aligned} & f(x+1) - f(x-1) \\ =\ & m(x+1) + c - (m(x-1) + c) \\ =\ & mx + m + c - mx + m - c \\ =\ & 2m, \text{ which is constant.} \end{aligned}$$

635. By considering intersections with the line $y = x$, show that the outputs of $g(x) = x^2 - 2x + 3$ are always greater than its inputs.

Sketch the graphs $y = x$ and $y = g(x)$.

Solving to find intersections, we have $x^2 - 3x + 3 = 0$, which has discriminant $\Delta = 9 - 12 < 0$. Hence, the positive parabola $y = g(x)$ is always above the line $y = x$. This means that the outputs of g (y values) are always bigger than the inputs of g (x values).

636. Factorise $680x^2 - 842x - 1207$.

Solve a quadratic equation.

Setting up a quadratic equation and solving using the formula, we find that this expression is zero at

$$\begin{aligned} x &= \frac{842 \pm \sqrt{842^2 + 4 \cdot 680 \cdot 1207}}{2 \cdot 680} \\ &= \frac{71}{34}, -\frac{17}{20}. \end{aligned}$$

So, by the factor theorem, the expression has factors of $(34x - 71)$ and $(20x + 17)$. Since $34 \cdot 20 = 680$, we have

$$680x^2 - 842x - 1207 \equiv (34x - 71)(20x + 17).$$

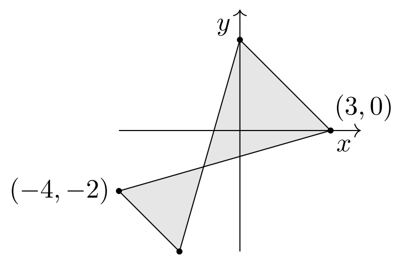
637. The quartic approximation to $\tan^2 x$, for x defined in radians, is $\tan^2 x \approx x^2 + \frac{2}{3}x^4$. Determine the percentage error in this approximation at $\theta = \frac{1}{6}\pi$.

...

The exact value of $\tan^2 \frac{\pi}{6}$ is $\frac{1}{3}$. So, the percentage error is given by

$$\frac{\frac{\pi}{6}^2 + \frac{2}{3} \cdot \frac{\pi}{6}^4 - \frac{1}{3}}{\frac{1}{3}} \approx -2.7\%.$$

638. Find the area of the shaded region depicted below, which has the line $y = x$ as a line of symmetry.



Find the coordinates of the intersection point, which must lie on $y = x$.

The intersection lies on $y = x$ and $7y = 2x - 6$. Solving simultaneously gives $(-1.2, -1.2)$. Using the fact that the triangles are isosceles, the total shaded area is given, then, by

$$\begin{aligned} & \tfrac{1}{2} \cdot 2\sqrt{2} \cdot 1.8\sqrt{2} + \tfrac{1}{2} \cdot 3\sqrt{2} \cdot 2.7\sqrt{2} \\ &= 3.6 + 8.1 \\ &= 11.7 \end{aligned}$$

639. Show that a rectangle with sides 20 and 28 cm will not fit inside a circle of diameter 34 cm.

Calculate the length of the diagonals of the rectangle.

The diagonals of the rectangle have length $\sqrt{20^2 + 28^2} = 4\sqrt{74} > 34$ cm. These will not fit inside a circle of diameter 34 cm.

640. Prove that, if x and y are related linearly and so are y and z, then so are x and z.

Express these facts algebraically, then combine them.

We know that $x = ay + b$ and $y = cz + d$, for some constants a, b, c, d. Substituting gives $x = a(cz+d)+b$, which simplifies to $x = (ac)z+(ad+b)$. This has the form of a linear relationship.

641. Assuming the formula $\log_a x + \log_a y = \log_a xy$, either prove or disprove the following:

$$\log_a x + \log_a y + \log_a z = \log_a xyz.$$

Start with the RHS.

Starting with the RHS, we have

$$\begin{aligned} & \log_a xyz \\ = \ & \log_a x + \log_a yz, \text{ by the assumed log rule} \\ = \ & \log_a x + \log_a y + \log_a z, \text{ again by the assumed log rule.} \end{aligned}$$

This proves the required result.

642. Write $x^2 + x + 6$ in terms of $(x + 1)$.

Start with $(x + 1)^2$.

Start with $(x + 1)^2$. This gives $2x$, where x is required. So, we subtract $(x + 1)$. This gives a constant term of 0, when we want 6. So,

$$x^2 + x + 6 \equiv (x + 1)^2 - (x + 1) + 6.$$

643. The cubic equation $(x + a)(x^2 + 4x + b) = 0$, for constants $a, b \in \mathbb{R}$ has exactly one real root. Find the set of possible values of b.

Since $x = -a$ is already a root, consider the right-hand bracket.

Since $x = -a$ is clearly a root, the right-hand bracket must have either no real roots, or it must be equal to $(x + a)^2$. The former is true if $b \in (4, \infty)$. The latter can only be true if $a = 2$, in which case b can be exactly 4. So, the set of possible values of b is $[4, \infty)$.

644. Evaluate the following sums:

(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots,$

(b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots,$

These are both GPs.

Both are GPs. So, we use $S_\infty = \dfrac{a}{1-r}$, which gives

(a) $S_\infty = \dfrac{1}{1-\frac{1}{2}} = 2,$

(b) $S_\infty = \dfrac{1}{1+\frac{1}{2}} = \dfrac{2}{3}.$

645. Give the acceleration of a lift if accurate weighing scales placed inside it underestimate mass by 15%.

Weighing scales measure downwards reaction force on them. This force is the Newton pair of the upwards reaction force on the object being weighed.

Weighing scales measure downwards reaction force. This force is the Newton pair of the upwards reaction force on the object being weighed. So, we know that, in the accelerating lift, the upwards reaction force on an object of weight mg is $0.85mg$. Hence, $F = ma$ is $0.15mg = ma$, and the acceleration is $0.15g$ downwards.

646. When divided by $(x-3)$, the quadratic px^2+x+p leaves no remainder. Find the value of p.

Use the factor theorem.

Since division by $(x-3)$ leaves no remainder, $(x-3)$ is a factor of the quadratic. Hence, by the factor theorem, $x = 3$ is a root. So $9p + 3 + p = 0$, which gives $p = -\frac{3}{10}$.

647. The *lazy caterer's sequence* describes the number of pieces into which a circular cake can be cut using n straight line cuts. Its formula is

$$P_n = \frac{n^2 + n + 2}{2}.$$

Verify this formula for $n = 1, 2, 3, 4$.

Sketch four cakes, with cuts drawn on them.

The formula gives $2, 4, 7, 11$. We can verify this (and sketch a formal proof) by seeing that the zeroth case $P_0 = 1$ is trivial, and that the $(n + 1)$th cut can cross a maximum of n lines, which gives creation of a maximum of $(n + 1)$ new pieces. So,

$$\begin{aligned} P_0 &= 1 \\ P_1 &= 1 + 1 = 2, \\ P_2 &= 2 + 2 = 4 \\ P_3 &= 4 + 3 = 7, \\ P_4 &= 7 + 4 = 11. \end{aligned}$$

This verifies the ordinal formula.

648. A straight line L is parallel to the line $2x+3y = 7$ and passes through the midpoint of $(-10, 0)$ and $(-2, 4)$. Show that L goes through $(6, -6)$.

Find the equation of L.

L has equation $2x + 3y = k$, and passes through $(-6, 2)$. Hence, it has equation $2x+3y = -6$. Substituting $x = 6$ gives $2 \cdot 6 + 3 \cdot -6 = -6$, so L goes through $(6, -6)$.

649. Provide counterexamples to the following:

(a) "All points of inflection are stationary."

(b) "No point of inflection is stationary."

(a) Find a curve with a point of inflection that is not stationary.

(b) Find a curve with a point of inflection that is stationary.

(a) The origin in $y = x^3 + x$ is a counterexample.

(b) The origin in $y = x^3$ is a counterexample.

650. Show that, for $a \neq 0$, $y = a(x - b) + c$ and $y = -\frac{1}{a}(x - b) + c$ intersect normally at (b, c).

These are two straight lines.

These are straight lines. Their gradients are a and $-\frac{1}{a}$, which are negative reciprocal, so they intersect normally. Furthermore, at $x = b$, each gives $y = c$. So, they intersect normally at (b, c).

651. A chemist is measuring acidity during a titration experiment. The pH value of the contents of a flask is modelled, for the first minute of the experiment, as having a rate of change, in units of s^{-1},

$$\frac{d}{dt}(\text{pH}) = 0.32 - 0.012t.$$

(a) State the initial rate of change of pH.

(b) Find the time at which pH begins to fall.

(c) Determine ΔpH over the first minute.

(d) Explain how you know that pH has no units.

Remember that the differential operator $\frac{d}{dt}$ has units of s^{-1}.

(a) At $t = 0$, $\dfrac{d}{dt}(\text{pH}) = 0.32\ s^{-1}$.

(b) The pH begins to fall from the moment at which the rate of change drops to zero. Solving $0.32 - 0.012t = 0$ gives $t = 26.7$ s (3sf).

(c) ΔpH is the total change in pH. This is given by the integral of the rate of change of pH:

$$\begin{aligned}\Delta\text{pH} &= \int_0^{60} 0.32 - 0.012t\,dt \\ &= \left[0.32t - 0.006t^2\right]_0^{60} \\ &= 0.32 \cdot 60 - 0.006 \cdot 60^2 \\ &= -2.4\end{aligned}$$

(d) Since $\frac{d}{dt}(\text{pH})$ has units s^{-1}, and $\frac{d}{dt}$ also has units s^{-1}, pH cannot have units.

652. Explain whether it is possible for the two forces of a Newton III pair to act on the same object.

The answer is yes and no.

Yes and no. If forces internal to an object are considered, then both parts of a Newton III pair may be seen to act on a single object. But, in this case, they cancel out, and have no resultant effect. If the two forces of a Newton III pair are to have a resultant effect, then they must be taken as acting on two different objects.

653. A sample is taken, with statistics as follows:

$$n = 25, \quad \sum x = 327, \quad \sum x^2 = 5229.$$

Find the mean and standard deviation.

Use $s = \sqrt{\frac{\sum x^2 - n\bar{x}^2}{n}}$.

The mean is $\frac{327}{25} = 13.08$. The standard deviation is

$$\sqrt{\frac{5229 - 25 \cdot 13.08^2}{25}} = 6.17 \text{ (3sf)}.$$

654. Show that, if the limits of a definite integral are switched, then the value of the integral is negated.

If $F'(x) = f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

If $F'(x) = f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$. Reversing the limits gives $F(a) - F(b)$, which is equal to $-(F(b) - F(a))$.

655. Prove that, if x and y are large positive integers which satisfy *Pell's equation* $x^2 - 2y^2 = 1$, then $\frac{x}{y}$ is an approximation for $\sqrt{2}$.

Rearrange to make $\sqrt{2}$ the subject.

Rearranging $x^2 - 2y^2 = 1$, we get

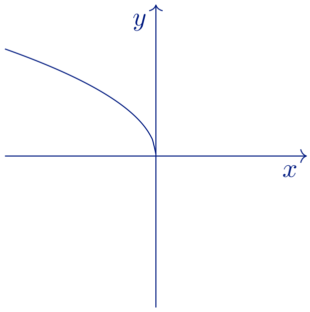
$$\begin{aligned} 2 &= \sqrt{\frac{x^2 - 1}{y^2}} \\ \implies \sqrt{2} &= \frac{\sqrt{x^2 - 1}}{y}. \end{aligned}$$

For any large positive integer x, we have $\sqrt{x^2 - 1} \approx x$, so $\sqrt{2} \approx \frac{x}{y}$.

656. Sketch the set of points which satisfy $y = \sqrt{-x}$.

Consider the graph (set of points) as a transformation of $y = \sqrt{x}$.

The graph $y = \sqrt{-x}$ is a reflection of $y = \sqrt{x}$ in the y axis:



657. Explain the difference in meaning, with regard to changes in t, between the notations Δt, δt, dt.

Consider the concept of a limit.

Upper-case "Delta t" Δt is a finite change in t, such as a period of 3 days. Lower-case "delta t" δt is also a finite change in t, but has the connotation of being small, such as a period of a microsecond. Latin dt is an infinitesimal change in t, and cannot be defined in isolation. It notates the limit of δt as $\delta t \to 0$, and only makes sense in either

- comparison to other infinitesimal quantities, such as in $\frac{dx}{dt}$, or
- when summed infinitely in $\displaystyle\int f(t)\,dt$.

658. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $x \in \mathbb{Q}'$,
- $x \in \mathbb{R}'$.

$\mathbb{Q}$ is the rationals.

Since the rationals $\mathbb{Q}$ are a subset of the real $\mathbb{R}$, the implication is $\Longleftarrow$. If a number is non-real, then it cannot be rational.

659. Determine the fraction of the area of a 2 by 4 cm rectangle which is within 1 cm of a vertex.

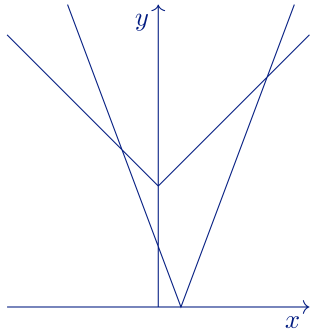
Consider circles.

Since two 1 cm radii fit into the shorter side of the rectangle, the region within 1 cm of a vertex is four quarter circles, i.e. one whole circle of radius 1 cm. Thus, the fraction is $\frac{\pi}{8}$.

660. Solve the equation $|3x - 1| = 2 + |x|$.

Sketch graphs of $y =$LHS and $y =$RHS.

The graphs of $y =$LHS and $y =$RHS are



The points of intersection occur when the mod signs are (active, active) and (passive, passive). So, the equations we require are $-(3x - 1) = 2 - (x)$, which gives $x = -\frac{1}{2}$, and $3x - 1 = 2 + x$, which gives $x = \frac{3}{2}$.

661. "The y axis is tangent to the curve $x = y^2 - 8x + 16$." True or false?

Factorise.

Factorising, we have $x = (y - 4)^2$, which shows that the curve is tangent to the y axis at $y = 4$.

662. A pair of events X and Y have probabilities, for constants m and n, given as follows:

$$P(X) = \frac{1}{m}, \quad P(Y) = \frac{1}{n}, \quad P(X \cup Y) = \frac{m + n - 1}{mn}.$$

Show that X and Y are independent.

Use $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$, which is always true, to show that, in this case, $P(X) \cdot P(Y) = P(X \cup Y)$.

Using $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$, we see that

$$\begin{aligned} P(X \cap Y) &= \frac{1}{m} + \frac{1}{n} - \frac{m + n - 1}{mn} \\ &= \frac{n}{mn} + \frac{m}{nm} - \frac{m + n - 1}{mn} \\ &= \frac{1}{mn} \\ &= P(X) \cdot P(Y). \end{aligned}$$

We have shown that $P(X \cap Y) = P(X) \cdot P(Y)$, which is equivalent to the statement that events X and Y are independent.

663. A function f has domain $[0, 1]$ (and cannot be defined outside this domain) and range $[0, 1]$. State, with a reason, whether the following are well-defined functions over the domain $[0, 1]$:

(a) $x \mapsto f(x) + 2$,

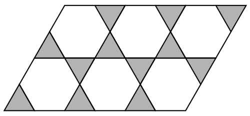
(b) $x \mapsto f(x + 2)$.

Consider, in each case, the set of values that are to serve as inputs to the function f.

(a) This is well defined. Adding one to the outputs of f does not affect its domain of definition.

(b) This is not well defined. Adding one to the inputs of f means that an original domain of $[0, 1]$ would produce inputs to f in the domain $[2, 3]$. The function f is not defined for these values.

664. A tiling pattern, consisting of regular hexagons and equilateral triangles, is as below. Determine the fraction of the total area that is covered by the triangles.



A hexagon consists of six equilateral triangles.

Since a hexagon consists of six equilateral triangles, the fraction covered is $\frac{1}{3}$.

665. A particle is modelled as having initial velocity u and constant acceleration a. By integrating a twice with respect to t, show that $s = ut + \frac{1}{2}at^2$.

Remember the $+c$!

Integrating once gives $v = at + c$. The constant is the value of v at $t = 0$, which is the initial velocity u. This gives $v = u + at$. Integrating again, we get $s = ut + \frac{1}{2}at^2 + c'$. Since, in suvat, displacement is measured from the initial position, c' is zero by definition.

666. A die is rolled. Determine whether knowing "The score is even" changes the probability that the score is prime.

Consider the restriction to the possibility space enacted by the piece of information given.

The probability that the score on a die is prime is usually $\frac{3}{6}$. Knowing that the score is even decreases the numerator to 1 and the denominator to 3. Hence, this knowledge decreases the relevant probability.

667. Evaluate $\left[\dfrac{x!}{1 + 2^{-x}}\right]_0^1$

0! is defined to be 1.

Given that $0! = 1$ by definition, this is

$$\left[\frac{x!}{1 + 2^{-x}}\right]_0^1 = \frac{1}{1 + \frac{1}{2}} - \frac{1}{1 + 1} = \frac{1}{6}.$$

668. The solution of the following equation is the same for all but two values of the constant k.

$$\frac{x^2 - 4}{x^2 - kx} = 0.$$

Write down those values.

Consider the fact that a fraction in its lowest terms is zero iff its numerator is zero.

Since the numerator is fixed, the solution can only be different if the denominator shares factors with the numerator. This is the case if $k = \pm 2$.

669. Find the interquartile range of the standardised normal distribution $N(0, 1)$.

We require $P(X < X_1) = 0.25$ etc.

Using a statistical function on a calculator, the quartiles are given by $\pm 0.674489...$. So, the IQR is the difference, which is $2 \cdot 0.674489... = 1.35$ (3sf).

670. (a) Find $\displaystyle\int_1^N \frac{1}{x^2}\,dx$.

(b) Hence, determine $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

Consider (b) as a limit of (a).

(a) $\displaystyle\int_1^N \frac{1}{x^2}\,dx = \left[-x^{-1}\right]_1^N = 1 - \frac{1}{N}$.

(b) The infinite integral is defined as

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x^2}.$$

From part (a), this is $\displaystyle\lim_{N\to\infty} 1 - \frac{1}{N} = 1$.

671. The first three terms of a geometric sequence are given as a, $a + 2$. $a + 3$. Find a.

Set up an equation encoding the fact "the ratio between terms 1 and 2 and terms 2 and 3 is the same".

Since this is a GP, we know that

$$\begin{aligned} &\frac{a+2}{a} = \frac{a+3}{a+2} \\ \Longrightarrow\ &(a+2)^2 = a(a+3) \\ \Longrightarrow\ &a^2 + 4a + 4 = a^2 + 3a \\ \Longrightarrow\ &a = -4. \end{aligned}$$

672. A car is turning around a corner. Explain how you know that the reaction forces exerted by the road on the wheels and the frictional forces exerted by the road on the wheels are perpendicular.

This is true by definition.

This is true by definition, regardless of the circumstances. Reaction forces are contact forces perpendicular to the relevant surfaces, while frictional forces are contact forces parallel to the relevant surfaces. These are at right angles to each other by definition. N.B. this modern usage of the word "reaction" does not tie in exactly with Newton's original usage; in modern parlance, a Newton III pair may consist of either zero or two reaction forces, but never one.)

673. You are told that "The rate of change of y with respect to x is proportional to x^2."

(a) Express the above statement as an equation, using a constant of proportionality k.

(b) You are also told that $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{3}} = 6$. Find k.

(c) By integrating, write y in terms of x.

(d) When $x = 0, y = 6$. Find y when $x = 3$.

...

(a) $\frac{dy}{dx} = kx^2$,

(b) $6 = 3k$, so $k = 2$,

(c) $y = \frac{2}{3}x^3 + c$,

(d) The information given tells us that $c = 6$. So, when $x = 3$, $y = \frac{2}{3} \cdot 3^3 + 6 = 24$.

674. It is given that the quadratic $px^2 + qx + r = 0$ has exactly two roots. Show that $px^2 - qx + r = 0$ also has exactly two roots.

Use the discriminant, or a graphical argument.

Since the discriminant is $\Delta = b^2 - 4ac$, a negation of the x term does not affect it. Hence, these two equations have the same Δ and thus the same number of roots.

Alternatively, the transformation from $y = px^2 + qx + r$ to $y = px^2 - qx + r$ is a reflection in the y axis. This does not affect the number of intersections with the x axis.

675. Without using calculus, show that $y = 3x + 2$ is a tangent to $y = x^3$.

Solve for intersections, and consider double roots.

Solving for intersections, we require $x^3 - 3x - 2 = 0$. By inspection, this is satisfied by $x = -1$. Hence $(x + 1)$ must be a factor. Taking it out, we get $(x+1)(x^2 - x - 2) = 0$, which then factorises again to $(x + 1)^2(x - 2) = 0$. Since this equation has a double root at $x = -1$, the line must be tangent to the cubic at $x = -1$.

676. A projectile is launched from ground level at speed 14 ms^{-1}, at an angle of 30° above the horizontal.

(a) State the assumptions of the projectile model.

(b) Find the horizontal and vertical components of the initial velocity.

(c) By considering the vertical motion, find the time taken for the projectile to land again.

(d) Hence, show that the range is $10\sqrt{3}$ m.

...

(a) A projectile is modelled as a particle (object of negligible size) with acceleration g ms^{-1} vertically downwards.

(b) $7\sqrt{3}$ and 7 ms^{-1}.

(c) Vertically, $0 = 7t - 4.9t^2$, which has roots $t = 0$ and $t = \frac{10}{7}$. The former is take-off, the latter is landing.

(d) Since horizontal acceleration is zero, the range is $7\sqrt{3} \cdot \frac{10}{7} = 10\sqrt{3}$ m.

677. Show that the equation of the locus of points equidistant from $y = x$ and $y = -x$ is $xy = 0$.

The set of points equidistant from a pair of lines consists of their angle bisectors.

The set of points equidistant from a pair of lines consists of their angle bisectors. In this case, this is the x and y axes, i.e. $y = 0$ or $x = 0$. This may be written as a single equation: $xy = 0$.

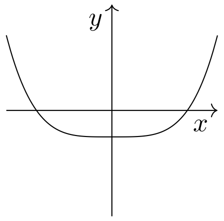
678. Using a Venn diagram, or otherwise, prove that, for any events A and B,

$$P(A \cup B) + P(A \cap B) = P(A) + P(B).$$

Consider the intersection as an overlap.

The LHS is the probability of the sum of the union and the intersection. This consists of elements in one of A or B counted once, and elements in both A and B counted twice. Split this "double counting" up, and we get precisely $P(A) + P(B)$.

679. Prove that $y = (x - 1)^4$ could not possibly be the equation of the following sketched graph:



Consider the range of $(x-1)^4$.

The quartic graph shown has range $[k, \infty)$, for some $k < 0$, while $y = (x-1)^4$ has range $[0, \infty)$. (There are lots of other ways of proving this.)

680. The tangent to the curve $y = x^2$ at point A passes through $(0, -4)$.

(a) Write down the equation of a line, with gradient m, through the point $(0, -4)$.

(b) Explain why, if such a line is to be tangent to the curve, then the equation $x^2 = mx - 4$ must have precisely one root.

(c) Hence, using the discriminant, or otherwise, show that P is $(\pm 2, 4)$.

...

(a) The equation of a general line, gradient m, through $(0, -4)$, is $y = mx - 4$.

(b) $x^2 - mx + 4 = 0$ is a quadratic equation in x. It must have a double root at a point of tangency, so cannot cross the curve again elsewhere.

(c) We require $\Delta = 0$, so $m^2 - 16 = 0$, whence $m = \pm 4$. Solving for x and substituting gives P as $(\pm 2, 4)$.

681. By setting $x = \cos y$, show that

$$\sin(\arccos x) = \sqrt{1 - x^2}.$$

Use the Pythagorean identity.

Setting $x = \cos y$, we have

$$\begin{aligned} & \sin^2 y + \cos^2 y = 1 \\ \Longrightarrow\ & \sin y = \pm\sqrt{1 - \cos^2 y} \\ \Longrightarrow\ & \sin y = \pm\sqrt{1 - x^2}. \end{aligned}$$

Since the range of arccos is positive, we can take the positive square root, giving $\sin(\arccos x) = \sqrt{1 - x^2}$.

682. The number of people x living in each flat in a particular block is given as follows:

x	1	2	3	4	5	6	7
f	12	22	14	17	15	8	3

Use statistical functions on your calculator to find

(a) $S_{xx} = \sum (x_i - \bar{x})^2$,

(b) the variance.

Use the formula $S_{xx} = \sum x^2 - \frac{1}{n}\left(\sum x\right)^2$.

(a) Using $S_{xx} = \sum x^2 - \frac{1}{n}\left(\sum x\right)^2$, the calculator gives $S_{xx} = 1308 - \frac{310^2}{91} = 252$, (3sf).

(b) The variance can be read off directly, as 2.77 (3sf), or calculated as $\frac{S_{xx}}{91}$.

683. Find the range of $f(x) = x^2 - 1$ over $[2, 3]$.

Sketch the function.

The domain does not include the vertex of the quadratic, which means that the function is increasing on $[2, 3]$. Hence, the range is simply $[f(2), f(3)]$, which is $[4, 9]$.

684. State that the following holds, or explain why not: "If there is no resultant force on an object, then it is in equilibrium."

Consider moments.

This is not true. According to the Newtonian model, for an object to be in equilibrium, it is required that it has not only no resultant force on it, but also no resultant moment. Objects with no resultant force acting on them may still experience angular acceleration.

685. By factorising, evaluate $\displaystyle\lim_{k \to 2} \frac{4k^2 - 16}{k^2 - 2k}$.

Reduce the fraction to its lowest terms before taking the limit.

We reduce the fraction to its lowest terms before taking the limit. This gives

$$\begin{aligned} & \lim_{k \to 2} \frac{4k^2 - 16}{k^2 - 2k} \\ =\ & \lim_{k \to 2} \frac{4(k+2)(k-2)}{k(k-2)} \\ =\ & \lim_{k \to 2} \frac{4(k+2)}{k}, \text{ since } k \neq 2 \\ =\ & 8. \end{aligned}$$

686. Two astronauts, each of mass m, are rotating around each other in space, holding onto the ends of a light cord of length d. The tension in the cord

is the only force acting on them. Each astronaut is moving in a circle at speed v. At time $t = 0$, one of them lets go of the cord. Show that the distance between them in the subsequent motion is given by $x = \sqrt{4v^2t^2 + d^2}$.

Consider that, when one lets go, the motion is symmetrical for both, by Newton's third law.

Newton III tells us that one astronaut letting go is equivalent to both astronauts letting go. So, the two will move off along diametrically opposite tangents to the circle of rotation. After t seconds, they will have separated, in the tangential direction, by $2vt$, and will therefore be a distance $\sqrt{4v^2t^2 + d^2}$ apart.

687. Describe all functions f for which f' is constant.

Consider integration.

If $f'(x) = a$, then $f(x) = ax + b$. This is a general formula for any linear function.

688. Using a double-angle identity, it can be shown that

$$2\sin 18° = 1 - 4\sin^2 18°.$$

Use this to determine the exact value of $\sin 18°$.

This is a quadratic in $\sin 18°$.

Rearranging and using the quadratic formula:

$$\begin{aligned} 4\sin^2 18° + 2\sin 18° - 1 &= 0 \\ \implies \sin 18° &= \frac{-2 \pm \sqrt{4 - 4 \cdot 4 \cdot -1}}{2 \cdot 4} \\ \implies \sin 18° &= \frac{-1 \pm \sqrt{5}}{4}. \end{aligned}$$

Since $18°$ is an acute angle, we need the positive root, so $\sin 18° = \frac{-1 \pm \sqrt{5}}{4}$.

689. A straight line is given parametrically as $x = 1 - \lambda$, $y = 3 + 2\lambda$, for $\lambda \in \mathbb{R}$. This line is then translated by the vector $2\mathbf{i} + 3\mathbf{j}$. Write down the equation of the new line, in the same form.

Add the relevant quantities to increase the value of x by 2 and the value of y by 3.

Equivalently, we either replace x by $x - 2$ and y by $y - 3$, or add 2 to the t-expression for x and add 3 to the t-expression for y. This gives the new equation as $x = 3 - \lambda$, $y = 6 + 2\lambda$, for $\lambda \in \mathbb{R}$.

690. Find the set of values of x for which the derivative of $y = x^2 - 10x + 2$ is non-negative.

Set up an inequality and solve.

We require $2x - 10 \geq 0$, so $x \in [5, \infty)$.

691. A function taking real inputs has instruction

$$g : x \mapsto \frac{1}{\sqrt{x^2 + px + q}}.$$

You are given that the largest domain over which this function is well defined is $(\infty, -3) \cup (4, \infty)$. Determine the values of the constants p and q.

The function is not well defined when $x^2 + px + q \leq 0$.

The function is ill defined when $x^2 + px + q \leq 0$. We are given that this is for $x \in [-3, 4]$. Hence, the quadratic must be $(x + 3)(x - 4) = x^2 - x - 12$. So $p = -1$ and $q = -12$.

692. It is given that $x + y$ is constant. Write down $\dfrac{dy}{dx}$.

Consider the line $x + y = k$.

If $x + y$ is constant, then $x + y = k$ for some $k \in \mathbb{R}$. Differentiating, we get $1 + \frac{dy}{dx} = 0$. So $\frac{dy}{dx} = -1$.

693. State, with a reason, whether the line $x - 2y = 0$ intersects the following curves:

(a) $y = |x| + 1$,

(b) $y = |x| - 1$,

(c) $x = |y| + 1$,

(d) $x = |y| - 1$.

Consider the fact that $x - 2y = 0$ passes through the origin with gradient 2.

(a) Yes.

(b) Yes.

(c) No.

(d) Yes.

694. The difference between integers p and q is six fifths of their arithmetic mean $\frac{p+q}{2}$. Find their difference in terms of their geometric mean $\sqrt{pq}$.

Find p in terms of q first.

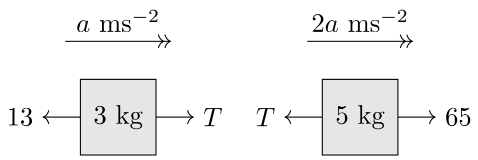
We know that $p - q = \frac{6}{10}(p + q)$,, which tells us that $p = 4q$. The difference $p - q$ is then $3q$, while the geometric mean $\sqrt{pq}$ is $2q$. So, $p - q = \frac{3}{2}\sqrt{pq}$.

695. Describe formally the transformation which takes the graph $y = x^2$ onto the graph $y = (x - 3)^2$.

Express the translation with a vector.

The transformation is a translation (not a "shift", which is informal language) by vector $3\mathbf{i}$.

696. Two objects are modelled as depicted:



(a) Find a and T.

(b) The forces with magnitude T are Newton III tensions exerted by a string. Explain how you know that the string must be extensible.

Solve simultaneously.

(a) $F = ma$ gives $T - 13 = 3a$ and $65 - T = 10a$. Adding gives $52 = 13a$, so $a = 4$ and $T = 25$ N.

(b) Since the accelerations of the two objects are different, the string connecting them must be capable of extension.

697. The interior angles of a triangle are in AP. Show that one of the angles must be $\frac{\pi}{3}$ radians.

Consider the average side length.

Three values in AP must be symmetrical around the central value. Since the sum of the three angles must be π radians, this means that the middle angle must have value $\frac{\pi}{3}$ radians.

698. Evaluate $1024 + 512 + 256 + \ldots$.

This is an infinite GP.

This is an infinite GP, with first term $a = 1024$ and common ratio $r = \frac{1}{2}$. Hence, the sum is

$$S_\infty = \frac{1024}{1 - \frac{1}{2}} = 2048.$$

699. A square is drawn inside $x^2 + 4x + y^2 - 6y = 0$, with its four vertices on the circumference.

(a) Complete the square in x and y, and hence find the radius of the circle.

(b) Show that the square has area 26.

For (b), there is no need to use coordinate geometry.

(a) Completing the square gives

$$(x + 2)^2 + (y - 3)^2 = 13.$$

So the radius is $\sqrt{13}$.

(b) The diagonals of the square are $2\sqrt{13}$, so its side lengths are $\frac{2}{\sqrt{2}}\sqrt{13}$. The area is then $\frac{4 \cdot 13}{2} = 26$.

700. Vertices of a polygon are named in order around the perimeter. Show that, under this convention, $ABCD$, with $A : (0, 0)$, $B : (5, 0)$, $C : (2, 1)$ and $D : (9, 4)$, is not convex.

A non-convex quadrilateral is one that intersects itself. Hence, show that two of the edges intersect.

Sketching shows that edges AD and BC are the ones that intersect. This can be shown formally by the fact that the the intersection of AD, which has equation $y = \frac{4}{9}x$, and BD, which has equation $x + 3y = 5$, is $(\frac{15}{7}, \frac{20}{21})$. This is between A and D, and between B and C.

701. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $(x - k)\cos x = 0 \implies x = k$,

(b) $(x - k)2^x = 0 \implies x = k$.

Consider the range of the functions $x \mapsto \cos x$ and $x \mapsto 2^x$.

(a) Since $\cos x = 0$ has roots, this is not true. $x = \frac{\pi}{2} \neq k$ is a counterexample.

(b) Since $2^x = 0$ has no roots, this is true.

702. The distribution of the means of samples of size 20 taken from a large population is $\bar{X} \sim N(50, 1.25)$. Give the mean and variance of the population.

Use the fact that the sample mean $\bar{X}$ of a distribution $X \sim N(\mu, \sigma^2)$ is $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

The sample mean $\bar{X}$ of a distribution $X \sim N(\mu, \sigma^2)$ is $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$. So, the population mean is 50, and the population variance is given by $\frac{\sigma^2}{20} = 1.25$. Hence, $\sigma^2 = 25$.

703. A function is given, for $x \neq 0$, by

$$f : x \mapsto \frac{4}{x^2} + x^3 + 4x + 7.$$

(a) Show that $f'(2) = 15$ and evaluate $f(2)$.

(b) Hence, show that the linear function that best approximates $f(x)$ at $x = 2$ is $g(x) = 15x - 6$.

"The best linear approximation at $x = 2$" is another way of saying "the tangent line at $x = 2$".

(a) Differentiating, $f'(x) = -8x^{-3} + 3x^2 + 4$. Evaluating, $f'(2) = -1 + 12 + 4 = 15$, and $f(2) = 24$.

(b) The best linear approximation is the tangent line $y = 15x + c$. Substituting $(2, 24)$ gives $y = 15x - 6$.

704. Solve $\cos^2 x - \cos x = 0$, for $x \in [-180°, 180°)$.

Factorise.

Factorising, we get $\cos x(\cos x - 1) = 0$. So $\cos x = 0, 1$. This gives $x = -90°, 0°, 90°$.

705. A student writes: "If the vertex of a positive parabola is at (a, b), then the parabola must have the equation $y = (x - a)^2 + b$." Correct the error.

Consider possible vertical stretches.

The student has ignored the fact that a parabola can be stretched in the y direction while maintaining the same vertex. Hence, the text should read "... must have the equation $y = k(x - a)^2 + b$."

706. A particle moves with position given by $r = t^3 - t^2$, for the time period $t \in [0, 5]$. Show that, at time $t = \frac{1}{3}(1+\sqrt{61})$, its instantaneous speed is the same as its average speed over the five seconds.

For the average speed, calculate the displacement over the five second period. For the instantaneous speed, differentiate.

The average speed is given by

$$\frac{\left[t^3 - t^2\right]_0^5}{5} = 20.$$

For instantaneous speed, we differentiate, obtaining $v = 3t^2 - 2t$. Setting this equal to 20 gives a quadratic:

$$\begin{aligned} & 3t^2 - 2t - 20 = 0 \\ \Longleftrightarrow\ & t = \frac{2 \pm \sqrt{4 - 4 \cdot 3 \cdot -20}}{2 \cdot 3} \\ & \quad = \frac{1 \pm \sqrt{61}}{3}. \end{aligned}$$

The implication going up the page gives the required result.

707. Showing detailed reasoning, solve the following simultaneous equations, giving exact solutions in $\mathbb{Q}$:

$$\begin{aligned} 4x^2 + y^2 &= 5, \\ 2x - 4y + 7 &= 0. \end{aligned}$$

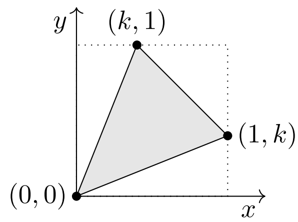
Substitute for x or y, and find an algebraic factorisation for the resulting quadratic. N.B. You can use a calculator to <u>find</u> the next step in a detailed piece of algebraic reasoning.

Substituting $x = 2y - \frac{7}{2}$ gives

$$\begin{aligned} & 4(2y - \frac{7}{2})^2 + y^2 = 5 \\ \Longrightarrow\ & 17y^2 - 56y + 44 = 0 \\ \Longrightarrow\ & (17y - 22)(y - 2) = 0 \\ \Longrightarrow\ & y = 2, \frac{22}{17}. \end{aligned}$$

This gives the simultaneous solution as $(\frac{1}{2}, 2)$, $(-\frac{31}{34}, \frac{22}{17})$.

708. An isosceles triangle is set up, inside a unit square, with one vertex at the origin.



(a) Show that the area of the shaded triangle is $A = \frac{1}{2}(1 - k^2)$.

(b) Find $\frac{dA}{dk}$, and hence prove that the maximal area of such a triangle is $\frac{1}{2}$.

In (a), subtract right-angled triangles from the square. In (b), set $\frac{dA}{dk} = 0$.

(a) Subtracting right-angled triangles from the square, the area of the shaded region is given by $A = 1 - \frac{1}{2}k - \frac{1}{2}k - \frac{1}{2}(1-k)^2$, which simplifies to $A = \frac{1}{2}(1 - k^2)$.

(b) Differentiating gives $\frac{dA}{dk} = -k$. So the area is stationary when $k = 0$. This gives $A = \frac{1}{2}$.

709. Disprove the following: "No term in the sequence $u_n = n^2 + 2$ is a perfect cube."

Look for a counterexample.

Counterexample: $n = 5$ gives $5^2 + 2 = 3^3$.

710. Two functions f and g are such that the indefinite integral of their difference is linear. Show that, if $f(x) = g(x)$ for any x, then $f(x) = g(x)$ for all x.

Translate the first sentence into an equation, and differentiate it.

The first sentence is

$$\int f(x) - g(x)\, dx = ax + b,$$

for some constants a, b. Differentiating both sides, $f(x) - g(x) = a$. So $f(x)$ and $g(x)$ differ by a constant. If $f(x) = g(x)$ for any x, then this constant must be zero, and therefore $f(x) = g(x)$ for all x.

711. The interior angles of an irregular pentagon are in AP. The smallest is $\frac{\pi}{4}$ radians. Find the largest.

Consider the average of the angles.

The total of the interior angles of a pentagon is 3π, so the average is $\frac{3\pi}{5}$. The average of the smallest and largest θ must therefore also be $\frac{3\pi}{5}$. This gives

$$\begin{aligned} \frac{1}{2}\left(\frac{\pi}{4} + \theta\right) &= \frac{3\pi}{5} \\ \Longrightarrow \theta &= \frac{19\pi}{20}. \end{aligned}$$

712. The following is an equation in the variable a:

$$\Big[a^x - a^{x-1}\Big]_{x=0}^{x=1} = 0.$$

(a) Show that $a - 2 + a^{-1} = 0$.

(b) Solve to find a.

The equation is a disguised quadratic.

(a) Substituting gives $(a^1 - a^0) - (a^0 - a^{-1}) = 0$. Since $a^0 = 1$, this is the required result.

(b) Multiplying up to get rid of the negative power, we have $a^2 - 2a + 1 = 0$, so $a = 1$.

713. Show that the functions $f(x) = 4x^2 + 8x + 2$ and $g(x) = x^2 + 6x + 7$ have the same range over $\mathbb{R}$.

Complete the square or differentiate.

Completing the square gives $f(x) = 4(x+1)^2 - 2$ and $g(x) = (x+3)^2 - 2$. These are both positive quadratic functions with minimum $f(x) = -2$. Hence, they have the same range $[-2, \infty)$.

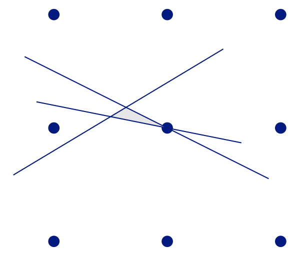
714. Three inequalities are given as

$$\begin{aligned} 3x - 5y &\geq 10, \\ x + 2y &\leq 15, \\ x + 5y &\geq 24. \end{aligned}$$

By solving pairs of equations, or otherwise, show that only one integer solution exists that satisfies all three inequalities.

Solve pairs of boundary equations, then sketch the triangular region which satisfies all three.

Solving pairs of boundary equations, we get pairwise intersections of straight lines at $(\frac{95}{11}, \frac{35}{11})$, $(\frac{17}{2}, \frac{31}{10})$, $(9, 3)$. Shading the relevant region on a lattice of integer coordinates:



So, the only integer point which satisfies all three inequalities simultaneously is $(9, 3)$.

715. The iteration $u_{n+1} = u_n + n$, $u_1 = 1$, defines a quadratic sequence.

(a) Write down the first five terms.

(b) The ordinal definition is $u_n = kn(n+1)$. Find the value of k.

...

(a)
$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1 + 1 = 2 \\ u_3 &= 2 + 2 = 4 \\ u_4 &= 4 + 3 = 7 \\ u_5 &= 7 + 4 = 11. \end{aligned}$$

(b) Using u_1, we have $1 = k \cdot 1 \cdot 2$. So $k = \frac{1}{2}$.

716. Either prove or disprove the following statement:

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx.$$

The statement is true. Think in terms of areas on a graph.

The result is true. Graphically, it is trivial: the signed area from a to b plus the signed area from b to c is the signed area from a to c.

Alternatively, an algebraic approach says that, defining $F'(x) = f(x)$, the LHS is $F(b) - F(a) + F(c) - F(b)$, which simplifies to $F(c) - F(a)$, which is the RHS.

717. By eliminating the variable t from the formulae $s = ut + \frac{1}{2}at^2$ and $v = u + at$, derive $v^2 = u^2 + 2as$.

Substitute for t.

Rearranging, we have $t = \frac{v-u}{a}$. Substituting gives

$$\begin{aligned} & s = u\frac{v-u}{a} + \tfrac{1}{2}a\left(\frac{v-u}{a}\right)^2 \\ \implies & as = uv - u^2 + \tfrac{1}{2}(v^2 - 2uv + u^2) \\ \implies & 2as = v^2 - u^2 \\ \implies & v^2 = u^2 + 2as. \end{aligned}$$

718. Write each of the following as a single interval:

(a) $(-\infty, 2] \cup (-4, 6]$,
(b) $[0, \infty) \setminus (1, \infty)$,
(c) $(-\infty, 1) \cap (-1, \infty)$.

...

(a) $(-\infty, 6]$,
(b) $[0, 1]$,
(c) $(-1, 1) \cap (-1, \infty)$.

719. The line $x = k$ is a normal to $y = x^4 - 32x^2$. Find the possible values of the constant k.

The normal to a curve is only vertical at stationary points.

The line $x = k$ is parallel to the y axis. So, if it is to be a normal, the tangent must be parallel to the x axis. Hence, set $\frac{dy}{dx} = 4x^3 - 64x = 0$. Solving yields $x = 0$ or $x = \pm 4$. So, $k = 0, \pm 4$.

720. Show that $0.\dot{2}3\dot{7}$ may be written as $\dfrac{237}{999}$.

Set $x = 0.\dot{2}3\dot{7}$, then calculate $1000x - x$.

Set $x = 0.\dot{2}3\dot{7}$. Then $1000x = 237.\dot{2}3\dot{7}$. So, $999x = 237$, and $x = \frac{237}{999}$.

721. The *impulse* on an object, moving from $x = a$ to $x = b$ under the action of a force F, is given by

$$I = \int_a^b F\,dx.$$

Show that the gravitational impulse on an object falling from a height h close to the Earth's surface has magnitude mgh.

Use $F_{\text{grav}} = mg$.

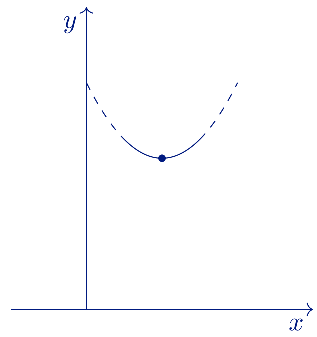
Defining x as positive downwards from the original height, the motion is from $x = 0$ to $x = h$, and the gravitational impulse is therefore

$$\begin{aligned} & \int_0^h mg\,dx \\ = \; & [mgx]_0^h \\ = \; & (mgh) - (0) \\ = \; & mgh. \end{aligned}$$

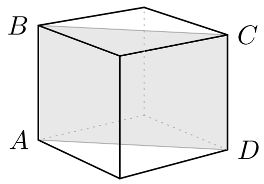
722. A function f has $f(2) = 3$, $f'(2) = 0$, $f''(2) < 0$. Sketch the graph $y = f(x)$ for x values close to 2.

Consider the fact that $x = 2$ is a stationary point of $f(x)$, and classify the stationary point with the second derivative.

Since the first derivative is zero at $x = 2$, this is a stationary point. Furthermore, since the second derivative is negative, the point $(2, 3)$ is a local minimum. Hence, in the vicinity of $x = 2$, the graph looks as follows:



723. The diagram shows a cube of unit side length.



Show that rectangle $ABCD$ has the dimensions of a sheet of A4 paper.

Show that both ratios are $1 : \sqrt{2}$.

A4 to A5 is designed so that the area scale factor is 2, which means the length scale factor is $1 : \sqrt{2}$. The longer sides of the rectangle are the hypotenuses of $(1, 1, \sqrt{2})$ triangles, so the rectangle has the dimensions of A4.

724. State, with a reason, whether these hold:

(a) $x, y \in \mathbb{Q} \implies xy \in \mathbb{Q}$,
(b) $x, y \in \mathbb{R} \setminus \mathbb{Q} \implies xy \in \mathbb{R} \setminus \mathbb{Q}$.

Look for counterexamples: values (x, y) that satisfy the "if" but not the "then".

(a) This holds. If the rationals x and y are quotients of integers, then their product is too.

(b) This doesn't hold. A pair of surds such as $x = y = \sqrt{2}$ is a counterexample.

725. To find the gradient formula $\dfrac{dy}{dx}$ of $y = x^n$, where $n \in \mathbb{N}$, we set up the limit
$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}.$$
(a) Using the binomial expansion, show that
$$(x+h)^n = x^n + nx^{n-1}h + ...$$
(b) Hence, prove that $\dfrac{dy}{dx} = nx^{n-1}$.

...

(a) Expanding using the standard formula, we have
$$\begin{aligned}(x+h)^2 = \ & x^n + {}^nC_1 x^{n-1}h \\ & + \text{ terms in } h^2 \text{ and higher.}\end{aligned}$$
Then ${}^nC_1 = n$ gives the required result.

(b) Simplifying the limit, we get
$$\begin{aligned}& \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ = \ & \lim_{h\to 0} \frac{x^n + nx^{n-1}h + ... - x^n}{h} \\ = \ & \lim_{h\to 0} \frac{nx^{n-1}h + ...}{h} \\ = \ & \lim_{h\to 0} nx^{n-1} + \text{ terms in } h \text{ and higher.} \\ = \ & nx^{n-1}.\end{aligned}$$

726. Show that $y = \dfrac{8x+3}{2x+1}$ has an asymptote at $y = 4$.

Consider the behaviour of the graph as $x \to \infty$.

Dividing top and bottom of the fraction by x,
$$y = \frac{8 + \frac{3}{x}}{2 + \frac{1}{x}}.$$
Now, as $x \to \infty$, the two inlaid fractions tend to zero, which means $y \to \frac{8}{2} = 4$. Hence, the curve has a horizontal asymptote at $y = 4$.

727. Two dice are rolled. State which, if either, of the following events has the greater probability:

- the total is six,
- the total is eight.

No calculations are needed: consider the symmetry of the possibility space.

Since the possibility space is symmetrical around 7, which is the most likely sum, no calculations are required: the probabilities are equal.

728. A particular cubic function $h(x) = x^3 + ax^2 + bx + c$ has $h(0) = 0$, $h'(0) = 0$, $h''(0) = 2$.

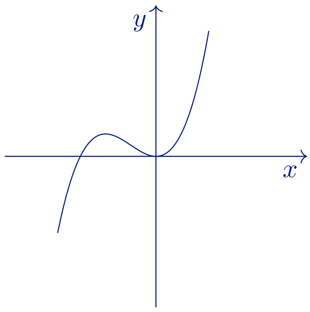
(a) Find a, b, c,

(b) Sketch the graph $y = f(x)$.

Find $f'(x)$ and $f''(x)$ in terms of a, b, c, then substitute the relevant values.

(a) Differentiating, we have $h'(x) = 3x^2 + 2ax + b$ and $h''(x) = 6x + 2a$. Since $h''(0) = 2$, we know that $a = 1$. Then, using $h'(0) = h(0) = 0$, we get $b, c = 0$. So, the function is $h(x) = x^3 + x^2$.

(b) First, we factorise the cubic: $y = x^2(x+1)$. This has a double root at $x = 0$ and a single root at $x = -1$. Hence, at $x = 0$ the curve looks like a parabola, and at $x = -1$ it looks like a straight line.



729. Two stationary cows are pushing each other. With reference to Newton's laws, explain how you know that the magnitudes of the following are equal:

(a) the frictional force of the ground on cow A; the force of cow B on cow A,

(b) the force of cow A on cow B; the force of cow B on cow A,

Quote Newton II and Newton III.

(a) We know these are equal due to Newton II. Since cow A is not accelerating, the resultant horizontal force on it must be zero. Hence, the forces in question must be equal in magnitude.

(b) We know these are equal due to Newton III. They are the two aspects of the single interaction between cow A and cow B. They would be equal in magnitude whatever the acceleration of the cows.

730. Complete the square on $\sqrt{2}x^2 + \sqrt{8}x + \sqrt{32}$.

You might want to take out a factor of $\sqrt{2}$ before you begin.

There is a common factor of $\sqrt{2}$. Taking it out, we have $\sqrt{2}(x^2 + 2x + 4)$. Completing the square inside the bracket gives $\sqrt{2}((x+1)^2+3)$, which we can then expand again as $\sqrt{2}(x+1)^2 + 3\sqrt{2}$.

731. A sample $\{x_i\}$ of size 12 has mean 10 and standard deviation 5. Find the mean value of

(a) $x_i - 10$,

(b) $(x_i - 10)^2$.

In (b), consider the quantity $S_{xx} = \sum(x_i - \bar{x})^2$.

(a) The mean signed deviation from the mean is zero by definition.

(b) Since $s = 5$, the variance $s^2 = 25$. The variance is defined as the mean squared deviation from the mean, which is the quantity required in the question.

732. Simplify $\dfrac{d(2x + 3y + 5)}{dx}$.

Differentiate term by term.

Differentiating term by term, we obtain

$$\frac{d(2x + 3y + 5)}{dx} = 2 + 3\frac{dy}{dx}.$$

733. Write down the largest real domains over which the following functions may be defined:

(a) $x \mapsto x^1$,

(b) $x \mapsto x^0$,

(c) $x \mapsto x^{-1}$.

Consider whether any real values of x may not appear in the domain (set of inputs) of each function.

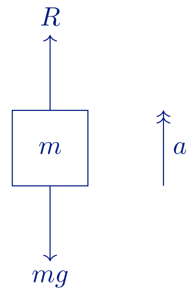
(a) $\mathbb{R}$,

(b) $\mathbb{R}$,

(c) $\mathbb{R} \setminus \{0\}$.

734. Give the acceleration of a lift if accurate weighing scales placed inside it overestimate mass by 20%.

Draw a force diagram.



Scales measure contact reaction force. So, if this is overestimated by 20%, then $R = 1.2mg$. Hence the resultant force is $0.2mg$, so the acceleration is $0.2g$.

735. Show, without using calculus, that the equation of the normal to $x^2 + 4x + y^2 - 2y = 0$ at $(-4, 0)$ is $2y = x + 4$.

Use circle geometry.

The equation is a circle. Completing the square, we have $(x+2)^2 + (y-1)^2 = 5$. The normal to a circle is the radius, which is from $(-2, 1)$ to $(-4, 0)$. It has gradient $\frac{1}{2}$, which gives $y = \frac{1}{2}x + 2$, and hence the required result.

736. True or false?

(a) $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$,

(b) $\displaystyle\int_a^b k + f(x)\,dx = k + \int_a^b f(x)\,dx$.

...

(a) True. Integration is linear (areas stretched in one direction stretch accordingly), so common factors can be taken out.

(b) False. The integral could be split up in this way, but the first term k would integrate to give $k(b - a)$.

737. A sequence is given iteratively by

$$u_{n+1} = 4u_n, \quad u_1 = 5.$$

Find the value of the first term over one billion.

This is a GP. Solve using logarithms.

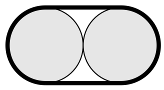
This is a GP with ordinal formula $u_n = 5 \cdot 4^{n-1}$. We set up the equation $10^9 = 5 \cdot 4^{n-1}$. Solving gives $n = \log_4 10^9 + 1 = 14.94$. So $n = 15$ gives the first term over 1 billion, which is $u_{15} = 5 \cdot 4^{14} = 1.34$ billion (3sf).

738. Simplify $\{x : 0 \le x \le 3\} \cup \{x : |x-3| < 2\}$.

Draw the intervals on a number line.

The latter set is the interval $(1, 5)$, so the union of the two sets is $[0, 5)$.

739. Two bamboo canes, each of radius 0.5 cm, are bound together, in equilibrium, with a rubber band, as shown below in plan view. The tension in the band is a constant 3 N throughout. In the plane shown, no external forces act.



(a) Determine the length, in cm, of the band.

(b) Explain why, even if the canes are rough, there can be no friction acting between them.

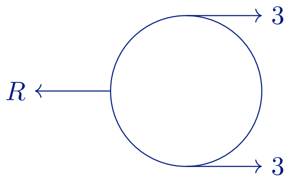
(c) Find the contact force between the two canes.

For (c), draw a force diagram for one cane in isolation, with two tension forces acting on it.

(a) The straight sections have total length 2 cm, and the curved sections have total length π cm. So the length is $2 + \pi$ cm.

(b) The situation is symmetrical, so the contact force between the canes can only act along the line of symmetry. Since this is perpendicular to the surfaces, it can only be a reaction force.

(c) Modelling the left-hand cane:

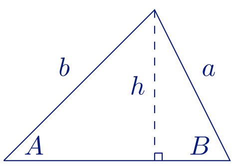


Hence, $R = 6$ N.

740. Prove the sine rule.

Calculate the length of a perpendicular in two different ways.

Dropping a perpendicular in a general triangle ABC, we have



The perpendicular h can be calculated in two different ways: $h = a \sin B$ and $h = b \sin A$. Hence

$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

By symmetry, the same holds for the third side/angle, which gives the sine rule.

741. Find the length of the line segment

$$x = 2 + s, \quad y = -1 + \tfrac{4}{3}s, \quad s \in [-1, 2].$$

Substitute the s values at the endpoints, and use Pythagoras.

The endpoints of this line segment have s values $s = -1$ and $s = 2$. Substituting these gives $(1, -\frac{7}{3})$ and $(4, \frac{5}{3})$. The distance between these is $\sqrt{3^2 + 4^2} = 5$.

742. A quadratic graph passes through the points $(-6, 0)$, $(0, 36)$, and $(3, 0)$. Find the equation of the graph, in the form $y = ax^2 + bx + c$.

Use the factor theorem.

Using the factor theorem, we know that the parabola has equation $y = k(x+6)(x-3)$. Substituting $(0, 36)$ gives $k = -2$. So $y = -2x^2 - 6x + 36$.

743. In a probability model, p and $q = 1 - p$ satisfy the equation $16p^2q^2 = 1$.

(a) Show that this equation can be factorised as a difference of two squares.

(b) Solve to find p and q.

...

(a) $16p^2q^2 - 1 = 0$, which can be factorised as $(4pq - 1)(4pq + 1) = 0$.

(b) From (a), $pq = \pm\frac{1}{4}$. Since $p, q \ge 0$, the negative root is not possible. Hence $pq = \frac{1}{4}$ and $p + q = 1$. Solving simultaneously gives $p = q = \frac{1}{2}$.

744. Show that $\cos^2 x + \cos x - 6 = 0$ has no real roots.

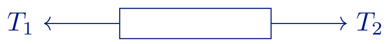
Factorise. The quadratic has roots, but the individual factors do not.

The quadratic has positive discriminant, and can be factorised as $(\cos x - 3)(\cos x + 2) = 0$. But the range of the cosine function is $[-1, 1]$, so neither of the factors can equal zero.

745. Explain, with reference to Newton's laws, why, if a string is modelled as light, then the resultant force on it must be modelled as negligible.

Draw a force diagram for the string, using T_1 and T_2 as the tensions on each end.

Drawing a force diagram for the string:



$F = ma$ for the string tells us that, if m is taken to be zero, then the resultant force F must also be taken to be zero. Hence, $T_1 - T_2 = 0$, and the tensions on either end of the string must be the same.

746. Find simplified expressions for the sets

(a) $\{x \in \mathbb{R} : |x| < 1\} \cap \{x \in \mathbb{R} : |x| < 2\}$,

(b) $\{x \in \mathbb{R} : |x| < 1\} \cap \{y \in \mathbb{R} : |y| < 2\}$.

Consider the sets defined with modulus signs as intervals.

(a) This is $(-1, 1) \cap (-2, 2)$, which is $(-1, 1)$.

(b) This is exactly the same as (a). The variables x and y are defined and used within their sets, so $\{x \in \mathbb{R} : |x| < 2\}$ and $\{y \in \mathbb{R} : |y| < 2\}$ are identical sets.

747. "The curves $x^2 + y^2 = 1$ and $(x+1)^2 + (y+1)^2 = 1$ are tangent to one another." True or false?

Use circle geometry.

False. The curves are unit circles centred on $(0, 0)$ and $(-1, -1)$. Since the distance between these two points is $\sqrt{2}$, which is less than 2, the circles must overlap and intersect twice.

748. Four cards are dealt, with replacement, from a standard deck. Find the probability that the first two are hearts, and the second two are diamonds.

Consider the cards one by one.

There are 13 hearts and 13 diamonds, so the probability is given by

$$\frac{13}{52} \cdot \frac{12}{52} \cdot \frac{13}{52} \cdot \frac{12}{52} = \frac{9}{2704}.$$

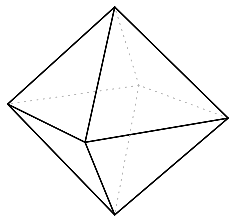
749. Explain why one of the following expressions is well-defined and the other is not:

$$\left.\frac{x^2 - 1}{x^2 - x}\right|_{x=1} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x^2 - x}$$

The left-hand expression is not well defined.

Evaluation at $x = 1$ is not well defined, because the denominator $x^2 - x$ is equal to zero at $x = 1$. However, since the numerator also has a factor of $(x - 1)$, the limit has a well-defined value, namely 2.

750. A regular octahedron is shown below.



State, with justification, the type of polyhedron which has vertices at the midpoints of each of the octahedron's faces. (This is called the *dual*.)

Count the vertices, edges and faces.

The dual of the octahedron is the cube (and vice versa).

751. Using the quadratic discriminant Δ, or otherwise, show that the line $y = 2px - p^2$, for constant p, is tangent to the curve $y = x^2$.

Start by solving for intersections.

Solving for intersections, we have $x^2 = 2px - p^2$. Rearranging gives $x^2 - 2px + p^2 = 0$, which has discriminant $\Delta = 4p^2 - 4 \cdot p^2 = 0$, for any p. Hence, since they have only one intersection, the line is tangent to the curve.

752. Show that a nanocentury is around π seconds.

"Nano" is $\times 10^{-9}$.

A nanocentury is $10^{-9} \cdot 100 \cdot 365 \cdot 24 \cdot 60 \cdot 60$ seconds. This is $3.1536 \approx \pi$.

753. Write down the range of $h(x) = (e^x - 1)^2 + 4$.

Consider the minimum value of the squared term.

The squared factor can be zero, at $x = 0$, which means that the minimum value of the squared term is zero. Hence, the range is $[4, \infty)$.

754. The vector equation of a line L is given, relative to an origin, by $\mathbf{r} = t\mathbf{a} + (1 - t)\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of points A and B respectively, and $t \in \mathbb{R}$ a parameter.

(a) Show, by giving values of the parameter, that L passes through both A and B.

(b) Write down the position vector of the point that divides AB in the ratio $1:3$.

...

(a) When $t = 0$, $\mathbf{r} = \mathbf{b}$, and when $t = 1$, $\mathbf{r} = \mathbf{a}$. Hence, the line L passes through A and B.

(b) If AB is divided $1:3$, $t = \frac{3}{4}$. So $\mathbf{r} = \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$.

755. Prove that no function has linear first and second derivatives that are non-constant.

Set up a general first derivative, and differentiate it.

If a function f has a linear and non-constant first derivative, then $f'(x) = ax + b$, for $a \neq 0$. Differentiating $f''(x) = a$. This is a linear function, but it is constant.

756. It is given that the equation $(ax+b)(bx+a) = 0$, for constants $a, b \in \mathbb{R}$, has precisely one root. Show that either $a = \pm b$, or one of a and b is zero.

Multiply out to use the discriminant, or consider repeated factors.

If the equation has precisely one root, then either the equation is linear, or it is quadratic and the factors $(ax + b)$ and $(bx + a)$ are multiples of one another. The former scenario requires that one of a or b is zero; the latter requires that $\frac{a}{b} = \frac{b}{a}$, which is solved by $a = \pm b$.

757. Simplify the following:

(a) $\dfrac{a-b}{b-a}$,

(b) $\dfrac{b^2-a^2}{a-b}$.

Remember that $(a - b)$ and $(b - a)$ are negatives.

(a) Since $(a - b)$ and $(b - a)$ are negatives, this simplifies to -1.

(b) $\dfrac{b^2-a^2}{a-b} = \dfrac{(b+a)(b-a)}{a-b} = -a - b.$

758. Show that the area of a triangle with sides length $9, 13, 14$ cm is $18\sqrt{10}$ cm^2.

Use the cosine rule (or Heron's formula if you know it!).

Using the cosine rule to find the angle opposite the 14 cm edge, we get

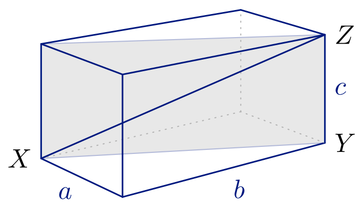
$$\cos\theta = \frac{9^2 + 13^2 - 14^2}{2 \cdot 9 \cdot 13} = \frac{3}{13}.$$

Taking the positive root, since this is an acute triangle, we get $\sin\theta = \frac{4\sqrt{10}}{13}$. Hence, the area is

$$\frac{1}{2} \cdot 9 \cdot 13 \cdot \frac{4\sqrt{10}}{13} = 18\sqrt{10}.$$

759. Prove Pythagoras's theorem in three dimensions from Pythagoras's theorem in two dimensions.

Start with a cuboid, side lengths (a, b, c).



By 2D Pythagoras, length $|XY|^2 = a^2 + b^2$. Then, using 2D Pythagoras again, we have

$$\begin{aligned} |XZ|^2 &= |XY|^2 + c^2 \\ &= a^2 + b^2 + c^2. \end{aligned}$$

760. The parabola $y = x^2 - 2x$ is reflected in the line $x = 6$. Find the equation of the new parabola.

Consider the effect of the transformation on the x intercepts.

The monic parabola $y = x^2 - 2x = x(x - 2)$ has roots at $x = 0$ and $x = 2$. Reflecting this in $x = 6$, the new parabola must be monic and have roots at $x = 10$ and $x = 12$. So, the equation of the new parabola is $y = (x - 10)(x - 12)$ or $y = x^2 - 22x + 120$.

761. Evaluate $\displaystyle\sum_{i=1}^{3} \cos\frac{\pi \text{ rad}}{i}$.

Write the sum out longhand, and calculate the terms individually.

The sum is $\cos\frac{\pi}{1} + \cos\frac{\pi}{2} + \cos\frac{\pi}{3}$. These can be written down without a calculator, giving $-1 + 0 + \frac{1}{2} = -\frac{1}{2}$.

762. A pirate stands in a lift, with a large red parrot on his shoulder. The pirate has mass 80 kg, the parrot has mass 5 kg, and the lift has upwards acceleration a ms^{-2}.

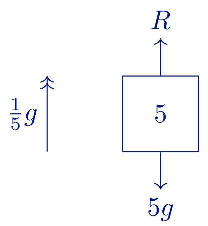
(a) In the case where $a = 0$, determine the force exerted by the pirate's feet on the lift floor.

(b) In the case where $a = \frac{1}{5}g$, determine the force exerted by the parrot's feet on the pirate's shoulder.

In part (a), write down the answer, considering pirate and parrot as one system. In (b), draw a force diagram for the parrot alone.

(a) The force exerted by the pirate's feet on the lift floor is the same as the reaction force exerted by the lift floor on the pirate's feet. Treating the pirate/parrot as one object of mass 85 kg, this force is 85 g, since the acceleration is zero.

(b) Modelling the parrot, we have



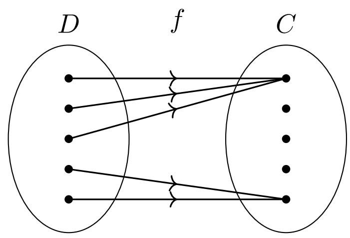
The force exerted by the parrot's feet on the pirate's shoulder is the same as the force exerted by the pirate's shoulder on the parrot's feet. This is R in the diagram above. $F = ma$ gives $R - 5g = 5 \cdot \frac{1}{5}g$, whence $R = 6g$.

763. By factorising, solve $\sin^2 x + \sin x = 0$, giving all values $x \in [0, 360°)$.

...

Factorising gives $\sin x(\sin x + 1) = 0$, so $\sin x = 0$ or -1. This yields $x \in \{0, 180, 270°\}$.

764. A function f maps a domain D to a codomain C, according to the following scheme:



Write down all possible numbers of roots of the equation $f(x) = a$, for $a \in C$.

A root of such an equation is a dot in the set D.

The number of roots of such an equation, listing in order from the uppermost element of the codomain C, is $3, 0, 0, 0, 2$. So, the possible numbers of roots are $\{0, 2, 3\}$.

765. Show that, for $k \in \mathbb{R}$, $y = x^2+kx+k$ and $y = x+k$ always have at least one point of intersection.

Use the discriminant.

Solving for intersections, $x^2 + kx + k = x + k$ gives $x^2 + (k-1)x = 0$. Since this has no constant term, $x = 0$ is always a root, so the line and the curve have at least one intersection.

766. A sample $\{x_i\}$ is taken. Afterwards, the values of twenty percent of the sample are reduced by twenty percent. Find the expected percentage change in $\bar{x}$.

Express this information algebraically.

The scale factor of this increase is given by $0.2 \times 0.8 + 0.8 \times 1 = 0.96$. So the expected percentage reduction is 4%.

767. In a quadrilateral, there are a pair of right angles at opposite vertices. Prove that there is a circle which passes through all four vertices.

Use a circle theorem.

A pair of right-angles at opposite vertices add up to 180°. Hence, by the standard circle theorem, the quadrilateral is a cyclic quadrilateral, meaning that there is a circle passing through its four vertices.

768. Prove that, if α is a fixed point of an invertible function f, then it is also a fixed point of f^{-1}.

Consider the relationship of the equations $f(x) = x$ and $x = f^{-1}(x)$.

If α is a fixed point of f, then $f(\alpha) = \alpha$. Applying f^{-1} to both sides of this equation, we get $\alpha = f^{-1}(\alpha)$, which tells us that α is also a fixed point of the inverse f^{-1}.

769. A heavy box of mass 60 kg is sitting on the ground. The coefficient of friction between it and the ground is $\mu = 0.4$. Find the acceleration of the box if a horizontal force is applied of magnitude

(a) 300 N,

(b) 200 N.

Remember that a negative acceleration, in such a static-frictional context, signifies zero acceleration.

(a) Since the applied force is horizontal, the reaction force is $60g$, and $F_{\text{max}} = 24g$. $300 - 24g = 60a$ gives $a = 1.08$ ms^{-2}.

(b) Again, $F_{\max} = 24g$. Then $200 - 24g = 60a$ gives $a = -0.587 \text{ ms}^{-2}$. But friction acting to oppose motion cannot generate a negative acceleration. The true frictional force is 200 N, and the acceleration is zero.

770. Three six-sided dice are stacked neatly on top of one another on a table. Show that the maximum possible number of dots visible is 48.

Opposite faces of a die add up to 7.

Since opposite faces of a die add up to seven, the vertical faces of each die total to $2 \times 7 = 14$. The maximum is then attained when the uppermost die has six showing on its top face, giving a maximum total of $3 \cdot 14 + 6 = 48$.

771. For a particular pair of events A and B, it is given that $P(A \cap B') = x$, $P(B) = 3x + \frac{1}{5}$, $P(A' \cap B') = 4x$.

(a) Find x.

(b) Assuming independence, find $P(A \cap B)$.

These events sum to 1.

(a) These events sum to 1. So $x + 3x + \frac{1}{5} + 4x = 1$, which gives $x = \frac{1}{10}$.

(b) The definition of independence gives $P(A \cap B) = P(A) \times P(B)$. We know that $P(A) = P(A \cap B') + P(A' \cap B') = 5x = \frac{1}{2}$. Therefore $P(A \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

772. Write 2^t in terms of 16^t.

Express 2 as 16^k.

Since $2 = 16^{\frac{1}{4}}$, $2^t = (16^t)^{\frac{1}{4}}$.

773. Simultaneous equations A, B, C are as follows:

$$\begin{aligned} A: &\quad x - 4y + 3z + 4 = 0, \\ B: &\quad 3x + y - 4z + 25 = 0, \\ C: &\quad 6x + 3y - 2z + 9 = 0. \end{aligned}$$

(a) Eliminate x from equation-pairs (A, B) and (B, C) to give two equations in y and z.

(b) Solve to find x, y, z.

Two equations in two unknowns is solved by reducing to one equation in one unknown. Likewise, three equations in three unknowns, is reduced to two in two, then further reduced to one in one, and solved.

(a)

$$\begin{aligned} 3A - B &: -13y + 13z - 13 = 0 \\ 2B - C &: -y - 6z + 41 = 0. \end{aligned}$$

(b) Solving, we get $y = 5, z = -6$, which then yields $x = -2$.

774. Prove that

$$\sin 60° = \frac{\sqrt{3}}{2}.$$

Consider an equilateral triangle.

In an equilateral triangle of side length 2, drop a perpendicular to split the triangle into two right-angled triangles. These have sides $(1, \sqrt{3}, 2)$. Hence, $\sin 60° = \frac{\sqrt{3}}{2}$.

775. You are told that there are infinitely many (x, y) pairs which satisfy both $2x - 3y = 7$ and $x = \frac{3}{2}y + k$. Find the value of k.

Consider how it is possible for two lines to intersect at infinitely many points.

For infinite intersections, the two lines must be identical. Rearranging, this gives $k = \frac{7}{3}$.

776. Give the meaning of the following adjectives used in mechanical modelling:

(a) “inextensible”,

(b) “uniform”,

(c) “flabelliform” (just kidding).

In each case, use “negligible”.

In the following, the phrase “A is negligible” means “A is close enough to zero that, compared with other quantities, its effects can be neglected by setting it to zero.”

(a) undergoes negligible extension,

(b) has negligible deviation from constant density

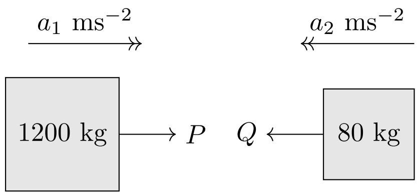
(c) “flabelliform” means “shaped like a fan”. This has nothing to do with mechanics, (unless you are a fan manufacturer).

777. A family of lines L_n is given by $y = mx + c_n$, where c_n forms an arithmetic sequence. Prove that the distance between adjacent lines is constant.

Sketch the lines.

Since c_n forms an AP, the vertical distance between successive pairs of lines is constant. And, because they are parallel, the shortest distance between them (perpendicular, rather than vertical) must also be constant.

778. A spacecraft is winching an astronaut in using a light, inextensible cable. Initially, the astronaut is floating at a constant 50 metres from the spacecraft.



(a) State, with a reason, whether the following pairs of magnitudes are equal:
   i. P and Q,
   ii. a_1 and a_2.

(b) The winch pulls for five seconds, exerting a constant tension of 60 N, and then cuts out. Find the total time taken for the astronaut to reach the spacecraft.

Consider Newton III.

(a) i. The cable is light, so these two tensions are an effective Newton III pair. Hence, they are equal by definition.
   ii. Since the masses are different, the magnitudes of the accelerations, by Newton II, will be different.

(b) For the five seconds of tension, $a_1 = 0.05$ and $a_2 = 0.75$. So, the relative acceleration is $a = 0.8$. In the first five seconds, the distance is reduced by $s = \frac{1}{2} \cdot 0.8 \cdot 5^2 = 10$ metres, and the relative speed is $v = 0.8 \cdot 5 = 4$ ms^{-1}. So the remaining 40 metres is covered in 10 seconds, giving 15 seconds overall.

779. Find $f(0)$, if f is a linear function such that

$$\int_{-2}^{2} f(x)\, dx = 1.$$

Consider that 0 is the midpoint of the interval $[-2, 2]$.

The information gives does not allow us to find f explicitly. However, we know that $f(x) = ax + b$, so that $\left[\frac{1}{2}ax^2 + bx\right]_{-2}^{2} = 1$. This yields $(2a + 2b) - (2a - 2b) = 1$, so $b = \frac{1}{4}$. Hence, regardless of the value of a, $f(0) = \frac{1}{4}$.

780. True or false? "Negating every data value negates...

(a) ...the mean."

(b) ...the IQR."

(c) ...the standard deviation,"

(d) ...the variance."

...

The mean is negated, as a measure of central tendency, but the measures of spread, all of which must be non-negative, are not.

(a) True,

(b) False,

(c) False,

(d) False.

781. You are told that the variables x, y, z satisfy

$$\frac{dy}{dz} = k\frac{dx}{dz},$$

for a constant $k \neq 0$. State, with a reason, whether the following are necessarily true:

(a) $y \propto x$,

(b) $y = kx + c$.

Integrate the given formula.

Integrating with respect to z, we have $y = kx + c$. Hence (a) is not necessarily true (though possible), and (b) is true.

782. To determine the solution of $2^{3x} = 5 \cdot 2^{2x}$, a pupil writes "We divide both sides by 2^{2x}, which gives $2^x = 5$. Hence, $x = \log_2 5$." Criticise the argument.

The argument gives a correct result, but without logical justification.

The result is correct, but a piece of justification has been left out, which is necessary if the solution is to be <u>determined</u> rather than merely <u>found</u>. When dividing by a function or expression, the possibility that that function or expression is zero should be considered. In this case, a better argument would begin "Since $2^{2x} \neq 0$, we can divide both sides by 2^{2x}, which...".

783. By integrating twice, find the general solution of the following differential equation, for constant a:

$$\frac{d^2y}{dx^2} = 2a.$$

Remember to include the constants of integration.

Integrating twice with respect to x gives

$$\begin{aligned}\frac{d^2y}{dx^2} &= 2a \\ \implies \frac{dy}{dx} &= 2ax + b \\ \implies y &= ax^2 + bx + c.\end{aligned}$$

The general solution is all quadratic graphs of the form $y = f(x)$.

784. Give the acceleration of a lift if accurate weighing scales placed inside it overestimate mass by $k\%$.

Consider that scales measure reaction force.

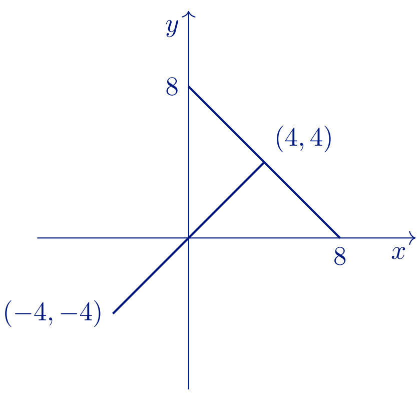
Scales measure reaction force. If mass is overestimated by $k\%$, then R is $k\%$ greater than mg. Hence, the acceleration upwards is $k\%$ of g, or $\frac{kg}{100}$ ms^{-2}.

785. The following line segments, when drawn together in the (x, y) plane, depict a capital letter:

$$\begin{aligned}\mathbf{r} &= s\mathbf{i} + (8 - s)\mathbf{j}, \quad s \in [0, 8], \\ \mathbf{r} &= t\mathbf{i} + t\mathbf{j}, \quad t \in [-4, 4].\end{aligned}$$

Sketch the line segments and identify the letter.

Consider the endpoints of the line segments.



Capital letter T.

786. For small values of θ in radians, the cosine function may be approximated by $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Find the percentage error when this approximation is used to estimate

$$\int_0^{\frac{\pi}{6}} \cos\theta \, d\theta.$$

Evaluate the integral with and without the small-angle approximation.

Evaluating the integral directly gives $\frac{1}{2}$. Evaluating it using the small-angle approximation gives

$$\left[\theta - \frac{\theta^3}{6}\right]_0^{\frac{\pi}{6}} = 0.499674...$$

So, the percentage error is $\frac{0.499674 - 0.5}{0.5} = 0.065\%$.

787. Solve $(x + 1) + \dfrac{(x + 1)^2}{x^2 + 1} = 0$.

Take out a common factor first.

There is a common factor of $(x+1)$. Hence, either $x = -1$, or

$$\begin{aligned}1 + \frac{x + 1}{x^2 + 1} &= 0 \\ \implies x^2 + 1 + x + 1 &= 0 \\ \implies x^2 + x + 2 &= 0.\end{aligned}$$

The discriminant of this quadratic is $\Delta = -7 < 0$, so it has not real roots. Hence, the solution of the original equation is $x = -1$.

788. State, with a reason, which, if any, of the implications $\implies$, $\impliedby$, $\iff$ links the following statements concerning a real number x:

- $x \in [0, 2]$,
- $x \in [1, 3]$.

...

Since neither interval is a subset of the other, there is no implication statement that joins these two. All four truth combinations TT, TF, FT, FF are possible.

789. From $v^2 = u^2 + 2as$ and $s = \frac{1}{2}(u + v)t$, derive the equation $s = ut + \frac{1}{2}at^2$.

Eliminate v from the equations.

The second equation may be written as $v = \frac{2s}{t} - u$. Substituting gives $\frac{4s^2}{t^2} - 4sut + u^2 = u^2 + 2as$. The u^2 terms cancel. Multiplying by $\frac{4t^2}{s}$, we get $s - ut = \frac{1}{2}at^2$, which rearranges to the required formula. (Potential divisions by zero, of t and s, are no problem. If either $t = 0$ or $s = 0$, the result is trivial.)

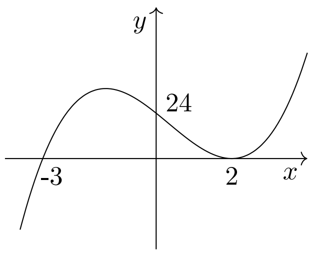
790. The linear expression $(2x - \beta)$ divides exactly into the quadratic expression $2x^2 + x + \beta$. Find all possible values of β.

Use the factor theorem.

Since $(2x - \beta)$ divides exactly into the quadratic, $x = \frac{\beta}{2}$ must be a root, by the factor theorem. Hence,

$$\begin{aligned}2\beta^2 + \frac{\beta}{2} + \beta &= 0 \\ \implies 3\beta^2 + \beta &= 0 \\ \implies \beta &= 0, -3.\end{aligned}$$

791. Find the equation of the cubic curve shown below, on which the axes intercepts have been marked, giving your answer in expanded polynomial form.



Use the fact that there is a double root at $x = 2$.

This is a positive cubic with a double root at $x = 2$ and a single root at $x = -3$. Hence, it has the form $y = k(x-2)^2(x+3)$. The y intercept tells us that $k = 2$. Multiplying out gives $y = 2x^3 - 2x^2 - 16x + 24$.

792. Determine whether the point $(0.8, 0.9)$ lies inside, on, or outside the closed curve $x^4 + y^4 = 1$.

Test the quantity $x^4 + y^4$.

Since this is a closed curve, we need only test the quantity $x^4 + y^4$, and compare it to 1. We get $x^4 + y^4\big|_{(0.8,0.9)} = 1.0657 > 0$, so the point lies outside the curve.

793. Prove that, if $\mathbf{p}$ and $\mathbf{q}$ are non-parallel vectors, then there are no non-zero real numbers a and b such that $a\mathbf{p} + b\mathbf{q} = 0$.

Prove this by contradiction.

Assume, for a contradiction, that there are non-zero real numbers a and b such that $a\mathbf{p} + b\mathbf{q} = 0$. Rearranging gives $\mathbf{p} = -\frac{b}{a}\mathbf{q}$. But $-\frac{b}{a}$ is a scalar, which means that $\mathbf{p}$ and $\mathbf{q}$ must be parallel. This is a contradiction. Hence, the result holds. $\square$

794. State, with a reason, whether getting two cards of different suits is more probable if the cards are picked

(a) with replacement,

(b) without replacement.

...

Different suits is more probable without replacement. Removal of the first card from the possibility means that more cards of other suits remain.

795. The following identity is proposed:

$$(x+1)^4 \equiv \left(A(x-1)^2 + Bx^2\right)^2.$$

Prove that there are no constants A, B which make the above identity hold.

Substitute in the values $x = 0, 1, 2$.

Substituting the values $x = 0, x = 1$, we get $A = \pm 1$, $B = \pm 4$. But $x = 2$ gives $A + 4B = \pm 9$, which is not compatible with the above values.

796. State, with a reason, whether $x = p$ intersects the following curves:

(a) $y = \dfrac{1}{x+p}$,

(b) $y = \dfrac{1}{x-p}$.

Sketch the curves.

The curves are reciprocal graphs, and $x = p$ is parallel to y. Such a line will intercept a reciprocal graph for any x value other than that of the asymptote. Hence $x = p$ intercepts the curve in (a), but not the curve in (b).

797. The scores A and B on two tests, out of 40 and 60 marks respectively, are to be combined into a single mark X, given out of a hundred. Each test (as opposed to each mark) is to carry equal weight. Find a formula for X in terms of A and B.

Scale each test to 50 marks.

For the tests to have equal weight, they must each be scaled to 50 marks. This requires scale factors of $\frac{5}{4}$ and $\frac{5}{6}$. Hence, the formula is $X = \frac{5}{4}A + \frac{5}{6}B$.

798. A positive geometric progression with n^{th} term u_n begins $a, ar, ar^2, \ldots$.

(a) Explain why $1 - 2r + r^2$ is never negative.

(b) Hence, prove that u_2 cannot be bigger than the mean of u_1 and u_3.

In (a), factorise.

(a) Factorising, we have $1 - 2r + r^2 = (1-r)^2$. Being a square, this is never negative.

(b) Since $1 - 2r + r^2 \geq 0$, and $a > 0$, we know that

$$\begin{aligned} & a - 2ar + ar^2 \geq 0 \\ \Longrightarrow\ & a + ar^2 \geq 2ar \\ \Longrightarrow\ & \frac{a + ar^2}{2} \geq ar. \end{aligned}$$

This is the required result.

799. Prove that the sum of three odd squares is odd.

A general odd number is $2a + 1$, for $a \in \mathbb{Z}$.

For $a, b, c \in \mathbb{Z}$, the sum of three odd squares is $(2a+1)^2 + (2b+1)^2 + (2c+1)^2$. This simplifies to $4(a^2+a+b^2+b+c^2+c)+3$, which is odd. Q.E.D.

800. Functions f and g are such that $x = a$ is a root of $f(x) = 0$, $x = b$ is a root of $g(x) = 0$, and $x = c$ is a root of $f(x) = g(x)$. State, with a reason, whether the following hold:

(a) If $a = b$, then $f(c) = 0$.

(b) If $a = c$, then $g(c) = 0$.

...

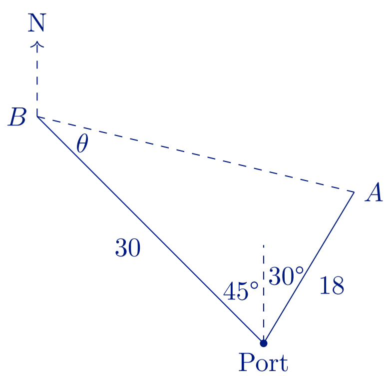
(a) This does not hold. The information is satisfied by $a = b \neq c$, which allows $f(c) \neq 0$. This is a counterexample.

(b) This holds. If $a = c$, then $f(c) = 0$. And, since c is a root of $f(x) = g(x)$, $f(c) = g(c)$.

801. Two ships leave port simultaneously. Ship A travels on bearing 030° at 12 mph; ship B travels on bearing 315° at 20 mph. Determine the bearing of A from B after 90 minutes.

Draw a diagram and use the sine rule.

After 90 minutes, the ships are as follows:



By the cosine rule, $AB = 30.732319...$. Then, by the sine rule, $\sin\theta = \frac{18 \sin 75}{30.732319...}$. This yields $\theta = 34.5°$. Hence, the bearing of A from B is given by $180 - 45 - \theta = 100.5°$ (1dp).

802. Show that $(\sqrt{2} + \sqrt{3})^4 = 49 + 20\sqrt{6}$.

Use the binomial expansion.

Expanding binomially, we have

$$\begin{aligned} & (\sqrt{2}+\sqrt{3})^4 \\ = \ & \sqrt{2}^4 + 4\sqrt{2}^3\sqrt{3} + 6\sqrt{2}^2\sqrt{3}^2 + 4\sqrt{2}\sqrt{3}^3 + \sqrt{3}^4 \\ = \ & 4 + 8\sqrt{6} + 36 + 12\sqrt{6} + 9 \\ = \ & 49 + 20\sqrt{6}. \end{aligned}$$

803. Three integers are chosen, without replacement, from $1, 2, 3, 4, 5$. Find the probability that

(a) the 1 is not chosen,

(b) all three odd numbers are chosen.

Use $p = \frac{\text{successful}}{\text{total}}$.

There are ${}^5C_3$ ways of choosing the three integers. This gives probabilities as

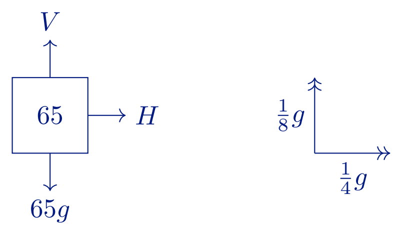
(a) $p = \frac{{}^4C_3}{{}^5C_3} = \frac{2}{5}$.

(b) $p = \frac{1}{{}^5C_3} = \frac{1}{10}$.

804. At takeoff, an airliner accelerates at $(\frac{1}{4}\mathbf{i} + \frac{1}{8}\mathbf{j})g$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in horizontal and vertical directions. Determine the magnitude of the contact force on a passenger of mass 60 kg.

Draw a force diagram for the passenger.

Modelling the passenger, on whom the seat exerts horizontal and vertical forces H and V:



Newton II gives $V - 65g = 65 \cdot \frac{1}{8}g$ and $H = 65 \cdot \frac{1}{4}g$. The total contact force is then

$$C = \sqrt{H^2 + V^2} = 74.9g.$$

805. Determine which of the points $O : (0,0)$ and $A : (4,0)$ is closer to the line $y = 2x - 3$.

Find the midpoint of OA, and test to see which side of the line it lies on.

The midpoint of OA is at $(2,0)$. This has $y < 2x - 3$, as does $(4,0)$ so the midpoint lies on the same side of the line as $(4,0)$. Hence, the origin is closer.

806. A parabola is given by $x = y^2 - y + 20$.

(a) Determine all axis intercepts.

(b) By finding $\frac{dx}{dy}$ and setting it to zero, find the coordinates of the point where the tangent is in the y direction.

(c) Hence, sketch the curve.

Switching the variables x and y does not affect the mathematics of the problem.

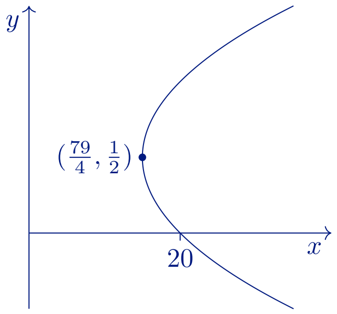
(a) Setting $y = 0$ gives $x = 20$. Setting $x = 0$ gives $y^2 - y + 20 = 0$, which has discriminant $\Delta = -79 < 0$. Hence, $(20, 0)$ is the only axis intercept.

(b) Differentiating with respect to y,

$$\begin{aligned} x &= y^2 - y + 20 \\ \implies \frac{dx}{dy} &= 2y - 1. \end{aligned}$$

When the tangent is parallel to y, $\frac{dx}{dy} = 0$, which occurs at $y = \frac{1}{2}$. Substituting gives the vertex as $(19.75, 0.5)$.

(c) This is a positive parabola, $x = f(y)$, with vertex at $(19.75, 0.5)$, so looks like



807. Determine whether, at $x = 1$, the function $f(x) = x\sqrt{x} + 3x$ is increasing, decreasing or neither.

The terms "increasing" and "decreasing" refer to the value of the first derivative being positive or negative.

Differentiating, we get $f'(x) = \frac{3}{2}x^{\frac{1}{2}} + 3$. Evaluating, $f(1) = \frac{9}{2} > 0$, so the function is increasing at this point.

808. A student writes: "Friction always acts to oppose motion or potential motion. When a car is accelerating forwards, the potential motion is the slipping of the tyres backwards across the surface of the road. Hence, friction acts forwards on the tyres, and drives the car forwards." True or false?

...

This is true.

809. Solve the equation $\displaystyle\sum_{r=1}^{3} \frac{r - x}{rx} = \frac{1}{2}$.

Write the sum out longhand.

Written longhand, this equation is

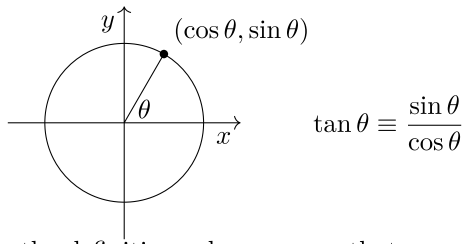
$$\begin{aligned} &\frac{1-x}{x} + \frac{2-x}{2x} + \frac{3-x}{3x} = \frac{1}{2} \\ \implies &\frac{6(1-x) + 3(2-x) + 2(3-x)}{6x} = \frac{1}{2} \\ \implies &18 - 11x = 3x \\ \implies &x = \frac{9}{7}. \end{aligned}$$

810. By sketching a cubic graph, or otherwise, solve the inequality $(x-a)(x-b)(x-c) > 0$, where $a < b < c$ are constants. Give your answer in set notation.

The cubic has three distinct factors, so it has three distinct x intercepts.

This is a positive cubic, with three distinct single roots. Hence, the sign of the cubic changes at each root. For large negative values of x, the cubic is $-$ve, then, moving rightwards, it is $+$ve beyond $x = a$, then $-$ve beyond $x = b$, then $+$ve beyond $x = c$. So the solution is $(a, b) \cup (c, \infty)$.

811. A unit circle is depicted, with the definitions of sin, cos and tan shown.



From the definitions above, prove that

(a) $\sin^2\theta + \cos^2\theta \equiv 1$,

(b) $\tan^2\theta + 1 \equiv \sec^2\theta$.

In (b), look to divide the identity in (a) by a suitable function of θ.

(a) This is Pythagoras's theorem.

(b) Dividing the above Pythagorean identity by $\cos^2\theta$ yields the required result, which is true whenever $\cos\theta \neq 0$.

812. Find the probability that three consecutive dice rolls give three different scores.

Consider the rolls one at a time.

The first roll can be anything. Then the second roll and third roll must be different. So, the probability is $p = 1 \times \frac{5}{6} \times \frac{4}{6} = \frac{5}{9}$.

813. Describe the transformation that takes the graph $y = ax^3 + bx$ onto the graph $x = ay^3 + by$.

Switching x and y is a reflection.

The roles of x and y have been reversed in the transformed graph, which corresponds to a reflection in the line $y = x$.

814. Two vector lines have parametric equations

$$\mathbf{r}_1 = \begin{pmatrix} 1+4s \\ 5-2s \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 6-2t \\ 13+t \end{pmatrix}.$$

Show that the lines are parallel.

The gradient of a parametrically defined line is given by the ratio of the coefficients of the parameter.

The gradient of a parametrically defined line is the ratio of the coefficients of the parameter. These are $\frac{4}{-2} = -2$ and $\frac{-2}{1} = -2$. Hence, the lines are parallel.

815. A sample of size 100 is taken, which has mean 21.2. Subsequently, a set of 10 data, whose mean is 19.4, is removed from the sample. Calculate the raised mean of the remaining 90 data.

Calculate the new $\sum x$.

The set of 10 data has $\sum x = 194$, and the sample has $\sum x = 2120$. Subtracting these, the remaining 90 have $\sum x = 1926$. Their mean is $\frac{1926}{90} = 21.4$.

816. Show that the maximum possible value of the expression $20x^2(40 - x^2)$ is 8000.

Differentiate to find any stationary points, or complete the square.

Since this is a negative quartic, it has a maximum which occurs at a stationary point. Differentiating and setting to zero, we get $-80x^3 + 1600x = 0$. This has three roots, at $x = 0, \pm\sqrt{20}$. By the shape of a quartic, the maximum must be at $x = \pm\sqrt{20}$. Substituting, we get $20^3 = 8000$.

817. You are given that invertible functions f and g have $f(a) = b$, $f(c) = d$, $g(a) = c$ and $g(b) = d$. Write down the values of

(a) $gf(a)$,

(b) $f^{-1}g^{-1}(d)$,

(c) $g^{-1}fg(a)$.

...

(a) $gf(a) = g(b) = d$,

(b) $f^{-1}g^{-1}(d) = f^{-1}(b) = a$,

(c) $g^{-1}fg(a) = g^{-1}f(c) = g^{-1}(d) = b$.

818. Prove that the sum of the exterior angles of a polygon is 2π radians.

Consider a journey around the perimeter.

A journey around the perimeter of a polygon involves a combined rotation of 2π radians (one full revolution). The turns at the individual vertices are exactly the exterior angles of the polygon.

819. Find the range of $h(x) = 64x^4 - 48x^2 + 5$, defined over the domain $\mathbb{R}$.

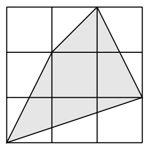
Complete the square.

Completing the square, we get

$$h(x) = (8x^2 - 3)^2 - 4.$$

Since the square term has a minimum value of zero, the range of the function is $\{y \in \mathbb{R} : y \geq -4\}$.

820. The grid shown below consists of unit squares.



Find the fraction of the total area which is shaded.

Subtract the unshaded regions from the total.

The grid has area 9. From this, we remove a square of area 1, and triangles of area $1.5, 1, 1, 0.5$. The shaded area is 4, so the fraction of the total area which is shaded is $\frac{4}{9}$.

821. Two objects are dropped from rest simultaneously. Neglecting air resistance, prove that the distance between them remains constant until they land.

Set up a vertical suvat.

Horizontally, there is no velocity, so the distance remains constant. Vertically, the positions are given by $y_1 = h_1 - \frac{1}{2}gt^2$ and $y_2 = h_2 - \frac{1}{2}gt^2$. The difference between this is a constant $h_1 - h_2$, which is the difference between the initial heights. Since both horizontal and vertical distances are constant, the overall distance between the objects is constant.

822. An arithmetic sequence has n^{th} term u_n. Show that, for any constants $p, q \in \mathbb{R}$, $w_n = pu_n + qu_{n+1}$ is also arithmetic.

Since the sequence is an AP, we know that $u_n = a + (n-1)d$, for some constants a, d. Substituting this into the expression for w_n.

Since the sequence is an AP, we know that $u_n = a + (n-1)d$, for some constants a, d. Substituting these into the expression for w_n, we get

$$\begin{aligned} w_n &= p(a + (n-1)d) + q(a + nd) \\ &= (ap + aq + dq) + (n-1)(p+q)d. \end{aligned}$$

This is an arithmetic progression with first term $(ap + aq + dq)$ and common difference $(p+q)d$.

823. The two quadratic graphs $y = 3(x-2)(x-4)$ and $y = 3x^2 + 24x + 45$ are drawn on the same set of axes. Show that one is a reflection of the other, and find the equation of the line of symmetry.

Factorise the second quadratic, and compare the roots.

The second quadratic is $y = 3(x+3)(x+5)$. Since the quadratics have the same leading coefficient and roots the same distance apart, they are reflections in a line of the form $x = k$. Taking the mean of the outermost roots, we see that $k = \frac{1}{2}(4-5) = -\frac{1}{2}$. So, the line of symmetry is $x = -\frac{1}{2}$.

824. Consider the equation $(a + b\sqrt{6})^2 = 58 - 12\sqrt{6}$, where a and b are integers.

(a) By multiplying out, find two simultaneous equations linking a and b.
(b) Hence, determine $\sqrt{58 - 12\sqrt{6}}$.

...

(a) Multiplying out, we have

$$a^2 + 6b^2 + 2ab\sqrt{6} = 58 - 12\sqrt{6}.$$

Comparing coefficients gives $a^2 + 6b^2 = 58$ and $ab = -6$.

(b) Substituting $a = -\frac{6}{b}$ gives

$$\begin{aligned} &\frac{36}{b^2} + 6b^2 = 58 \\ \Longrightarrow\ &6b^4 - 58b^2 + 36 = 0 \\ \Longrightarrow\ &b^2 = \frac{58 \pm \sqrt{58^2 - 4 \cdot 6 \cdot 36}}{2 \cdot 6} \\ \Longrightarrow\ &b^2 = 9, \frac{2}{3}. \end{aligned}$$

For $b \in \mathbb{Z}$ and the positive square root, we want $b = 3$. Hence, $a = -2$. This gives

$$\sqrt{58 - 12\sqrt{6}} = 3 - 2\sqrt{6}.$$

825. Explain why it is not possible to use the factor theorem, over the real numbers, to establish whether $(2x^2 + x + 1)$ is a factor of $8x^5 + 12x^2 + 3$.

Consider the discriminant of the quadratic.

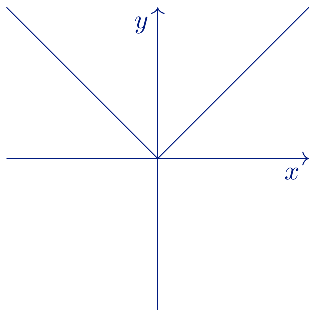
The discriminant of the quadratic is $\Delta = -7 < 0$, so it has no real roots. Hence, we cannot, while working in $\mathbb{R}$, use the factor theorem to establish whether it is a factor of the given quintic.

826. Sketch the following graphs:

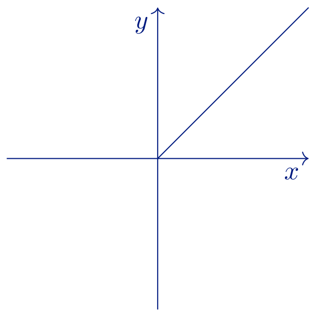
(a) $y = \sqrt{x^2}$,
(b) $y = (\sqrt{x})^2$.

These are parts of the $y = |x|$ graph.

(a) The graph $y = \sqrt{x^2}$ is defined for all real x, and is the same as $y = |x|$.



(b) The graph $y = (\sqrt{x})^2$ is only defined for positive x, over which domain it is the same as $y = x$:



827. A parachutist of mass 55 kg is descending, under canopy, with a constant speed of 1.5 ms^{-1}. When she lands, she comes to rest in 0.4 seconds. Motion may be assumed to be purely vertical throughout.

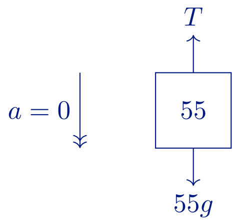
(a) Draw a force diagram for the parachutist in the period before landing.
(b) Show that, at this stage, the parachute exerts an upwards force of 539 Newtons.

(c) Assuming the force exerted by the parachute remains constant throughout the motion, draw a force diagram during the landing.

(d) Determine the average force exerted on the parachutist's feet by the ground during the 0.4 seconds of the landing.

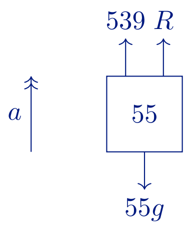
"Under canopy" means that an upwards tension is being exerted by cords attached to a canopy (the main part of a parachute).

(a) Before landing:



(b) At this stage, $T - 55g = 0$, so $T = 539N$.

(c) During landing:



(d) During the landing, the average acceleration is $a = \frac{1.5}{0.4} = 3.75$ ms^{-2}. Substituting into $F = ma$, we get $R + 55g - 55g = 55 \cdot 3.75$, which gives an average force of 206.25 N.

828. Simplify $(2x+3)^4 + (2x-3)^4$.

Use the binomial expansion, nothing that odd powers of x will cancel.

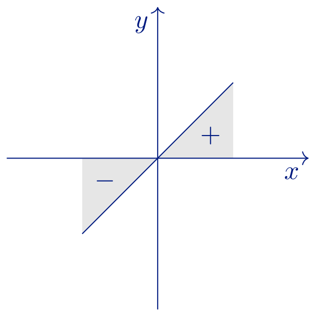
Since the two expansions are symmetrical, the odd powers of x will cancel, leaving only even powers. These are $2 \cdot (2x)^4 + 2 \cdot 6(2x)^2(-3)^2 + 2(-3)^4$. This simplifies to $32x^4 + 432x^2 + 162$.

829. The following definite integrals have value zero. In each case, shade the relevant regions whose areas cancel to give zero, marking them either + or − according to their contribution.

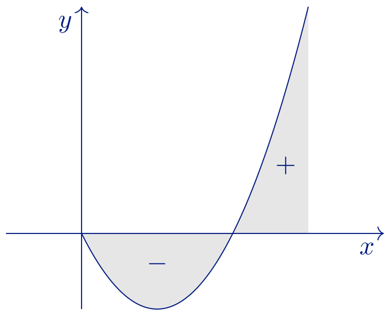
(a) $\displaystyle\int_{-1}^{1} x\,dx$, (b) $\displaystyle\int_{0}^{3} x^2 - 2x\,dx$.

Remember that an integral is a sum (the integral symbol is a big curly S for Sum) of the values of a function. This includes the sign of the function.

(a) $\displaystyle\int_{-1}^{1} x\,dx = 0$,



(b) $\displaystyle\int_{0}^{3} x^2 - 2x\,dx = 0$,



830. Solve $\sin x + \sqrt{3}\cos x = 0$ for $x \in [-180, 180°]$.

Use $\tan x \equiv \dfrac{\sin x}{\cos x}$.

Rearranging gives

$$\begin{aligned} \frac{\sin x}{\cos x} &= -\sqrt{3} \\ \Longrightarrow \tan x &= -\sqrt{3} \\ \Longrightarrow x &= -60°, 120°. \end{aligned}$$

831. Separate the variables in the following differential equation, writing it in the form $f(y)\frac{dy}{dx} = g(x)$ for some functions f and g:

$$\frac{dy}{dx} - 4xy = 0.$$

Add $4xy$ to both sides first.

$$\begin{aligned} &\frac{dy}{dx} - 4xy = 0 \\ \Longrightarrow\ &\frac{dy}{dx} = 4xy \\ \Longrightarrow\ &\frac{1}{y}\frac{dy}{dx} = 4x. \end{aligned}$$

832. The interior angles of a triangle are in AP. Find the sum of the smallest and largest angles.

Consider the value of the middle angle.

The middle angle must be the average of the three, so 60°. Hence, the sum of the smallest and largest angles is 120°.

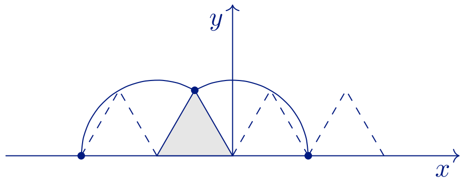
833. A shape is produced by rolling an equilateral triangle of side length 1 along the x axis, and marking the path of one of its vertices. This shape is known as a *cyclogon*.

(a) Sketch this cyclogon.

(b) Show that each period has arc length $\frac{4\pi}{3}$.

Search "cyclogon" if you can't visualise this...

(a) Each of the sections of the path of the relevant vertex is a circular arc:



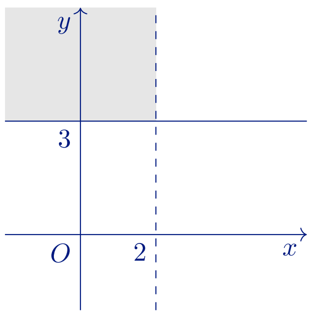
(b) The two circular arcs depicted are a single period of the cyclogon. Together, they form $\frac{2}{3}$ of a circle, with length $l = \frac{2}{3} \cdot 2\pi = \frac{4\pi}{3}$.

834. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$x < 2, \qquad y \geq 3.$$

This is a quadrant-like region.

The boundary equations are lines $x = 2$ and $y = 3$. We require the region to the left of $x = 2$, and above and including $y = 3$. For this, we draw the line $y = 3$ as solid, and the line $x = 2$ as dashed:



835. Write down the largest real domains over which the following functions may be defined:

(a) $x \mapsto \sqrt{x}$,

(b) $x \mapsto \sqrt{x^2}$,

(c) $x \mapsto \sqrt{x^3}$.

Consider the odd and even cases separately.

The odd and even cases are different. Since the square renders all numbers positive, the second function may be defined over $\mathbb{R}$, unlike the other two.

(a) $\mathbb{R}^+$,

(b) $\mathbb{R}$,

(c) $\mathbb{R}^+$.

836. Prove that no three distinct points on a parabola are collinear.

Assume, for a contradiction, that a line intersects the parabola at three distinct points.

Assume, for a contradiction, that such points exist. If the three points are collinear, then there is a straight line which intersects the parabola at three distinct points. But, if we solve for intersections between a line and a parabola, we get a quadratic equation, which has a maximum of two roots. This is a contradiction. Hence, no three distinct points on a parabola are collinear.

837. Two functions f and g have constant derivatives. Determine the possible numbers of roots of the equation $f(x) = g(x)$.

Translate into algebra and integrate.

Translating these facts into algebra, we have $f'(x) = a$ and $g'(x) = b$, for constants a, b. Integrating gives $f(x) = ax + c$, $g(x) = bx + d$. These are two linear functions. So, $f(x) = g(x)$ can have

0 roots, if $a = b$ and $c \neq d$,

1 root, if $a \neq b$,

infinitely many roots, if $a = b$ and $c = d$.

838. The parabola P_1, given by the equation $y = x^2$, is translated by the vector $k\mathbf{i}$ onto parabola P_2. The curves P_1 and P_2 intersect at $y = 16$. Find all possible values of k.

Parabola P_2 is $y = (x - k)^2$.

Since parabola P_2 has been translated by $k\mathbf{i}$, its equation is $y = (x - k)^2$. The curves intersect at $(\pm 4, 16)$, so we know that $16 = (\pm 4 - k)^2$. Solving gives $\pm 4 = \pm 4 - k$, where the two $\pm$ signs are independent. Hence, $k \in \{-8, 0, 8\}$.

839. A student claims that $x(x^2 + 1)$ can be integrated factor by factor, to give $\frac{1}{2}x^2(\frac{1}{3}x^3 + x) + c$. Show that this claim is incorrect.

Multiply out the proposed integral and differentiate it.

Multiplying out the proposed integral gives $\frac{1}{6}x^5 + \frac{1}{2}x^3 + c$. The derivative of this is $\frac{5}{6}x^4 + \frac{3}{2}x^2$, which was not the original expression.

840. According to an ancient legend, an Indian vizier, who made a chessboard for the king, requested the following payment: 1 grain of rice on the first square, 2 on the second, 4 on the third, and so on. The king readily agreed to this. Taking the mass of a grain of rice to be a hundredth of a gram, show that, unbeknownst to the king, the vizier was, in fact, asking for 184 billion tonnes of rice.

This is a geometric series.

The number of grains of rice is given by a geometric series, with first term 1, common ratio 2, and number of terms 64. The total is therefore

$$N = \frac{1(2^{64}-1)}{2-1} \approx 2^{64}.$$

One grain of rice is 10^{-2} grams, which is 10^{-5} kg, which is 10^{-8} tonnes. So, the total mass demanded is approximately $2^{64} \times 10^{-8}$, which is 1.84×10^{11}, or 184 billion tonnes.

841. Show that the following equation, with the square root function defined over the domain $\mathbb{R}^+$, has no roots:

$$\frac{1}{1+\sqrt{x}} + \frac{1}{1-\sqrt{x}} = 1.$$

Multiply up by the denominators.

Multiplying up by the denominators, we get

$$\begin{aligned} 1-\sqrt{x}+1+\sqrt{x} &= 1-x \\ \Longrightarrow x &= -1 \end{aligned}$$

But the square root function is undefined for negative x, so the equation has no roots.

842. Two distinct integers x_1 and x_2 are chosen from $\{x \in \mathbb{Z} : 1 \leq x \leq 5\}$. Find $P(x_1x_2 < 10)$.

The possibility space is a 5×5 grid.

The possibility space is a 5×5 grid:

	1	2	3	4	5
1	✓	✓	✓	✓	✓
2	✓	✓	✓	✓	
3	✓	✓	✓		
4	✓	✓			
5	✓				

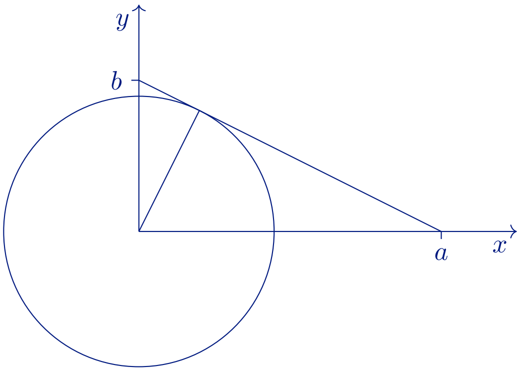
$$p = \frac{\text{successful}}{\text{total}} = \frac{15}{25} = \frac{3}{5}.$$

843. On a right-angled triangle (a, b, c), a circle is drawn. It is tangent to the hypotenuse, and its centre is at C. Show that it has area

$$A = \frac{\pi a^2 b^2}{c^2}.$$

The circle is centred on the right-angle. Set up axes, and solve an equation for the position of the point of tangency.

Setting up x and y axes, we have



The equation of the hypotenuse is $y = b - \frac{b}{a}x$; the equation of the radius is $y = \frac{a}{b}x$. Substituting to find the point of tangency, we get

$$\begin{aligned} & b - \tfrac{b}{a}x = \tfrac{a}{b}x \\ \Longrightarrow\ & ab^2 = (a^2+b^2)x \\ \Longrightarrow\ & x = \frac{ab^2}{c^2}, \quad y = \frac{a^2b}{c^2} \end{aligned}$$

This gives the area as

$$\begin{aligned} & \pi\frac{a^2b^4 + a^4b^2}{c^4} \\ =\ & \frac{\pi a^2b^2(a^2+b^2)}{c^4} \\ =\ & \frac{\pi a^2 b^2}{c^2}. \end{aligned}$$

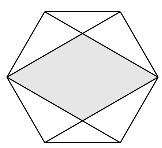
844. State, with a reason, whether the following gives a well-defined function:

$$g : \begin{cases} \mathbb{R} \mapsto \mathbb{R} \\ x \mapsto \dfrac{1}{x^2+x+1}. \end{cases}$$

Consider whether the denominator can equal zero.

This is a well defined function. The discriminant of x^2+x+1 is $\Delta = -3 < 0$, so the denominator cannot equal zero. Hence, $f(x)$ is well-defined for all $x \in \mathbb{R}$.

845. The regular hexagon below has side length 1.



Show that the area of the shaded region is $\dfrac{2\sqrt{3}}{3}$.

The angles of the shaded rhombus are 60° and 120°.

The shaded rhombus consists of two equilateral triangles. The height of the hexagon is $\sqrt{3}$, and the height of each of the small isosceles triangles at the top and bottom is $\frac{1}{2\sqrt{3}}$. Hence, the side length of the rhombus is

$$\sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

The area is then

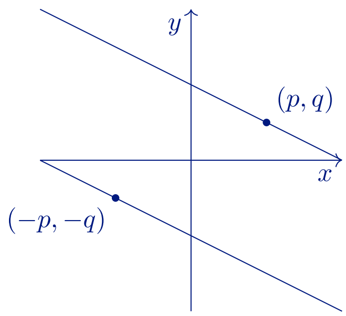
$$2 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{2\sqrt{3}}{3}\right)^2 = \frac{2\sqrt{3}}{3}.$$

846. On the same axes, for constants $p, q > 0$ and $k < 0$, sketch the graphs

(a) $\dfrac{y-q}{x-p} = k$, (b) $\dfrac{y+q}{x+p} = k$.

These are two straight lines.

These are two straight lines, with negative gradient k, passing through the points (p, q) and $(-p, -q)$:



847. Show that $x^2 + 1$ is a factor of $1 + x + x^2 + x^3$.

The factor theorem, over the real numbers, cannot be used here. Factorise explicitly.

Since $x^2 + 1$ is an irreducible quadratic over the reals, we must factorise explicitly:

$$1 + x + x^2 + x^3 \equiv (x^2 + 1)(x + 1).$$

848. You are given that $\mathbf{a}$ and $\mathbf{b}$ are two perpendicular unit vectors. Prove that $\frac{3}{5}\mathbf{a} + \frac{4}{5}\mathbf{b}$ and $-\frac{4}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$ are also perpendicular unit vectors.

Assume, without loss of generality, that $\mathbf{a}$ and $\mathbf{b}$ are the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

We may assume, without loss of generality, that $\mathbf{a}$ and $\mathbf{b}$ are the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$. The gradient of $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$ is $\frac{4}{3}$, while the gradient of $-\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$ is $-\frac{3}{4}$. These are negative reciprocals, so the vectors are perpendicular. And, since $(\frac{3}{5}, \frac{4}{5}, 1)$ is a Pythagorean triad, we know that these are unit vectors.

849. Find all possible values of k satisfying

$$\int_1^k \frac{x^2+1}{x^2}\,dx = 4.8.$$

Split the fraction up before integrating.

Splitting the fraction up, we get

$$\begin{aligned}
&\int_1^k 1 + \frac{1}{x^2}\,dx = 4.8 \\
\Longrightarrow\ & \left[x - \frac{1}{x}\right]_1^k = 4.8 \\
\Longrightarrow\ & (k - \tfrac{1}{k}) - (1 - 1) = 4.8 \\
\Longrightarrow\ & 5k^2 - 24k - 5 = 0 \\
\Longrightarrow\ & (5k+1)(k-5) = 0 \\
\Longrightarrow\ & k = -\tfrac{1}{5}, 5.
\end{aligned}$$

850. A projectile is launched from ground level at speed u, at an angle that will attain maximum range. In terms of u, find the greatest height achieved by the particle in this motion.

The angle of projection which attains maximum range over flat ground is 45°.

The angle of projection which attains maximum range over flat ground is 45°. At this angle, the maximum height h is given by

$$0 = \left(\frac{\sqrt{2}}{2}\right)^2 - 2gh.$$

This gives $h = \dfrac{u^2}{4g}$.

851. Two of the following statements are true; the other is not. Prove the two and disprove the other.

(a) $x^3(2^x + 4) = 0 \implies x = 0$,

(b) $x^3(2^x - 1) = 0 \implies x = 0$,

(c) $x^3(2^x - 3) = 0 \implies x = 0$.

The implication is only valid if the only root of the second factor is $x = 0$.

(a) This is true, since $2^x + 4$ is never zero.

(b) This is true; both factors are zero at $x = 0$.

(c) This is false; $x = \log_2 3$ is a counterexample.

852. Solve $\sin^4 x + \sin^2 x = 0$ for $x \in [0, 360°)$.

Factorise.

Factorising, we have

$$\begin{aligned} & \sin^4 x + \sin^2 x = 0 \\ \Longrightarrow\; & \sin^2 x(\sin^2 x + 1) = 0. \end{aligned}$$

The latter factor has no real roots, so we require $\sin x = 0$, which gives $x = 0, 180°$.

853. A function g has domain $[0, 1]$ and range $[0, 1]$. State, with a reason, whether the following are well-defined functions over the domain $[0, 1]$:

(a) $x \mapsto g(2x)$,

(b) $x \mapsto 2g(x)$.

In each case, consider the numbers that serve as inputs to the function g.

(a) This is not well-defined. Over $[0, 1]$, the expression $2x$ has range $[0, 2]$, which contains values not in the domain of g.

(b) This is well-defined. It has domain $[0, 1]$ and range $[0, 2]$.

854. Disprove the following statement: "Every pair of linear simultaneous equations in two unknowns has a unique solution."

Give a counterexample: two linear simultaneous equations which do not have a unique solution.

Any pair of lines $ax + by = c$ and $ax + by = d$ form a counterexample. If $c = d$, then there are infinitely many solutions; if $c \neq d$, there are none.

855. Show that the following equation defines a circle minus a point, and determine its centre and radius:

$$\frac{x + y}{x^2 + y^2} = 1.$$

Multiplication by $x^2 + y^2$ adds a point back in, for which the original equation is undefined.

Multiplying by $x^2 + y^2$ gives

$$\begin{aligned} & x + y = x^2 + y^2 \\ \Longrightarrow\; & x^2 - x + y^2 - y = 0 \\ \Longrightarrow\; & (x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{2}. \end{aligned}$$

The above is a circle, centre $(\frac{1}{2}, \frac{1}{2})$, radius $\frac{\sqrt{2}}{2}$. However, the origin, which lies on this circle, does not satisfy the original equation, since $\frac{0}{0}$ is undefined. Hence, the locus of points satisfying the original equation is a circle minus a point.

856. A die has been rolled. Determine whether the fact "The score is even" increases, decreases or doesn't affect the probability that the score is at least four.

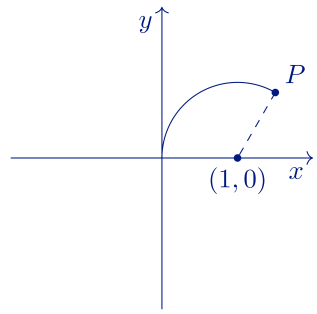
Consider the restriction of the possibility space brought about by the information given.

Without this information, the probability is $\frac{1}{2}$. With it, the possibility space is reduced to $\{2, 4, 6\}$, of which $\frac{2}{3}$ are at least 4. Hence, the information increases the relevant probability.

857. Point $P : (\frac{3}{2}, \frac{\sqrt{3}}{2})$ is rotated anticlockwise around the point $(1, 0)$ by $\frac{2\pi}{3}$ radians. Find the image of P under this transformation.

Sketch the scenario.

Sketching the scenario gives



The image under this rotation is the origin.

858. Without a calculator, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sin x \, dx$.

Use the standard trig values.

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} \sin x \, dx \\ =\; & \Big[-\cos x\Big]_0^{\frac{\pi}{3}} \\ =\; & -\cos\tfrac{\pi}{3} + \cos 0 \\ =\; & -\tfrac{1}{2} + 1 \\ =\; & \tfrac{1}{2}. \end{aligned}$$

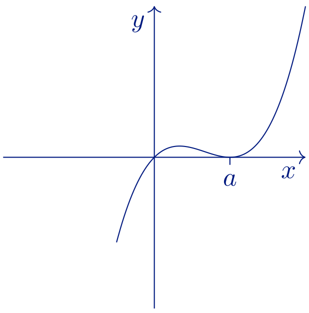
859. The graph $y = 12x(x - a)^2$, for $a > 0$, is a positive cubic with a double root at $x = a$.

(a) Sketch the graph.

(b) Show that the area enclosed by the graph and the x axis is a^4.

The double root implies a point of tangency with the x axis.

(a) The curve crosses the x axis at 0 (single root) and touches it at $x = a$ (double root):



(b) The area enclosed by the graph and the x axis is given by the definite integral

$$\begin{aligned}&\int_0^a 12x^3 - 24ax^2 + 12a^2x \, dx \\ =\ & \left[3x^4 - 8ax^3 + 6a^2x^2\right]_0^a \\ =\ & (3a^4 - 8a^4 + 6a^4) - (0) \\ =\ & a^4.\end{aligned}$$

860. The line $y = 2x + 5$ is closest to $(0, 0)$ at point P. Find the coordinates of point P.

The shortest path from a point to a line is perpendicular to the line.

The shortest path from a point to a line is perpendicular to the line. So, the line OP is $y = -\frac{1}{2}x$. Solving simultaneously gives $(-2, 1)$.

861. Find simplified expressions for the following sets, in which $a < b < c < d$ are constants:

(a) $(a, c) \cap [b, d]$,

(b) $(a, c] \cup [b, d)$,

(c) $(a, b] \cap [c, d)$.

To visualise these, sketch a number line.

(a) $(a, c) \cap [b, d] = [b, c)$,

(b) $(a, c] \cup [b, d) = (a, d)$,

(c) $(a, b] \cap [c, d) = \varnothing$.

862. Solve the inequality $t^3 - 4t \geq 0$.

Sketch the cubic $y = t^3 - 4t$.

The boundary equation $t^3 - 4t = 0$ has roots $t = -2, 0, 2$. We require a positive cubic to be above the t axis, which gives the solution

$$t \in [-2, 0] \cup [2, \infty).$$

863. Variables x and y are related as follows:

$$y = \frac{1}{1+x} + \frac{1}{1-x}.$$

Make x the subject of this relationship.

...

$$\begin{aligned} & y = \frac{1}{1+x} + \frac{1}{1-x} \\ \implies & y = \frac{1}{1-x^2} \\ \implies & 1 - x^2 = \tfrac{1}{y} \\ \implies & x = \pm\sqrt{1 - \tfrac{1}{y}}. \end{aligned}$$

864. The tangent to the curve $y = \frac{1}{2}x^2$ at $x = a$ makes an angle θ with the x axis.

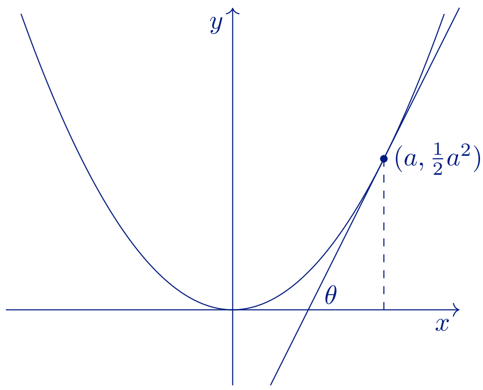
(a) Sketch the information above.

(b) Differentiate $y = \frac{1}{2}x^2$.

(c) Hence, show that $\tan\theta = a$.

...

(a) Tangent at $x = a$:



(b) $\dfrac{dy}{dx} = x$.

(c) The gradient of the tangent depicted is a. So, in the right-angled triangle produced by the dashed perpendicular, $\tan\theta = a$.

865. Show that the following ratio cannot be expressed as $1 : p(x)$, where $p(x)$ is a polynomial:

$$x + 1 : x^3 - 4x^2 + 2x + 1.$$

This is equivalent to saying that $(x + 1)$ is not a factor of the cubic.

This is equivalent to saying that $(x + 1)$ is not a factor of the cubic. So, we need only evaluate the cubic at $x = -1$. This gives -6. Hence, by the factor theorem, $(x+1)$ is not a factor of the cubic.

866. A triangle has shortest side 4, and its larger two angles are $\arccos\frac{1}{8}$ and $\arccos\frac{9}{16}$.

(a) Find the smallest angle of the triangle.
(b) Hence, find its side lengths.

Use the sine rule.

(a) The smallest angle is $180° - \arccos\frac{1}{8} - \arccos\frac{9}{16} = 41.4096...°$.
(b) By the sine rule, the other lengths are given by
$$\frac{4\sin(\arccos\frac{9}{16})}{\sin 41.4} = 5$$
$$\frac{4\sin(\arccos\frac{1}{8})}{\sin 41.4} = 6.$$

867. "The coordinate axes are normal to the curve $x^2 + 4y^2 = 1$." True or false?

The curve is an ellipse.

This is true. The curve is an ellipse, centred on the origin. A circle centred on the origin is normal to the coordinate axes, and the stretch factor $\frac{1}{2}$ in the y direction does not change this fact.

868. An object is in equilibrium, acted on by exactly three forces $\mathbf{F}, \mathbf{G}, \mathbf{H}$. Show that, if $\mathbf{F} = p\mathbf{G}$ for some $p \in \mathbb{R}$, then $\mathbf{H} = q\mathbf{G}$ for some $q \in \mathbb{R}$.

Consider whether the forces are parallel or not.

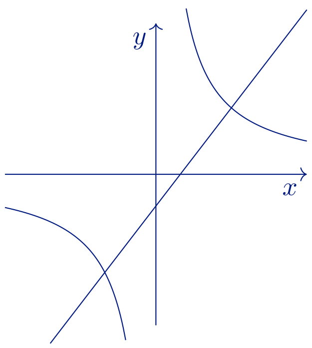
If $\mathbf{F} = p\mathbf{G}$ for some $p \in \mathbb{R}$, then the forces $\mathbf{F}$ and $\mathbf{G}$ are parallel. The third force cannot then have a component perpendicular to these. So, it must also be parallel to $\mathbf{G}$ (and $\mathbf{F}$). This is expressed in the statement "$\mathbf{H} = q\mathbf{G}$ for some $q \in \mathbb{R}$."

869. Solve $1.30x - 0.42 < \dfrac{0.88}{x}$.

The boundary equation is a quadratic.

Setting up the boundary equation,
$$\begin{aligned} & 1.30x - 0.42 = \frac{0.88}{x} \\ \Longrightarrow\ & 65x^2 - 21x - 44 = 0 \\ \Longrightarrow\ & (65x + 44)(x - 1) = 0 \\ \Longrightarrow\ & x = 1, -\frac{44}{65}. \end{aligned}$$

Sketching the LHS and RHS



So, the set of x values satisfying the inequality is $(-\infty, -\frac{44}{65}) \cup (0, 1)$.

870. The discriminant Δ of a quadratic equation Q satisfies $\Delta^2 - \Delta = 0$. Find all possible numbers of real roots of equation Q.

Solve for Δ.

The information we are given is a quadratic equation in Δ. Factorising gives $\Delta(\Delta - 1) = 0$, so $\Delta = 0$ or $\Delta = 1 > 0$. These two possibilities mean that the original quadratic Q could have 1 or 2 real roots.

871. In each case, state the value of the limit:

(a) $\displaystyle\lim_{x\to+\infty} \frac{x}{|x| + 1}$,
(b) $\displaystyle\lim_{x\to-\infty} \frac{x}{|x| + 1}$.

In each case, compare the size and sign of the numerator and denominator, for large (in magnitude) values of x.

In each case, the ratio of the magnitudes of numerator and denominator approaches 1. The only difference is the sign of the numerator. This gives

(a) $\displaystyle\lim_{x\to+\infty} \frac{x}{|x| + 1} + 1$,
(b) $\displaystyle\lim_{x\to-\infty} \frac{x}{|x| + 1} - 1$.

872. Show that the normal to $x = y^2$ at $(9, 3)$ crosses the x axis at $x = \frac{19}{2}$.

Since $(9, 3)$ is in the positive quadrant, you can use $y = \sqrt{x}$.

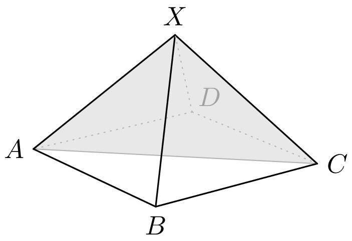
Since $(9, 3)$ is in the positive quadrant, we can use $y = \sqrt{x}$. This has $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$. The gradient of the tangent at $(9, 3)$ is $\frac{1}{6}$, so the normal gradient is -6. The equation of the tangent is therefore $y = -6x + c$. Substituting $(9, 3)$ gives $c = 57$. Then, solving for $y = 0$, we get $x = \frac{19}{2}$.

873. A projectile is launched from a height h above the ground, with fixed initial speed u. Prove that the landing speed will be the same if the projectile is launched vertically upwards or downwards.

Use suvat.

The relevant constant acceleration formula is $v^2 = u^2 + 2as$. Since the initial velocity u is squared in this equation, it makes no difference whether it is positive or negative.

874. The square-based pyramid shown below is formed of eight edges of unit length.



Find the area of the shaded triangle AXC.

AXC is a right-angled triangle.

AXC is a right-angled triangle, so its area is $\frac{1}{2}$.

875. Verify that the quartic curve $y = x^4$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = 12\sqrt{y}.$$

Calculate the second derivative and substitute.

Differentiating twice, we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= 12x^2 \\ &= 12\sqrt{x^4} \\ &= 12\sqrt{y}.\end{aligned}$$

This is the required result.

876. A number generator in a computer produces the periodic sequence $1, 2, 3, 4, 5, 1, 2, ...$

(a) Find the probability that 1 is chosen if a digit is randomly selected from among the first

i. 5 terms,

ii. 6 terms,

iii. $5k$ terms, $k \in \mathbb{N}$,

iv. $5k + 1$ terms, $k \in \mathbb{N}$,

(b) Show that iv. tends to $\frac{1}{5}$ as $k \to \infty$.

Use $p = \frac{\text{successful}}{\text{total}}$.

(a) The probabilities are

i. $\frac{1}{5}$,

ii. $\frac{1}{3}$,

iii. $\frac{1}{5}$,

iv. $\frac{k+1}{5k+1}$.

(b) Dividing top and bottom by k, the limiting probability is

$$p = \lim_{k\to\infty} \frac{1 + \frac{1}{k}}{5 + \frac{1}{k}} = \frac{1}{5}.$$

877. Show that $(x^2+4x+4)(x^2+x+3) = 0$ has exactly one root.

Consider the roots of each factor.

The discriminants of each quadratic factor are $\Delta_1 = 0$ and $\Delta_2 = -11$, giving 1 and 0 roots respectively. Hence, the quartic has exactly 1 root.

878. Solve the following simultaneous equations:

$$\begin{aligned}x^2 + y^2 &= 10, \\ 2x^2 - 3y^2 &= 15.\end{aligned}$$

Solve as a pair of linear equations in x^2 and y^2.

These can be considered as a pair of linear equations in x^2 and y^2. Solving simultaneously gives $x^2 = 9$, $y^2 = 1$. Either can go with either, so there are four solutions $(\pm 3, \pm 1)$, where the $\pm$ signs are independent.

879. Functions f and g are said to *commute* if

$$f(g(x)) \equiv g(f(x)).$$

(a) Show that, if the functions $f(x) = x^2$ and $g(x) = kx$ commute, then $k = 0$ or 1.

(b) Prove that $f(x) = x^a$ and $g(x) = x^b$ always commute, for all $a, b \in \mathbb{R}$.

In (b), the result follows directly from an index law.

(a) Since f and g commute, we know that $(kx)^2 \equiv kx^2$. This requires $k^2 = k$, which has roots $k = 0, 1$.

(b) This follows directly from an index law:

$$f(g(x)) = (x^b)^a = x^{ab} = (x^a)^b = g(f(x)).$$

So $f(x) = x^a$ and $g(x) = x^b$ always commute.

880. A sector has arc length l and subtends an angle θ, measured in radians, at the centre. Show that the area is given by the formula

$$A = \frac{l^2}{2\theta}.$$

Quote the formulae for A and l, and eliminate θ from the pair of them.

We know that $l = r\theta$ and $A = \frac{1}{2}r^2\theta$. Substituting the former into the latter, we get

$$A = \frac{1}{2}\left(\frac{l}{\theta}\right)^2\theta = \frac{l^2}{2\theta}.$$

881. A student is attempting to solve the equation $(x-2)^4-(x-2)^3=0$. His first step is to multiply out, using the binomial expansion. Explain why this is a long way round, and solve succinctly.

Consider a common factor.

There is a common factor of $(x-2)^3$. It is best to take this factor out first:

$$\begin{aligned} &(x-2)^4-(x-2)^3=0\\ \Longrightarrow\ &(x-2)^3(x-2-1)=0\\ \Longrightarrow\ &(x-2)^3(x-3)=0\\ \Longrightarrow\ &x=2,3. \end{aligned}$$

882. Given that $\cos 36° = \frac{1}{4}(1+\sqrt{5})$, find a simplified expression for $\sec 36°$.

Reciprocate the expression, and rationalise its denominator.

Since $\cos 36° = \frac{1}{4}(1+\sqrt{5})$, we know that

$$\begin{aligned} \sec\theta &= \frac{4}{1+\sqrt{5}}\\ &= \frac{4(1-\sqrt{5}}{1-5}\\ &= \sqrt{5}-1. \end{aligned}$$

883. Determine the solution of $x^3+4x^2-27x-90=0$.

Find a root/factor by inspection or numerical methods.

Setting up a Newton-Raphson iteration, we have

$$x_{n+1}=x_n-\frac{x^3+4x^2-27x-90}{3x^2+8x-27}.$$

Running this iteration with $x_0=0$ yields $x_n \to -3$. So $x=-3$ is a root, and $(x+3)$ is a factor. This gives

$$\begin{aligned} &x^3+4x^2-27x-90=0\\ \Longrightarrow\ &(x+3)(x^2+x-30)=0\\ \Longrightarrow\ &(x+3)(x+6)(x-5)=0\\ \Longrightarrow\ &x=-6,-3,5. \end{aligned}$$

884. A particle is placed on a smooth slope of angle of elevation 45° and released from rest. Find the time taken for the particle to move 1 metre.

The acceleration under gravity on a smooth slope of angle θ is $g\sin\theta$.

The acceleration under gravity on a smooth slope of angle θ is $g\sin\theta$. In this case, $a=\frac{\sqrt{2}}{2}g$. Using $s=ut+\frac{1}{2}at^2$, we get

$$\begin{aligned} 1 &= \frac{\sqrt{2}}{4}gt^2\\ \Longrightarrow t &= \pm 0.537229... \end{aligned}$$

So $t=0.537$ s (3sf).

885. A region of the (x,y) plane is defined by

$$(x^2+y^2-1)(x^2+y^2-4)\le 0.$$

(a) Explain, referring to the signs of the factors, why the region is an *annulus*, i.e. a thick ring bounded by two concentric circles.

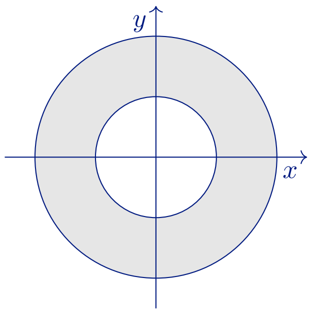
(b) Sketch and shade the region.

Concentric circles have the same centre. In this case, the centre is the origin.

(a) The boundary equation is solved when one of the factors is zero, i.e. $x^2+y^2=1,4$. These are two circles, centred on the origin, with radii 1 and 2.

For the original LHS to be negative, exactly one of its factors must be negative. So the region is the annulus $1\le x^2+y^2\le 4$ between the circles.

(b) Sketch of $1\le x^2+y^2\le 4$



886. Simplify $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}+\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.

Rationalise the denominators.

Rationalising the denominators, we get

$$\begin{aligned} &\frac{(\sqrt{2}+1)^2}{1}+\frac{(\sqrt{2}-1)^2}{1}\\ =\ &(2+2\sqrt{2}+1)+(2-2\sqrt{2}+1)\\ =\ &6. \end{aligned}$$

887. The tangent to the curve $y=x^2+2x$ at point P passes through $(0,-4)$.

(a) Write down the equation of a general line through the point $(0,-4)$.

(b) Hence, using $\Delta = b^2 - 4ac$, or otherwise, find the possible coordinates of point P.

In (b), the tangent line has exactly one intersection with the quadratic.

(a) A general line through $(0, -4)$ is $y = mx - 4$.

(b) Intersections are given by $x^2 + 2x = mx - 4$, which simplifies to $x^2 + (2 - m)x + 4 = 0$. We require this quadratic to have exactly one root, so the discriminant must be zero:

$$\Delta = (2 - m)^2 - 16 = 0.$$

Solving for m gives $m = -2, 6$. Substituting this back in and solving the resulting quadratic gives the possible coordinates of P as $(-2, 0)$ or $(2, 8)$.

888. A rational number can be expressed as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$. Prove that the sum of two rational numbers is rational.

Express the two rational numbers as quotients of integers.

The sum of two rational numbers can be written in terms of integers a, b, c, d as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Since a, b, c, d are integers, we know that $ad + bc$ and bd are integers, which makes the sum a rational number.

889. State, with a reason, which, if any, of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $x^2 + x = 0$,
- $x + 1 = 0$.

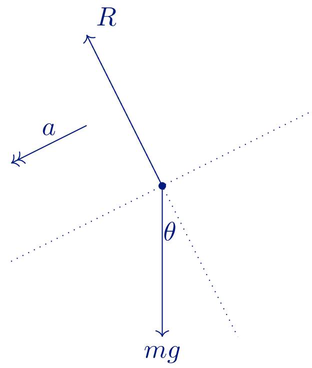
Consider the value $x = 0$.

The implication is backwards $\Longleftarrow$. $x = -1$ is a root of the original equation, but so is $x = 0$; the latter is a counterexample to a forwards implication.

890. Prove that, when a particle is placed on a smooth slope of angle of inclination θ, its acceleration is given by $a = g \sin\theta$ ms^{-2}.

Draw a force diagram, and give the particle an unknown mass m.

The particle has forces as follows:



Resolving parallel to the slope, Newton II gives $mg\sin\theta = ma$, which gives the required result $a = g\sin\theta$ ms^{-2}.

891. Find the probability that, when two dice are rolled, at least one is a six.

Draw the possibility space, and count outcomes.

The possibility space is:

	1	2	3	4	5	6
1						✓
2						✓
3						✓
4						✓
5						✓
6	✓	✓	✓	✓	✓	✓

Counting outcomes gives $p = \frac{11}{36}$.

892. You are given that the cubic function $f(x) = x^3 + x^2 - 16x - 20$ has a double root at $x = 2$.

(a) Factorise $f(x)$ as $(x - 2)^2(x + a)$, where a is to be determined.

(b) Find the gradient function $f'(x)$.

(c) Show that $f'(x) = 0$ at the double root and $f'(x) \neq 0$ at the single root.

(d) Interpret this result.

...

(a) $\quad x^3 + x^2 - 16x - 20$
$\equiv (x - 2)^2(x + 5)$.

(b) Differentiating, $f'(x) = 3x^2 + 2x - 16$.

(c) $f'(2) = 0$, $f'(-5) = -105$.

(d) Since $f(x) = 0$ has a double root at $x = 2$, the root is also a turning point, hence $f'(2) = 0$. However, $x = -5$ is a single root, which means the curve $y = f(x)$ crosses the x axis, with non-zero gradient.

893. Exactly three forces act on an object, which is in equilibrium. The three forces have magnitudes $50, 60$ and F Newtons respectively. Determine the set of possible values of F.

Consider the greatest and smallest values of the magnitude of F, as determined by the directions of the 50 and 60 Newton forces.

The magnitude of F is largest/smallest if the 50 and 60 Newton forces are parallel/antiparallel. Hence, the set of possible values is $[10, 110]$ N.

894. Primes of the form $n! \pm 1$, for $n \in \mathbb{N}$, are known as *factorial primes*. Disprove the following statement: "Every number of the form $n! \pm 1$ is prime."

Find a counterexample.

The factorials are $1, 2, 6, 24, ...$ Hence $n = 4$ is a counterexample to this claim, since $n! + 1 = 5 \times 5$.

895. Three sets A, B, C are defined as $A : \{1, 2, ..., 10\}$, $B : \{1, 2, 3, 4, 5\}$, and $C : \{2, 4, 6, 8, 10\}$. An element x is picked at random from set A.

(a) Find $P(x \in B)$, $P(x \in C)$ and $P(x \in B \cap C)$.

(b) By comparing these values, determine if the events represented by the sets B and C are independent.

Use $p = \frac{\text{successful}}{\text{total}}$.

(a) Counting successful outcomes, $P(x \in B) = \frac{1}{2}$, $P(x \in C) = \frac{1}{2}$, and $P(x \in B \cap C) = \frac{1}{5}$.

(b) Since the probability of the intersection is $\frac{1}{5} \neq \frac{1}{2} \times \frac{1}{2}$, events $x \in B$ and $x \in C$ are not independent.

896. Describe all functions f for which f' is linear.

Translate this into algebra, and integrate.

If f' is linear, then $f'(x) = px + q$, for some constants p and q. Integrating, we get $f(x) = ax^2 + bx + c$, for constants a, b, c. This is all quadratic ($a \neq 0$) or linear ($a = 0$) functions.

897. You are given that

$$\int_0^1 x^n(1 + x)(1 - x)\,dx = \frac{2}{35}.$$

Determine all possible values of n.

Multiply out first.

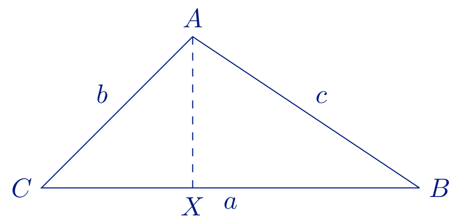
Multiplying out, the integrand is $x^n(1-x^2)$, which simplifies to $x^n - x^{n+2}$. Hence, we require

$$\begin{aligned}
&\left[\frac{1}{n+1}x^{n+1} - \frac{1}{n+3}x^{n+3}\right]_0^1 = \frac{2}{35} \\
\Longrightarrow\ &\frac{1}{n+1} - \frac{1}{n+3} = \frac{2}{35} \\
\Longrightarrow\ &35(n+3) - 35(n+1) = 2(n+1)(n+3) \\
\Longrightarrow\ &35n + 105 - 35n - 35 = 2n^2 + 8n + 6 \\
\Longrightarrow\ &n^2 + 4n - 32 = 0 \\
\Longrightarrow\ &(n+8)(n-4) = 0 \\
\Longrightarrow\ &n = -8, 4.
\end{aligned}$$

898. Prove the cosine rule.

Drop a perpendicular in an acute triangle, and calculate lengths using Pythagoras.

We set up a general triangle ABC as follows, dropping a perpendicular from the vertex A and labelling its endpoint X:



Working from left to right, $CX = b\cos C$, $AX = b\sin C$, and $XB = a - b\cos C$. Pythagoras in triangle ABX then gives

$$\begin{aligned}
&c^2 = (a - b\cos C)^2 + (b\cos C)^2 \\
\Longrightarrow\ &c^2 = a^2 - 2ab\cos C + b^2\cos^2 C + b^2\sin^2 C.
\end{aligned}$$

By the Pythagorean identity, the last two terms sum to b^2, which gives the cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

899. Car A sets off from O with constant acceleration 0.4 ms^{-2}. Car B sets off two seconds later with constant acceleration 0.6 ms^{-2}, from a point 10 metres back from O. Both cars start from rest.

(a) Show that the time t at which car B overtakes satisfies $t^2 - 12t - 88 = 0$.

(b) Find the distance from O at which B overtakes.

Set up simultaneous equations, using $t - 2$ to express the fact "sets off two second later".

(a) Translating into algebra, we have $x_A = 0.2t^2$ and $x_B = 0.3(t-2)^2 - 10$. The positions are the same when

$$\begin{aligned} & 0.2t^2 = 0.3(t-2)^2 - 10 \\ \Longrightarrow\ & 2t^2 = 3t^2 - 12t + 12 - 100 \\ \Longrightarrow\ & t^2 - 12t - 88 = 0. \end{aligned}$$

(b) Solving, we get $t = 6+2\sqrt{31}$. Substituting this back into either expression for position gives 58.7 m (3sf).

900. Prove, by considering interior angles, that regular heptagons do not tessellate.

Find the interior angle of a regular heptagon, and show that it does not divide exactly into 360°.

The interior angle of a regular heptagon is given, in degrees, by $180 - \frac{360}{7} = \frac{900}{7}$. Dividing 360 by this value gives 2.8. Since this is not an integer, regular heptagons do not tessellate.

901. A student tries to find the intersection of the graphs $x^2 + y^2 = 1$ and $5x + 6y = 8$, and receives an error message from his calculator. Show that this is correct, and explain the result.

Substitute to find a quadratic and consider the discriminant.

Substituting $y = \frac{1}{6}(8-5x)$ gives

$$\begin{aligned} & x^2 + \frac{1}{36}(8-5x)^2 = 1 \\ \Longrightarrow\ & 61x^2 - 80x + 28 = 0. \end{aligned}$$

This has $\Delta = 80^2 - 4 \cdot 61 \cdot 28 = -432 < 0$, so there are no intersections.

902. Determine the value of $\ln \sqrt{e} + \ln \sqrt[3]{e} + \ln \sqrt[6]{e}$.

The natural logarithm is defined as $\ln x := \log_e x$. Translated, this is "whatever you need to raise e by to get x".

Since $\ln x$ is $\log_e x$, this simplifies, using log rules, as

$$\begin{aligned} & \ln \sqrt{e} + \ln \sqrt[3]{e} + \ln \sqrt[6]{e} \\ =\ & \tfrac{1}{2}\ln e + \tfrac{1}{3}\ln e + \tfrac{1}{6}\ln e \\ =\ & \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{6} \\ =\ & 1. \end{aligned}$$

903. Show that the best linear approximation to the function $f(x) = 3x^3 - 5x$, for x values close to 1, is given by $f(x) \approx 4x - 6$.

Finding "the best linear approximation" is equivalent to finding a tangent line.

The best linear approximation to $f(x)$ at $x = 1$ is tangent to $y = f(x)$ at $x = 1$. So, we differentiate to get $f'(x) = 9x^2 - 5$, whence $f'(1) = 4$. Substituting the point $(1, -2)$ gives the tangent line as $y = 4x - 6$. Hence, close to $x = 1$, $f(x) \approx 4x - 6$.

904. State, with a reason, whether variables related in the following ways are likely to display correlation if a sample of values is taken. In each case, $\bar{x} = 0$.

(a) $x + y$ is approximately constant,
(b) $x^2 + y$ is approximately constant,
(c) $x^3 + y$ is approximately constant.

In e.g. (a), sketch $x + y = k$.

(a) Since $x + y \approx k$, x and y are related linearly. Hence, a sample is likely to show strong correlation.

(b) Since $x + y^2 \approx k$, the relationship is parabolic. And, since $\bar{x} = 0$, sampled values will be approximately symmetrical around $x = 0$. Since correlation is closeness to a linear relationship, samples are unlikely to show correlation.

(c) Such a cubic relationship is likely to show some correlation, depending on the spread of the sample, but it may well be weak. The behaviour of x^3, as an odd power, resembles that of x^1 more than that of x^2 does.

905. A quadratic sequence Q_n begins $1, 8, 21, \ldots$. By considering second differences, or otherwise, find an n^{th} term formula.

For a quadratic sequence $u_n = an^2 + bn + c$, the second difference is $2a$.

For a quadratic sequence $u_n = an^2 + bn + c$, the second difference is $2a$. In this case, the second difference is $(21 - 8) - (8 - 1) = 6$, so $a = 3$. Substituting $n = 1, 2$ then gives $1 = 3 + b + c$ and $8 = 12 + 2b + c$. Subtracting, we get $b = -2$ and then $c = 0$. So the ordinal n^{th} term formula is $u_n = 3n^2 - 2n$.

906. Prove that the gradients of perpendicular lines are negative reciprocals.

Draw gradient triangles.

A rotation by 90°, i.e. perpendicular, is equivalent to a reflection in $x = 0$ followed by a reflection in $y = x$. The former negates gradients; the latter switches x and y, which reciprocates gradients. In combination, this gives a negative reciprocal.

907. A parabola is given by the equation $x = 2y^2 + 1$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{1}{4y}$.

(b) Hence, show that the tangent at $y = \frac{1}{8}$ has equation $32x = 16y + 31$.

Differentiate with respect to y, then reciprocate.

(a) Differentiating with respect to y, we get $\frac{dx}{dy} = 4y$. Reciprocating gives $\frac{dy}{dx} = \frac{1}{4y}$.

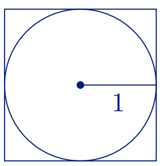
(b) At $y = \frac{1}{8}$, the gradient is 2, and $x = \frac{33}{32}$. So the tangent has equation

$$\begin{aligned} 2 &= \frac{y - \frac{1}{8}}{x - \frac{33}{32}} \\ \implies 2 &= \frac{32y - 4}{32x - 33} \\ \implies 32x - 33 &= 16y - 2 \\ \implies 32x &= 16y + 31. \end{aligned}$$

908. A square has a circle inscribed. A point is then chosen at random in the square. Show that the probability that the point is inside the circle is $\frac{\pi}{4}$.

In this problem, the square is the possibility space, and area corresponds to probability.

In this problem, the square is the possibility space, and area corresponds to probability. So, we need only calculate the relevant areas. Giving the radius of the circle length 1:

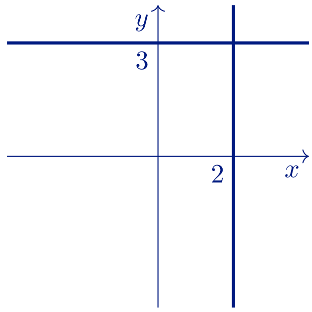


The areas of square and circle are 4 and π. So the probability that a randomly chosen point is inside the circle is $\frac{\pi}{4}$.

909. On a set of Cartesian axes, sketch the locus of points which satisfy $(x - 2)(y - 3) = 0$.

If two factors multiply to give zero, then...

The two brackets multiply to give zero, so at least one of them must be zero. This gives the locus as a pair of intersecting lines, $x = 2$ and $y = 3$:



910. Kepler's third law of planetary motion states that the orbital period T and radius r of a planetary orbit are related by $T^2 \propto r^3$.

(a) Assuming that the Earth orbits the sun once every 365 days at a distance of 152 million km, find r in terms of T, using units of millions of km and years.

(b) Determine the distance from the Sun of an asteroid whose orbital period is 2 years.

In (a), set up $r = kT^{\frac{2}{3}}$, substitute to find k, and then rearrange.

(a) Kepler's third law, in the form we require, is $r = kT^{\frac{2}{3}}$. Substituting $T = 1$ and $r = 152$, we get $k = 152$. This gives

$$r \approx 152T^{\frac{2}{3}}.$$

(b) Substituting $T = 2$ gives an orbital radius of 241 million km (3sf).

911. If $\log_9 y = x$, write 3^x in terms of y.

Rewrite as an index equation.

Rewriting the logarithmic equation as an index equation, we have $y = 9^x$. Taking the square root gives $3^x = \sqrt{y}$.

912. Prove that the area of a kite with diagonals of length a and b is given by $\frac{1}{2}ab$.

Consider the kite as four triangles.

A kite may be considered as four right-angled triangles, with perpendicular sides of length $\frac{1}{2}a$ and $\frac{1}{2}b$. So the area of the kite is given by

$$A = 4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}a \cdot \tfrac{1}{2}b = \tfrac{1}{2}ab.$$

913. Collectively, the graphs $x + y = 4$, $x + y = -4$ and $xy = -5$ meet at a total of four points. Show that these points lie at the vertices of a rectangle.

You do not need to find the coordinates of the points explicitly to prove the required result. You can use symmetry.

This set of three graphs has both $y = x$ and $y = -x$ as lines of symmetry. Hence, the four intersection points must have those lines of symmetry, which means they must form a rectangle.

914. A mathematician sets up a function to answer the question "How many real roots does the cubic equation $ax^3 + bx^2 + cx + d = 0$ have?" Write down

(a) the largest possible domain,

(b) a suitable codomain,

(c) the range, with the domain given.

The function sends cubic graphs to natural numbers.

(a) The largest possible domain is all cubic equations with real coefficients, or equivalently all quadruples $(a, b, c, d) \in \mathbb{R}^4$.

(b) Suitable codomains are $\mathbb{N}$, $\mathbb{Z}$ or $\{0, 1, 2, 3\}$.

(c) The range is $\{0, 1, 2, 3\}$.

915. Show that, if a square, a regular pentagon and a regular hexagon share vertex V, then at least 42° of angle around V is exterior to all three shapes.

The "at least" here refers to the fact that the polygons might overlap. To find the boundary situation, assume they do not.

The minimal case is when the polygons do not overlap. In this case, the angle exterior to all three shapes is given, in degrees, by

$$360 - (180 - \tfrac{360}{4}) - (180 - \tfrac{360}{5}) - (180 - \tfrac{360}{6}).$$

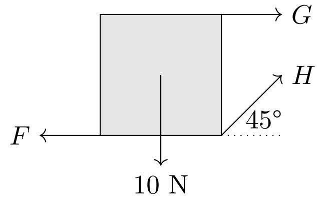
This comes to 42°. Hence, at least 42° must be exterior to all three shapes.

916. Find the double root of $(x^3 + a^3)(x^2 - a^2) = 0$.

Use the factor theorem.

The roots of the quadratic factor are $x = \pm a$. So, we need only test whether either of these are roots of the cubic factor: $x = a$ is not, but $x = -a$ is. So $x = -a$ is double root of the equation.

917. In the following force diagram of a square object in equilibrium, find the force magnitudes F, G, H.



Find three equations: vertical equilibrium, horizontal equilibrium, and moments around the bottom right corner.

Labelling the bottom-right corner A and calling the side length of the square 1, we have

$$\begin{aligned} \updownarrow &: \frac{\sqrt{2}}{2}H - 10 = 0 \\ \leftrightarrow &: \frac{\sqrt{2}}{2}H + G - F = 0 \\ \overset{\frown}{A} &: G - \tfrac{1}{2} \cdot 10 = 0. \end{aligned}$$

Hence, $G = 5$, $H = 10\sqrt{2}$, and lastly $F = 15$.

918. By first squaring the equations, find all possible values of R satisfying $R \sin\theta = 6$ and $R\cos\theta = 8$.

Use the Pythagorean identity.

After squaring the equations, we add, which gives $R^2(\sin^2\theta + \cos^2\theta) = 100$. By the Pythagorean identity, the trigonometric factor is 1, so $R = \pm 10$.

919. Show that the tangent to the curve $y = x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}$ at $x = 1$ has equation $3x + 2y = 13$.

...

Differentiating gives $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}}$. Evaluating at $x = 1$ gives a gradient of $-\frac{3}{2}$. So the tangent has equation $3x + 2y = k$. Substituting $(1, 5)$ gives $k = 13$, which is the required result.

920. The numbers 1 to 4 are placed at random in a two-by-two grid. Find the probability that the numbers are in ascending order when read clockwise around the grid.

It makes no difference where the 1 goes. So, place it, and then consider the possible arrangements of $2, 3, 4$.

Since the square is symmetrical, we can place the 1 without loss of generality. Then, there are 6 possible arrangements of the remaining three integers, only one of which yields ascending order when read clockwise. So, $p = \frac{1}{6}$.

921. A particle has initial velocity $5\mathbf{i} + 3\mathbf{j}$ ms^{-1}, and constant acceleration $2\mathbf{i} - 4\mathbf{j}$ ms^{-2}. Determine the average velocity over the first five seconds.

Use a vector suvat.

Using a vector suvat equation, and converting into column vectors for ease, we have

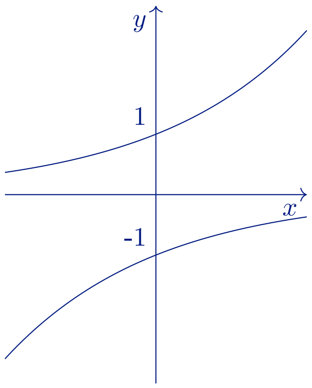
$$\mathbf{s} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \times 5 + \tfrac{1}{2}\begin{pmatrix} 2 \\ -4 \end{pmatrix} \times 5^2 = \begin{pmatrix} 50 \\ -35 \end{pmatrix}.$$

Dividing by the duration 5, the average velocity is $10\mathbf{i} - 7\mathbf{j}$ ms^{-1}.

922. Sketch the graphs $y = e^x$ and $y = -e^{-x}$.

Consider the second graph as the first, having undergone two reflections.

The first is a standard exponential. The second graph has undergone two reflections, in x and y, which amounts to rotation 180° around the origin:



923. Show that the lines $x + y = 1$ and $y - 4x = 7$ intersect inside the circle $x^2 + 2x + y^2 - 3y + 2 = 0$.

Rearrange the circle equation into the standard form, and test the intersection.

Completing the square, the circle is has equation

$$(x+1)^2 + (y - \tfrac{3}{2})^2 = \tfrac{5}{4}.$$

Solving simultaneously, the intersection of the lines is at $(-\frac{6}{5}, \frac{11}{5})$. To test this, we evaluate the LHS of the circle equation at this point, which gives

$$(x+1)^2 + (y - \tfrac{3}{2})^2 \Big|_{\left(-\frac{6}{5}, \frac{11}{5}\right)} = 0.53 < \tfrac{5}{4}.$$

Hence, the lines intersect inside the circle.

924. In each case, describe the symmetry of the graph $y = f(x)$, when f has the given property.

(a) $f(-x) \equiv f(x)$,

(b) $f(-x) \equiv -f(x)$.

The symmetries are "even" and "odd".

(a) If f satisfies $f(-x) \equiv f(x)$, then the line $x = 0$ is a line of symmetry. This is an even function.

(b) If f satisfies $f(-x) \equiv -f(x)$, then the origin is a point of rotational symmetry, order 2. This is an odd function.

925. A four-sided die and a six-sided die are rolled at the same time. Find the probability that the scores on the two dice are the same.

Visualise the possibility space, and count outcomes.

The possibility space is

	1	2	3	4	5	6
1	✓					
2		✓				
3			✓			
4				✓		

So, the probability is $\frac{4}{24} = \frac{1}{6}$.

926. Expand and simplify $(2^x + 1)^3 + (2^x - 1)^3$.

Use the binomial expansion.

Since the expression is symmetrical, the even-powered terms will cancel. This leaves

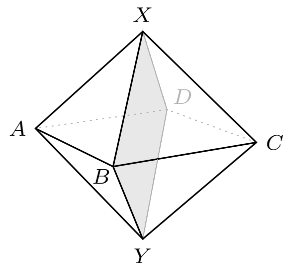
$$\begin{aligned} & (2^x+1)^3 + (2^x-1)^3 \\ \equiv\ & 2 \cdot (2^x)^3 + 6 \cdot 2^x \\ \equiv\ & 2 \cdot 2^{3x} + 6 \cdot 2^x. \end{aligned}$$

927. Prove that the curves $y = f(x)$ and $y = g(x)$, where $f(x)$ is a polynomial of degree m and $g(x)$ is a polynomial of degree n, can have at most $\max(m, n)$ intersections.

Consider the type of polynomial equation solved by the intersections.

If an equation is formed to find intersections, it can be a polynomial of degree at most $\max(m, n)$. And a polynomial of degree k has at most k roots. This gives the required result.

928. A regular octahedron is shown below.



State, with proof and in the most precise terms possible, what type of polygon $BXDY$ is.

Consider the symmetry of the octahedron.

Quadrilateral $BXDY$ is a square. By symmetry, its sides are equal and its angles are equal.

929. Complete the square in $5a^4b^2 + 100a^2b + 1$.

The variable is a^2b.

Completing the square in a^2b, we have

$$\begin{aligned} & 5a^4b^2 + 100a^2b + 1 \\ \equiv\ & 5(a^2b + 10)^2 + 1 - 500 \\ \equiv\ & 5(a^2b + 10)^2 - 499. \end{aligned}$$

930. Two cards are chosen from a standard deck, with replacement. State which, if either, of the following events has the greater probability:

- a jack, then another jack,
- a king, then a queen.

Consider the number of successful outcomes in each event.

The probabilities are the same. Since the order of king-then-queen is fixed, and since there is replacement, each event has $4 \times 4 = 16$ successful outcomes.

931. Prove that the area of an equilateral triangle may be expressed, in terms of its perimeter, as

$$A = \frac{\sqrt{3}P^2}{36}.$$

Calculate both quantities in terms of the side length l, and then substitute for l.

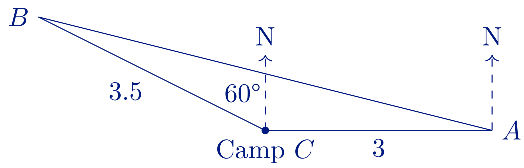
In an equilateral of side length l, the perpendicular height is given by $\frac{\sqrt{3}}{2}l$, so the area is given by $A = \frac{\sqrt{3}}{4}l^2$. The perimeter is given by $P = 3l$, so $l = \frac{1}{3}P$. Substituting gives

$$\begin{aligned} A &= \frac{\sqrt{3}}{4}l^2 \\ &= \frac{\sqrt{3}}{4}\left(\frac{1}{3}P\right)^2 \\ &= \frac{\sqrt{3}P^2}{36}. \end{aligned}$$

932. Two hikers leave camp simultaneously. A walks on bearing 90° at 3 mph; B walks on bearing 300° at 3.5 mph. Determine the bearing of B from A once they have left camp, and show that it is constant.

Consider the situation after 1 hour, and show that it represents the situation at any time.

After one hour, the situation is as follows:



The distance AB is given by

$$\begin{aligned} & AB^2 = 3^2 + 3.5^2 - 2 \cdot 3 \cdot 3.5 \cos 150 \\ \implies & AB = 6.27985... \end{aligned}$$

Then the sine rule tells us that

$$\begin{aligned} CAB &= \arcsin\left(\frac{3.5 \sin 150}{6.27985}\right) \\ &= 16.180...^\circ. \end{aligned}$$

So, the bearing of B from A is $270 + 16.2 = 286.2°$ (1dp). This bearing is constant, as varying the time $t > 0$ will scale the entire triangle ABC, and similar triangles contain the same angles.

933. Express $3t^2 + 4t - 6$ as a quadratic in $(t - 1)$.

Begin with $3(t-1)^2$, then deal with the term in x.

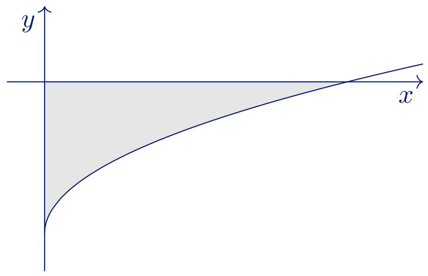
To produce the term in t^2, we need $3(t-1)^2$. This gives a byproduct of $-6t$ (and a constant term). So, we need $10(t - 1)$ to produce the term in t. The constant term is then $+1$, which yields

$$3t^2 + 4t - 6 \equiv 3(t - 1)^2 + 10(t - 1) + 1.$$

934. The curve $y = \sqrt{x} - 2$ crosses the x axis at $x = 4$. Determine the area enclosed by the curve, the x axis and the y axis.

Sketch the curve before setting up the appropriate integral.

Sketching, we have



So the area, with a negative sign to render the negative signed area positive, is given by

$$\begin{aligned} A &= -\int_0^4 \sqrt{x} - 2\,dx \\ &= -\left[\tfrac{2}{3}x^{\frac{3}{2}} - 2x\right]_0^4 \\ &= -\tfrac{2}{3}\cdot 4^{\frac{3}{2}} + 2\cdot 4 \\ &= \tfrac{8}{3}. \end{aligned}$$

935. In this question, f is a polynomial function. Prove that, if $f'(x)$ is given but $f(x)$ is not, then it is possible to calculate $f(p) - f(q)$ for any $p, q \in \mathbb{R}$, but it is not possible to calculate $f(p) + f(q)$.

Consider the integral of $f'(x)$.

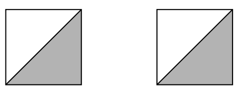
If $f'(x)$ is given, then, by integrating, $f(x)$ is given, up to an additive constant. In the expression $f(p) - f(q)$, the additive constant cancels; in the expression $f(p) + f(q)$, it does not. Hence, it is possible to calculate the former, but not the latter.

936. State, with a reason, whether the following holds: "Friction and reaction, if both act at the same point on the same object, are perpendicular."

This is true.

With the modern usage of the word "reaction", which is not Newton's original usage, this is true by definition. Reaction forces are defined to be contact forces acting perpendicular to surfaces, while frictional forces are defined to be contact forces acting parallel to surfaces.

937. The two tiles depicted below are placed side by side, in a random orientation.



Find the probability that the shading forms one continuous region.

Use $p = \frac{\text{successful}}{\text{total}}$.

There are four possible orientations of each tile. For the shading to form one continuous region, each must have a shaded edge at the centre. For each tile, there are two of four orientations which achieve this. Hence, the probability is $p = \frac{4}{16} = \frac{1}{4}$.

938. In each case, a list of quantities is given, which refers to a finite arithmetic series. State whether knowing the relevant quantities would allow you to evaluate the sum of the series, giving a reason if your answer is "No."

(a) Number of terms; mean.

(b) First term; last term; mean.

(c) Number of terms; first term; last term.

Consider the formula $S_n = \dfrac{1}{2}(a + l)n$.

(a) Yes. The sum is equal to the mean multiplied by the number of terms.

(b) No. The combination of first and last term gives the mean, and this set of information does not give the number of terms.

(c) Yes. The formula is $S_n = \frac{1}{2}(a + l)n$.

939. Find the angle, in radians, between the hands of a clock at twenty to six.

Work in degrees first if necessary.

At 5:40, the minute hand is at $\frac{4}{3}\pi$ radians clockwise from 12. The hour hand is at $\frac{17}{18}\pi$ radians clockwise from 12. So the required angle is $\frac{7}{18}\pi$.

940. A student writes: "Friction always acts to oppose motion. So, when a car is not moving, there can be no friction acting on it." Explain carefully why this statement is incorrect.

Consider the validity of the first sentence.

The first sentence is wrong. It is imprecise to say that friction always acts to oppose motion. Kinetic friction opposes motion, but static friction opposes potential motion, i.e. motion that would take place were the friction not present. When a car is parked on a hill, it is static friction on the tyres that stops it sliding downhill.

941. Solve the equation $2n^2(2n - 1) - (2n - 1)^2 = 0$.

Take out a common factor first.

Taking out the common factor $(2n-1)$, we get

$$\begin{aligned} & 2n^2(2n-1)-(2n-1)^2=0 \\ \Longrightarrow\ & (2n-1)(2n^2-(2n-1))=0 \\ \Longrightarrow\ & (2n-1)(2n^2-2n+1)=0 \end{aligned}$$

The quadratic factor has $\Delta=-4<0$, so the only root is $n=\frac{1}{2}$.

942. Simplify the following, for an invertible function f:

(a) $f^{-1}ff^{-1}(x)$,

(b) $f^{-1}f^2(x)$.

Remember that f^{-1} is the inverse of f, and f^2 is the function applied twice.

In each of these, we can simply add the indices. f^{-1} means "undo f", while f^2 means "apply f twice". In each case, the index means "the number of times f is applied". This gives

(a) $f^{-1}ff^{-1}(x) \equiv f^{-1}(x)$,

(b) $f^{-1}f^2(x) \equiv f(x)$.

943. The definite integral below gives the displacement, over a particular time period, for an object moving with constant acceleration:

$$s=\int_2^6 1+\tfrac{2}{5}t\,dt.$$

(a) Write down the acceleration and duration.

(b) Find the initial and final velocities.

(c) Calculate the displacement by integration.

(d) Verify the answer to part (c) using a constant acceleration formula.

The integrand is an expression for the velocity at time t.

(a) $a=\frac{2}{5}$, $\Delta t=4$.

(b) $u=1+\frac{2}{5}t\,\big|_2=1.8$, $v=1+\frac{2}{5}t\,\big|_6=3.4$.

(c) $s=\left[t+\frac{1}{5}t^2\right]_2^6=10.4$.

(d) Using $s=ut+\frac{1}{2}at^2$, we get

$$s=1.8\cdot 4+\frac{1}{2}\cdot\frac{2}{5}\cdot 4^2=10.4.$$

944. Find the length of the line segment

$$x=3+6\lambda, \quad y=-1+8\lambda, \quad \lambda\in[-1,1].$$

Either consider the vector displacement when $\Delta\lambda=1$, or find the endpoints explicitly.

When the parameter λ changes by 1, the point (x,y) moves by vector $6\mathbf{i}+8\mathbf{j}$. By Pythagoras, this has length 10. So, over the interval $[-1,1]$, the displacement is 20.

945. Two six-sided dice are rolled, and the scores summed. A student suggests that the total could be modelled with $B(12,\frac{1}{6})$. Explain why this is incorrect.

To disprove this, quote the assumptions necessary for the use of a binomial distribution.

There are many reasons why this is incorrect. To prove that it is, we could consider the fact that the score 1 is not attainable in the experiment, but has non-zero probability in the proposed model $B(12,\frac{1}{6})$, or say that the experiment does not consist of independent Y/N trials, which is necessary for the application of a binomial model.

946. Forces $\mathbf{F}=a\mathbf{i}+6b\mathbf{j}$ and $\mathbf{G}=(2b+1)\mathbf{i}+(4+3a)\mathbf{j}$ act on an object. Show that, unless other forces act, the object cannot remain in equilibrium.

For equilibrium, the resultant force in both x and y must be zero. So, set up simultaneous equations.

For equilibrium, the resultant force in both x and y must be zero. So, we set up and simplify simultaneous equations:

$$\begin{aligned} x &: a+2b+1=0 \\ y &: 3a+6b+4=0. \end{aligned}$$

These have no solutions, as $a+2b$ cannot be simultaneously -1 and $-\frac{4}{3}$. Hence, there are no values of a and b for which the object will remain in equilibrium.

947. Show that the line $y=456x-3600$ is a tangent to the curve $y=x^3+x^2$.

Differentiate and find the point(s) at which the gradient of the cubic is 456.

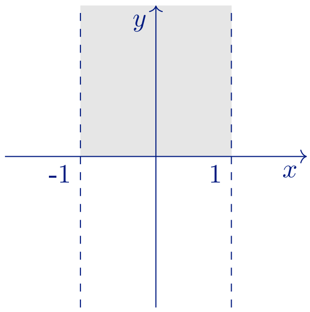
To find the point at which the gradient is 456, we set up the equation $3x^2+2x-456=0$. This has roots $x=12$ and $x=-\frac{38}{3}$. Testing the y values, we see that $(12,1872)$ is the point of tangency.

948. Shade the region of the (x,y) plane which satisfies both of the following inequalities:

$$|x|<2, \qquad y\geq 0.$$

Sketch the boundary equations, with dashed lines for strict inequality, and then shade regions.

The boundary equations are $|x| = 2$, which is the pair of lines $x = \pm 2$, and $y = 0$. Using dashed lines for strict inequality, the region is as follows:



949. State true or false for the following:

(a) "Every square is a rectangle."

(b) "Not all rectangles are parallelograms."

(c) "Some kites are trapezia."

Remember that a square is every type of quadrilateral.

(a) "Every square is a rectangle." True, by definition.

(b) "Not all rectangles are parallelograms." False. Every rectangle is also a parallelogram.

(c) "Some kites are trapezia." True. A square is both a kite and a trapezium.

950. Given $f(x) = x - 1$, solve for a in

$$\int_0^{f(a)} f(x)\,dx = 0.$$

The notation is unfamiliar, but just substitute for f and carry out the integral to set up a standard equation in a.

Substituting the definition of the function f,

$$\begin{aligned} &\int_0^{f(a)} f(x)\,dx = 0 \\ \implies &\int_0^{a-1} x - 1\,dx = 0 \\ \implies &\left[\tfrac{1}{2}x^2 - x\right]_0^{a-1} = 0 \\ \implies &\tfrac{1}{2}(a-1)^2 - (a-1) = 0 \\ \implies &(a-1)(a-3) = 0 \\ \implies &a = 1, 3. \end{aligned}$$

951. Write the following in simplified interval notation:

$$\{x \in \mathbb{R} : -1 < x < 3\} \cap \{x \in \mathbb{R} : |x+2| > 2\}.$$

Write the latter set in interval notation.

The latter set is all x values which differ from -2 by more than 2. This is $(-\infty, -4) \cup (0, \infty)$. The intersection of this set with $(-1, 3)$ is the interval $(0, 3)$.

952. State, with a reason, whether $y = x^3$ intersects the following curves:

(a) $y = x^2 + 1$,

(b) $y = x^3 + 1$,

(c) $y = x^4 + 1$.

If in doubt, sketch the graphs.

(a) Yes. A cubic equation always has a root.

(b) No. The two graphs are vertical translations of one another.

(c) No. For $x < 1$, so intersection is possible as $x^3 < 1$ and $x^4 + 1 > 1$. And for $x \geq 1$, $x^4 \geq x^3$, so $x^4 + 1 > x^3$.

953. Show that the largest angle in a triangle with sides $10, 12, 15$ is approximately 1.5 radians.

Use the cosine rule, working in radians. The largest angle is always opposite the longest side.

The largest angle is opposite the longest side. So, using the cosine rule, we have

$$\begin{aligned} \theta &= \arccos\left(\frac{10^2 + 12^2 - 15^2}{2 \cdot 10 \cdot 12}\right) \\ &= 1.49154... \\ &\approx 1.5 \text{ rad}. \end{aligned}$$

954. The solution of the following equation is the same for all but one value of the constant b.

$$\frac{(x^2+2)(x^2-3)}{x^2+b} = 0.$$

Write down that value.

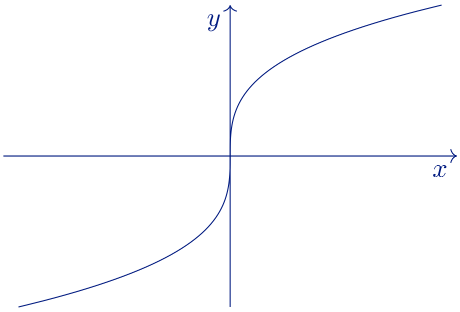
Consider common factors in the numerator and denominator.

The numerator has roots of ± 3, since $(x^2 + 2)$ is never zero. These will be the roots of the equation, unless these are also roots of the denominator. So $b = -3$ is the required value.

955. Sketch $y = \sqrt[3]{x}$.

Consider the graph $y = x^3$, with x and y changing roles.

Cubing both sides, we have $y^3 = x$, which is the graph $y = x^3$ reflected in the line $y = x$:



956. The interior angles of a quadrilateral are in AP. Give, in radians, the set of possible values for the smallest angle.

Consider the average value of the angles.

The interior angles must sum to 2π, so their mean is $\frac{\pi}{2}$. Hence, the set of values for the smallest is $(0, \frac{\pi}{2})$. (Or $(0, \frac{\pi}{2}]$, if you consider the wording of the question as permitting the constant AP case, which is a square.)

957. Two fair coins are tossed. Given that at least one head is observed, find the probability that two heads are observed.

Construct a possibility space, and restrict it using the information given.

The possibility space is $\{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$. The information given restricts this, however, ruling out TT and giving $\{\text{HH}, \text{HT}, \text{TH}, \cancel{\text{TT}}\}$. Hence, the probability is $\frac{1}{3}$.

958. Two vectors $\mathbf{a}$ and $\mathbf{b}$ can be expressed in terms of the standard perpendicular unit vectors as

$$\begin{aligned}\mathbf{a} &= 4\mathbf{i} + 3\mathbf{j}, \\ \mathbf{b} &= 2\mathbf{i} - 5\mathbf{j}.\end{aligned}$$

Express $16\mathbf{i} + 25\mathbf{j}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Set up and solve simultaneous equations.

We want $16\mathbf{i} + 25\mathbf{j} = p\mathbf{a} + q\mathbf{b}$, for some $p, q \in \mathbb{R}$. Expanding this and equating coefficients of $\mathbf{i}$ and $\mathbf{j}$ gives

$$\begin{aligned}\mathbf{i} &: 16 = 4p + 2q \\ \mathbf{j} &: 25 = 3p - 5q.\end{aligned}$$

Solving simultaneously gives $p = 5, q = -2$, so $16\mathbf{i} + 25\mathbf{j} = 5\mathbf{a} - 2\mathbf{b}$.

959. Find the equation of the normal to the curve $(x-1)^2 + y^2 = 25$ at the point $(4, 4)$, giving your answer in the form $ax + by + c = 0$, for $a, b, c \in \mathbb{Z}$.

Use circle geometry, rather than calculus.

Since the equation is a circle, centre $(1, 0)$, radius 5, the normal at $(4, 4)$ is a radius through $(4, 4)$ and $(1, 0)$. The equation of the line is $4x - 3y - 4 = 0$.

960. Prove that, if a sequence is quadratic with n^{th} term $u_n = an^2 + bn + c$, then the second difference of the sequence is constant, with value $2a$.

Use two consecutive terms to find teh first difference, $u_{n+1} - u_n$.

The first difference is given by

$$\begin{aligned}& u_{n+1} - u_n \\ =\ & \big(a(n+1)^2 + b(n+1) + c\big) - \big(an^2 + bn + c\big) \\ =\ & 2an + a + b.\end{aligned}$$

Increasing n by one increases this value by $2a$, so the second difference is a constant $2a$.

961. Show that $f(x) = x^2 + 4x + 2$ has two fixed points.

Solve a quadratic equation.

Fixed points satisfy $f(x) = x$. This is

$$\begin{aligned}& x^2 + 4x + 2 = x \\ \implies & x^2 + 3x + 2 = 0 \\ \implies & x = -1, -2.\end{aligned}$$

So, f has two fixed points.

962. The curve $y = f(x)$ is to be differentiated from first principles.

(a) Explain the significance of the numerator and denominator in the expression

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(b) Explain why the same result would be attained by using the expression

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

Sketch gradient triangles.

(a) The numerator and denominator are δy and δx, small but finite lengths in the gradient triangle of a chord. That chord is constructed, from x to $x + h$, to the right of the x value at which the tangent gradient is sought.

(b) This expression sets up a chord from points either side of the x value at which the tangent gradient is sought. These points are $(x-h, f(x-h))$ and $(x+h, f(x+h))$. The infinitesimal limit (gradient of chord $\to$ gradient of tangent) is the same.

963. Show that ${}^nC_1 + {}^nC_2 = \dfrac{n^2+3n}{2}$.

Use the factorial definition ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

Using the factorial definition, we have

$$\begin{aligned}&{}^nC_1 + {}^nC_2\\ \equiv\ &\frac{n!}{1!(n-1)!} + \frac{n!}{2!(n-2)!}\\ \equiv\ &n + \frac{n(n+1)}{2}\\ \equiv\ &\frac{n^2+3n)}{2}.\end{aligned}$$

964. The graph $y = a(x+b)^2(x+c)^3 = 0$ touches the x axis at $x = 1$, crosses the x axis at $x = -4$, and crosses the y axis at $y = 16$. Find a, b, c.

Consider the behaviour of a curve at double and triple roots, i.e. whether it crosses the x axis or not.

The curve has a double factor of $(x+b)$ and a triple factor of $(x+c)$, so it has a double root at $x = -b$ and a triple root at $x = -c$. Since the curve crosses the x axis at $x = -4$, this must be the triple root (odd powers change sign at a root, even powers don't), so $c = 4$. This leaves $b = -1$. Then, substituting the value $x = 0$ yields $64a = 16$, so $a = 4$.

965. For events A and B, write down the values of the following probabilities:

(a) $P(A \mid A)$,

(b) $P(A \cap B' \mid B)$,

(c) $P(A \cap B \mid A' \cup B')$.

$P(A \mid B)$ means "The probability of event A occurring, given the fact that event B is known to have occurred."

(a) $P(A \mid A) = 1$. If A is given, then occurrence of A is certain.

(b) $P(A \cap B' \mid B) = 0$. If B is given, then B' cannot have occurred.

(c) $P(A \cap B \mid A' \cup B') = 0$. These two events $A \cap B$ and $A' \cup B'$ are complementary and thus mutually exclusive. If the latter has occurred, the former cannot have done.

966. Prove that the curve $y = x^6 + x^3 + 1$ does not cross the x axis.

Consider a quadratic equation in x^3.

Setting $x^6 + x^3 + 1 = 0$, we have a quadratic in x^3. Its discriminant is $\Delta = 1 - 4 = -3 < 0$, so no values of x^3 satisfy this equation. Hence, no values of x do, and the curve does not cross the x axis.

967. A function f is defined over the reals, and has range $[-a, a]$. Give the ranges of the following:

(a) $x \mapsto f(x) + a$,

(b) $x \mapsto f(x) - a$,

(c) $x \mapsto a - f(x)$.

Consider transformations of the graph $y = f(x)$.

(a) The range is translated by a, giving $[0, 2a]$.

(b) The range is translated by $-a$, giving $[-2a, 0]$.

(c) This is the negative of the function in (b), so the range is $[0, 2a]$.

968. The inequality $x^2 + px + q > 0$ has solution set $(\infty, 4) \cup (5, \infty)$. Find p and q.

Consider the boundary values $x = 4, 5$.

Since $x = 4$ and $x = 5$ are the boundary values, the inequality must be $(x-4)(x-5) > 0$. Hence $p = -9, q = 20$.

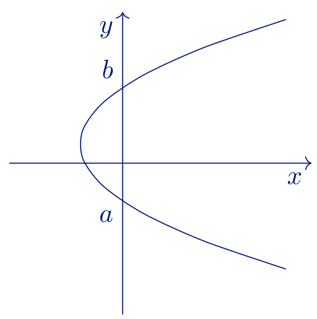
969. Sketch the following graphs, where $a < b$,

(a) $x = (y-a)(y-b)$,

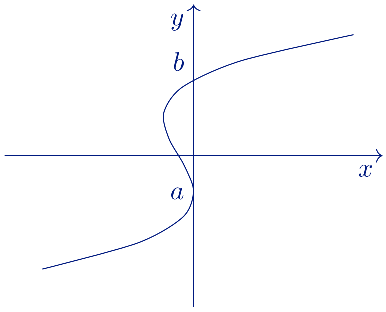
(b) $x = (y-a)^2(y-b)$.

In (b), consider the double root.

(a) $x = (y-a)(y-b)$ is a positive parabola with y intercepts at $y = a$ and $y = b$:



(b) $x = (y-a)^2(y-b)$ is a positive cubic with a double root (just touches) at $y = a$ and a single root (crosses) at $y = b$:



970. Show that $\displaystyle\int_2^4 \frac{3(2x+1)^3 - 3}{x}\,dx = 700$.

Expand the numerator and simplify.

Expanding the numerator gives $24x^3 + 36x^2 + 18x$. Since $x \neq 0$, we can then divide top and bottom by x, which gives

$$\begin{aligned}&\int_2^4 24x^2 + 36x + 18\,dx\\ =\ &\left[8x^3 + 18x^2 + 18x\right]_2^4\\ =\ &(8\cdot 4^3 + 18\cdot 4^2 + 18\cdot 4) - (8\cdot 2^3 + 18\cdot 2^2 + 18\cdot 2)\\ =\ &700.\end{aligned}$$

971. At the point with x coordinate a, the tangent line to $xy = 1$ has equation

$$y = -\frac{1}{a^2}x + c.$$

(a) Explain the presence of "$-\dfrac{1}{a^2}$".

(b) By substituting, show that a general tangent line to the curve $xy = 1$ has equation

$$y = -\frac{1}{a^2}x + \frac{2}{a}.$$

In (a), differentiate. In (b), substitute $(a, \frac{1}{a})$.

(a) Differentiating $y = \frac{1}{x}$, we get $\frac{dy}{dx} = -\frac{1}{x^2}$. Evaluating this at $x = a$ gives $-\frac{1}{a^2}$, which is why this appears as the gradient of the tangent line.

(b) Substituting the point $(a, \frac{1}{a})$ gives

$$\frac{1}{a} = -\frac{1}{a^2}a + c.$$

Simplifying yields $c = \frac{2}{a}$, the required result.

972. Solve the equation $\displaystyle\sum_{i=0}^{2}(1 + 4i - 3i^2)x^i = 0$.

Write the sum out longhand.

Writing the sum out longhand, we have

$$\begin{aligned}&1 + 2x - 3x^2 = 0\\ \Longrightarrow\ &(1-x)(1-3x) = 0\\ \Longrightarrow\ &x = \tfrac{1}{3}, 1.\end{aligned}$$

973. The variables x and y are defined, in terms of the variables a and b, by

$$\begin{aligned}x &= \sqrt{3}a - b,\\ y &= a + \sqrt{3}b.\end{aligned}$$

Express a and b in terms of x and y.

This is equivalent to solving simultaneously for a and b.

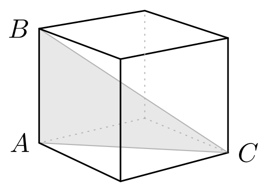
Adding $\sqrt{3}$ times the first equation to the second equation eliminates b, giving $\sqrt{3}x + y = 4a$. Hence, $a = \frac{1}{4}(\sqrt{3}x + y)$. Substituting this back in gives $b = \frac{1}{4}(\sqrt{3}y - x)$.

974. Determine the least n such that the product of n consecutive integers necessarily ends in 0.

"Ending in 0" is equivalent to "having a factor of 10".

To end in zero, a number requires factors of 2 and 5. A product of five consecutive integers necessarily has both. A product of four does not, however: $4! = 24$ is a counterexample. Hence, $n_{\text{min}} = 5$.

975. The diagram shows a cube of unit side length.



Find angle ACB, giving an exact answer.

ABC is a right-angled triangle.

The shaded triangle ABC is right-angled, and has lengths $(1, \sqrt{2}, \sqrt{3})$. So $\angle ACB = \arcsin\frac{1}{\sqrt{3}}$ (or equivalently $\arccos\frac{\sqrt{2}}{\sqrt{3}}$ or $\arctan\frac{1}{\sqrt{2}}$).

976. Find the unknown constant p, if the following may be expressed as a linear function of x:

$$\frac{x^2 + px + 2}{x - 6}.$$

For the fraction to be expressible as a linear function of x, $x-6$ must be a factor of the numerator.

For the fraction to be expressible as a linear function of x, $x-6$ must be a factor of the numerator. So, $x=6$ must be a root, giving $36+6p+2=0$. Hence, $p=-\frac{38}{6}=-\frac{19}{3}$.

977. A child mounts a small fan on the back of a toy sailing boat, and sets it to blow air forwards into the sail. Describe, with reference to Newton's laws, what will happen when the fan is turned on.

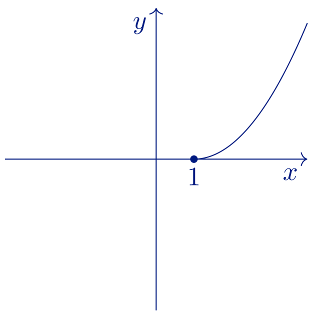
The boat will go backwards. Explain why.

The boat will go backwards. While the blown air will exert a force forwards on the sail, this cannot exceed the backwards force exerted on the fan by the air. Newton's third laws guarantees this. And, in fact, the forwards force will be less than the backwards, because some air is bound to escape around the sail. This will cause the boat to accelerate backwards.

978. Sketch $\sqrt{y}=x-1$.

Square both sides, but remember that this will introduce extra solutions.

We can square both sides to give $y=(x-1)^2$, but this will only be valid for $x-1\geq 0$. So the graph is half of a parabola:



979. Find the probability that, when two dice are rolled, the scores differ by less than two.

Draw the possibility space and count outcomes.

The possibility space is

	1	2	3	4	5	6
1	✓	✓				
2	✓	✓	✓			
3		✓	✓	✓		
4			✓	✓	✓	
5				✓	✓	✓
6					✓	✓

So, $p=\frac{16}{36}=\frac{4}{9}$.

980. Two functions f and g are such that the indefinite integral of their sum is quadratic. Determine the number of roots of the equation $f(x)+g(x)=0$.

Translate the first sentence into algebra, and differentiate it.

We are told that, for $a\neq 0$,

$$\int f(x)+g(x)\,dx=ax^2+bx+c.$$

Differentiating both sides with respect to x gives

$$f(x)+g(x)=2ax+b.$$

Setting to zero we get $2ax+b=0$, which, since $a\neq 0$, will always have exactly one root $x=-\frac{b}{2a}$.

981. Prove or disprove the following statement:

$$P(A \mid B)=P(A) \iff P(A)P(B)=P(A\cap B).$$

This is true. Each side is a definition of independence.

This is true. Each side is an equivalent definition of independence. To prove it, begin with the right-hand equation, and substitute the definition $P(A\cap B)=P(A\mid B)P(B)$:

$$\begin{aligned}
& P(A)P(B)=P(A\cap B)\\
\iff & P(A)P(B)=P(A\mid B)P(B)\\
\iff & P(A)=P(A\mid B).
\end{aligned}$$

This is the required result. (The case $P(B)=0$ is not relevant, as a conditional probability based on B is not defined when $P(B)=0$. A very formal statement of the implication might point this out to begin with.)

982. State, with a reason, whether the following gives a well-defined function:

$$h:\begin{cases}[0,1]\mapsto\mathbb{R}\\ x\mapsto \dfrac{1}{(x-2)(x-3)}.\end{cases}$$

Consider any values where the denominator is zero.

The instruction is well defined when $x\notin\{2,3\}$. But since neither of those values are in the given domain $[0,1]$, this function is well defined as it is.

983. An arithmetic progression has common difference 4, and the product of its first and third term is -7. Find all possible values of the second term.

Express this algebraically, and solve.

Expressing this algebraically, we get

$$\begin{aligned} & a(a+8)=-7 \\ \Longrightarrow\ & a^2+8a+7=0 \\ \Longrightarrow\ & a=-1,-7. \end{aligned}$$

So the second term has value ± 3.

984. Prove that the difference between two consecutive odd squares is divisible by 8.

Simplify $(2n+1)^2-(2n-1)^2$

The sum of two consecutive odd squares may be expressed algebraically as

$$\begin{aligned} & (2n+1)^2-(2n-1)^2 \\ =\ & 4n^2+4n+1-(4n^2-4n+1) \\ =\ & 8n. \end{aligned}$$

Since n is an integer, this is divisible by 8.

985. Find the area of the region of the (x,y) plane whose points simultaneously satisfy the following inequalities:

$$\begin{aligned} & y \le x^2, \\ & y \ge 4x^2-2x. \end{aligned}$$

This is equivalent to saying "Find the area enclosed by the two curves."

The inequalities are simultaneously satisfied in the region enclosed by the two boundary curves. The intersections are at $x^2=4x^2-2x$, which gives $x=0,\frac{2}{3}$. So, we integrate the difference between the curves, between those intersections:

$$\begin{aligned} & \int_0^{\frac{2}{3}} x^2-(4x^2-2x)\,dx \\ =\ & \int_0^{\frac{2}{3}} 2x-3x^2\,dx \\ =\ & \left[x^2-x^3\right]_0^{\frac{2}{3}} \\ =\ & \left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^3 \\ =\ & \frac{4}{27}. \end{aligned}$$

986. On a standard 8×8 chessboard, find an expression for the number of ways of placing

(a) 8 identical pawns,

(b) 8 identical white and 8 identical black pawns.

Use ${}^nC_r$.

(a) There are 64 squares on which to place 8 identical pawns. So, there are ${}^{64}C_8$ ways.

(b) There are ${}^{64}C_8$ ways of placing the white pawns, and then ${}^{56}C_8$ ways of placing the black. This gives ${}^{64}C_8 \times {}^{56}C_8$ ways in total. (Equivalently, ${}^{64}C_{16} \times {}^{16}C_8$.)

987. Show that, if vectors $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}$ are perpendicular, then $a_1b_1+a_2b_2=0$.

Consider the gradients of the vectors.

The gradients of the vectors, which are $\frac{a_2}{a_1}$ and $\frac{b_2}{b_1}$, must multiply to give -1:

$$\begin{aligned} & \frac{a_2}{a_1}\cdot\frac{b_2}{b_1}=-1 \\ \Longrightarrow\ & a_2b_2=-a_1b_1 \\ \Longrightarrow\ & a_1b_1+a_2b_2=0. \end{aligned}$$

988. Describe the transformation which takes the graph $y=f(x)$ onto the graph $y=2f(2x)$.

Consider the input and output transformations separately.

There are really two transformations here. There is an input transformation, a stretch factor $\frac{1}{2}$ in the x direction, and an output transformation, a stretch factor 2 in the y direction.

989. Show that there are no simultaneous solutions to the equations $2x+3y=7$, $4x=1-6y$.

Show that the equations represent distinct parallel lines.

Scaling and rearranging, the equations are

$$\begin{aligned} & 4x+6y=14 \\ & 4x+6y=1. \end{aligned}$$

These are a pair of distinct parallel lines, and do not intersect.

990. In a regular polygon, the *apothem* a is the radius of the largest circle which can be inscribed in the polygon. The *circumradius* R is the radius of the smallest circle within which the polygon can be inscribed. Determine the ratio of the apothem to the circumradius for

(a) an equilateral triangle,

(b) a square,

(c) a regular hexagon.

In each case, the answer corresponds to a standard exact trigonometric value.

(a) The ratio is $\sin 30$, or $1 : 2$.
(b) The ratio is $\sin 45$, or $1 : \sqrt{2}$.
(c) The ratio is $\sin 60$, or $1 : \frac{\sqrt{3}}{2}$.

991. Simplify $(k-4, k+3] \cap (k-1, k+6]$.

Sketch the regions on a number line.

Since $k-1 < k+3$ for all k, the interval simplifies to $(k-1, k+3]$.

992. Opposite faces of a six-sided die add up to seven. Prove that, with this restriction, there are only two different ways of arranging the faces on a die.

Place the 1 and 6, without loss of generality. Then consider placements of the 2 and 5.

The 1 and the 6 may be placed without loss of generality. Likewise, the 2 and the 5 may be placed w.l.o.g., since the four remaining faces are symmetrical. There are then two ways of placing the 3 and the 4, which will yield two arrangements with different handedness.

993. Find the area of a triangle with sides length $4, 5, 6$, giving your answer as a surd.

Use the cosine rule.

Using the cosine rule, the angle between the sides of length 4 and 5 is given by

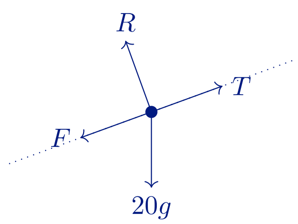
$$\cos\theta = \frac{6^2 - 4^2 - 5^2}{2 \cdot 4 \cdot 5} = -\frac{1}{8}.$$

Substituting into $A_\triangle = \frac{1}{2}ab\sin C$ gives $\frac{1}{2} \cdot 4 \cdot 5 \cdot \sqrt{1 - \frac{1}{64}}$. This simplifies to $A_\triangle = \frac{15\sqrt{7}}{4}$.

994. A packing case of mass 20 kg is being hauled up a rough ramp of inclination 7° with a light rope and a winch. The coefficient of friction is 0.215, and the winch moves the packing case at a constant speed of 0.15 ms^{-1}. Find the tension in the rope.

Draw a force diagram, and use $F_{\max} = \mu R$.

Acceleration is zero, and the forces are as follows (the speed is not relevant):



Resolving perpendicular to the slope, we get $R - 20g\cos 7 = 0$, which gives $R = 20g\cos 7$. Resolving parallel to the slope, using $F_{\max} = \mu R$ since the case is in motion, we get

$$\begin{aligned} & T - 0.215 \cdot 20g\cos 7 - 20g\sin 7 = 0 \\ \implies & T = 65.7 \text{ N (3sf)}. \end{aligned}$$

995. The monic parabola $y = x^2 + 2x + 3$ has a minimum at point (p, q). Find the equation of the monic parabola which has a minimum at point $(p, -q)$.

Find (p, q) first.

Completing the square gives $y = (x+1)^2 + 2$, so (p, q) is $(-1, 2)$. The required monic parabola has a vertex at $(-1, -2)$, so its equation is $y = (x+1)^2 - 2$.

996. Variables x and y take the following values:

x	0	1	2	3
y	0	2	6	10

Show that the relationship cannot be quadratic.

A quadratic relationship has a constant second difference.

A quadratic relationship has a constant second difference. Treating y as a sequence (which is permitted since x increases linearly) the first differences are $2, 4, 4$, which gives second differences of 2 and 0. Since these are not equal, the relationship cannot be quadratic.

997. A set of n letters, all different, has 40320 anagrams. Find n.

Use the fact that n objects have $n!$ arrangements.

Testing values gives $8! = 40320$, so $n = 8$.

998. A student is attempting to find constants A, B which will make the following identity hold:

$$\frac{1}{x^3 - x^2} \equiv \frac{A}{x^2} + \frac{B}{x-1}.$$

(a) Show that this can be written as

$$1 \equiv Bx^2 + Ax - A.$$

(b) Show that there are no constants A, B which make this an identity.

(c) Suggest an alteration to the RHS which would allow an identity.

In (b), compare coefficients. In (c), consider the repeated factor x^2.

(a) Multiplying by $x^3 - x^2 = x^2(x-1)$ gives

$$\begin{aligned} &1 \equiv A(x-1) + Bx^2 \\ \Longrightarrow\ &1 \equiv Bx^2 + Ax - A. \end{aligned}$$

(b) The coefficients of x^2 and x require $A, B = 0$. This renders the entire RHS zero.

(c) Because x^2 is a repeated factor, the correct form for these partial fractions is

$$\frac{1}{x^3 - x^2} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1}.$$

999. Determine which of the following algorithms will attain 1000 in the fewest iterations:

$$\begin{aligned} &a_1 = 1, a_{n+1} = 1.01 \times a_n, \\ &b_1 = 1, b_{n+1} = b_n + 1. \end{aligned}$$

These are a GP and an AP respectively.

Using the general formulae for GPs and APs, the n^{th} terms are

$$\begin{aligned} &a_n = 1.01^{n-1} \\ &b_n = n. \end{aligned}$$

Sequence a_n will reach 1000 when

$$\begin{aligned} &1.01^{n-1} = 1000 \\ \Longrightarrow\ &n - 1 = \log_{1.01} 1000 \\ \Longrightarrow\ &n = 695.22... \end{aligned}$$

Sequence a_n will reach 1000 first, at $n = 696$ as opposed to $n = 1000$.

1000. Solve the equation $2x^{0.4} + x^{0.2} = 1$.

This is a quadratic.

This is a quadratic in $x^{0.2}$. So, we could substitute, or it is simpler to factorise directly, giving

$$\begin{aligned} &2x^{0.4} + x^{0.2} - 1 = 0 \\ \Longrightarrow\ &(2x^{0.2} - 1)(x^{0.2} + 1) = 0 \\ \Longrightarrow\ &x^{0.2} = \tfrac{1}{2}, -1. \end{aligned}$$

Raising both sides to the power 5 gives $x = \frac{1}{32}, -1$.

1001. You are given that $-\frac{2}{3}$ is a root of f, where

$$f(x) = 6x^3 + 37x^2 - 41x - 42.$$

(a) Explain why $(3x+2)$ must be a factor of $f(x)$.

(b) Factorise $f(x)$ fully.

Use the factor theorem.

(a) The factor theorem tells us that $(x + \frac{2}{3})$ must be a factor. Any constant multiple of this must also be factor, so $3(x+\frac{2}{3}) = (3x+2)$ is a factor.

(b) Taking out this factor, we get

$$\begin{aligned} & 6x^3 + 37x^2 - 41x - 42 \\ \equiv\ & (3x+2)(2x^2 + 11x - 21) \\ \equiv\ & (3x+2)(2x-3)(x+7). \end{aligned}$$

1002. A die has been rolled. The score is X. Determine whether the fact "X is prime" increases, decreases or does not affect $P(X \geq 3)$.

List, and then restrict, the possibility space.

Without the information, $P(X \geq 3) = \frac{4}{6}$. With it, the possibility space is restricted to $\{2, 3, 5\}$. Over this space, $P(X \geq 3) = \frac{2}{3}$. So the information given does not affect $P(X \geq 3)$.

1003. Separate the variables in the following differential equation, writing it in the form $f(y)\frac{dy}{dx} = g(x)$ for some functions f and g:

$$x\frac{dy}{dx} + y^2 = 1.$$

Subtract y^2 from both sides, then divide.

$$\begin{aligned} & x\frac{dy}{dx} + y^2 = 1 \\ \implies & x\frac{dy}{dx} = 1 - y^2 \\ \implies & \frac{1}{1-y^2}\frac{dy}{dx} = \frac{1}{x}, \quad \text{for } x, 1 - y^2 \neq 0. \end{aligned}$$

1004. Prove that $\log_a b \times \log_b a = 1$, for all $a, b > 0$.

Express the factors on the LHS as x and y, and rewrite the log statements as index statements.

Defining $x = \log_a b$ and $y = \log_b a$, we can rewrite as $a^x = b$ and $b^y = a$. Substituting the former into the latter gives $(a^x)^y = a$, which, by an index law, is $a^{xy} = a$. Hence, $xy = 1$, the required result.

1005. Determine whether the line $x = t, y = 2 - t$, for $t \in [0, 4]$ intersects the circle $x^2 + y^2 = 10$.

Test the endpoints to see whether they lie inside or outside the circle.

The endpoints of the line are $(0, 2)$ and $(4, -2)$. Evaluating $x^2 + y^2$ at these points gives 4 and 20. Comparing these to 10, one endpoint lies inside the circle, and one lies outside. Hence, the line must intersect the circle.

1006. Write down the angles between the vectors

(a) $\mathbf{i}$ and $\mathbf{j} + \mathbf{k}$,

(b) $\mathbf{j} + \mathbf{k}$ and $\mathbf{k}$.

The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are perpendicular unit vectors. So visualise/draw the vectors as running along the edges of a unit cube.

(a) 90°. Since $\mathbf{i}$ is perpendicular to both $\mathbf{j}$ and $\mathbf{k}$, it is perpendicular to any linear combination of them.

(b) 45°. This is, in fact, a 2D problem: we are looking for the angle between the edge of a square and its diagonal.

1007. Show that, if the equation $y = 10x - 21 - x^2$ holds, then the inequality $y < x$ is always satisfied.

Show that the curve $y = 10x - 21 - x^2$ is always below the line $y = x$.

Solving for intersections between $y = 10x - 21 - x^2$ and $y = x$, we get $-x^2 + 9x - 21 = 0$, which has discriminant $\Delta = 81 - 4 \cdot 21 = -3 < 0$. So, the parabola $y = 10x - 21 - x^2$ does not intersect $y = x$. Therefore, since it is a negative parabola, it must also be below it, thus satisfying $y < x$.

1008. State, with a reason, whether the following claims are true or false:

(a) "If two triangles have the same three side lengths, then they are congruent."

(b) "If two parallelograms have the same four side lengths and the same two diagonal lengths, then they are congruent."

Part (b) uses part (a).

(a) True. It's a standard result: the SSS condition for congruency.

(b) True. Such a parallelogram can be divided up into two triangles along one diagonal, and these triangles then have known SSS.

1009. Prove that, if a quadratic graph $y = f(x)$ has a stationary point on the y axis, then it has no term in x.

Translate this into algebra, and complete the square or differentiate.

A general quadratic graph is $y = ax^2 + bx + c$, for $a \neq 0$. For stationary points, we require $2ax + b = 0$. This gives $x = -\frac{b}{2a}$. If the stationary point is on the y axis, then $x = 0$, so $b = 0$. This is equivalent to saying that the quadratic has no term in x.

1010. Is $\dfrac{dy}{dx} \cdot \dfrac{dx}{dy} \equiv 1$ true? Explain why.

Consider reflections on a graph.

This can be seen as a direct algebraic result of the chain rule, or, equivalently, a graphical argument can be used. Reflection in the line $y = x$ transforms $\frac{dy}{dx}$ to $\frac{dy}{dx}$ by switching x and y, and such a reflection reciprocates gradient triangles.

1011. Solve the to find all (x, y) points which satisfy both $x^2 + y = 7$ and $(x + y)^2 = 25$.

Solve simultaneously, to find four such points.

Taking the square root of the second equation, we get $y = \pm 5 - x$. Substituting this into the first equation gives $x^2 \pm 5 - x = 7$. These are two quadratics, $x^2 - x - 12 = 0 \implies x = 4, -3$ and $x^2 - x - 2 = 0 \implies x = 2, -1$. This gives four distinct points of intersection:

$$(4, -9), \quad (-3, -2), \quad (2, 3), \quad (-1, 6).$$

1012. Sketch the following graphs:

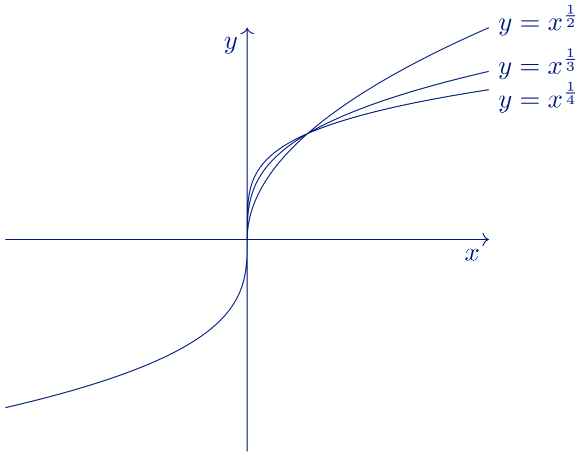
(a) $y = x^{\frac{1}{2}}$,

(b) $y = x^{\frac{1}{3}}$,

(c) $y = x^{\frac{1}{4}}$.

The graphs in (a) and (c) are similar to each other, but (b) is different. Negative numbers have cube roots.

The graph $y = x^{\frac{1}{2}}$ is a half-parabola, and $y = x^{\frac{1}{4}}$ is similar, though more snub-nosed. The shape of $y = x^{\frac{1}{3}}$ is between the two, but it is defined for negative x as well:



1013. Describe fully the rotation which transforms the parabola $y = 2x^2 + x$ onto the parabola $y = -2x^2 + 3x + 5$.

Complete the square on both.

Completing the square gives $y = 2(x - \frac{1}{4})^2 - \frac{1}{8}$ and $y = -2(x - \frac{3}{4})^2 - \frac{9}{8}$. The vertices (minimum and maximum respectively) are at $(\frac{1}{4}, -\frac{1}{8})$ and $(\frac{3}{4}, -\frac{9}{8})$. The midpoint of these is $(\frac{1}{2}, -\frac{5}{8})$. The rotation, then, is by 180° around this point.

1014. Find simplified expressions for the sets

(a) $\{x \in \mathbb{R} : |x| < 2\} \cap [1, 3]$,

(b) $\{x \in \mathbb{R} : |x| > 2\} \cap [1, 3]$,

(c) $\{x \in \mathbb{R} : |x - 1| \leq 1\} \cap [1, 3]$.

In each case, pay careful attention to the inclusion/exclusion of the endpoints.

(a) $[1, 2)$,

(b) $(2, \infty)$,

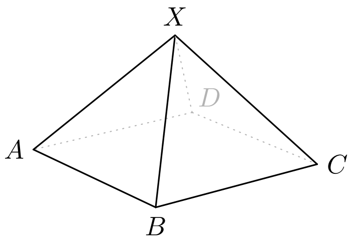
(c) $[1, 2]$.

1015. Explain the meanings of "census" and "sample".

The census is the biggest possible sample. How big is it?

A sample is a subset of the population, which is smaller than the population and is intended to be representative of it. A census is the same as a sample, except that a census consists of the entire population, not just a representative subset of it.

1016. The square-based pyramid shown below is formed of eight edges of unit length.



Determine angle BXD.

Triangle BXD is a right-angled isosceles triangle.

Triangle BXD is a right-angled isosceles triangle, so angle BXD is $45°$.

1017. Find the sum of the first 100 multiples of 3.

Consider these as an AP.

Multiples of 3 can be considered as an AP. Using the standard formula for the partial sum of an AP,

$$S = \frac{100}{2}(2 \cdot 3 + 99 \cdot 3) = 15150.$$

1018. True or false?

(a) $x \in A' \iff x \notin A$.

(b) $x \in A \iff x \notin A'$.

A' is the complement of A, i.e. the negation, in set terms, of A: everything that is not in A.

Both true, by definition.

1019. Simplify $\lim\limits_{p \to q} \dfrac{p^2 - q^2}{p - q}$.

Express the fraction in its lowest terms, then take the limit.

The numerator is a difference of two squares. So, we divide top and bottom by $p - q \neq 0$, giving

$$\lim_{p \to q}(p + q).$$

Taking the limit, this is $2q$.

1020. The graph $y = ax^2(x - b)$ has a local maximum at $(2, 4)$. Find the values of the constants a, b.

Expand and differentiate, or use the product rule.

Using the product rule, and setting to zero to find stationary points, we get

$$\begin{aligned} \frac{dy}{dx} &= 2ax(x - b) + ax^2 = 0 \\ \implies& x(2x - 2b + x) = 0 \\ \implies& x = 0, \frac{2b}{3}. \end{aligned}$$

Since the graph has a maximum at $x = 2$, we know that $\frac{2b}{3} = 2$. Hence, $b = 3$. Substituting $(2, 4)$ gives $a = -1$.

1021. Give the technical meaning of the following nouns used in mechanical modelling:

(a) "particle",

(b) "rod",

(c) "projectile".

Use the word "negligible".

(a) A particle is an object of negligible size, which can therefore be considered as existing at a zero-dimensional point.

(b) A rod is a rigid object of negligible size in two dimensions, which can therefore be considered as a one-dimensional line segment.

(c) A projectile is a particle on which the only force acting is weight. Equivalently, a particle whose acceleration is g ms^{-2} vertically downwards.

1022. "The curves $x^2+y^2 = 2$ and $(x+2)^2+(y+2)^2 = 2$ are tangent to one another." True or false?

Sketch and use circle geometry.

This is true. The curves are circles of radius $\sqrt{2}$, and, by Pythagoras, their centres are $2\sqrt{2}$ away from each other. So, they are tangent at $(-1, -1)$.

1023. By factorising, find all $x \in [0, 360°)$ such that

$$\cos^3 x = \cos x.$$

Rearrange to $f(x) = 0$ first.

Rearranging and factorising gives

$$\begin{aligned} & \cos x(\cos^2 x - 1) = 0 \\ \implies & \cos x(\cos x + 1)(\cos x - 1) = 0 \\ \implies & \cos x = -1, 0, 1. \end{aligned}$$

The solution set is $\{0, 90, 180, 270°\}$.

1024. The equations of two parabolae are given as $y = x^2$ and $y = -x^2 + 8x - 15$.

(a) Show that $y = \frac{3}{2} - \frac{1}{2}x$ is normal to both.

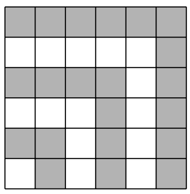
(b) Hence, determine the distance between the parabolae.

The distance between two smooth curves is distance along a path which is normal to both.

(a) Solving simultaneously for intersections gives $x = -\frac{3}{2}, 1$ for the first curve, and $x = 3, \frac{11}{2}$ for the second. The line has gradient $-\frac{1}{2}$, so we are looking for a tangent gradient of 2. This occurs at $x = 1$ for the first graph, and $x = 3$ for the second.

(b) The (shortest) distance between the curves runs along this normal line. So, we need only find the distance between the relevant points $(1, 1)$ and $(3, 0)$. By Pythagoras, the distance is $\sqrt{5}$.

1025. A *proof without words* is a proof which is self-evident visually. The following is one such:



Use the above to simplify $\displaystyle\sum_{r=1}^{n}(2r+1)$.

The terms of the sum are the odd numbers, which are the shaded/unshaded chevrons.

The terms of the sum are the odd numbers, which are the shaded/unshaded chevrons. So, the sum is the number of squares in the grid, which is n^2.

1026. Explain why any contact forces exerted on a smooth sphere must have lines of action that pass through the sphere's centre.

Remember that contact forces which are not friction are reaction forces.

The sphere is smooth, so a contact force cannot be frictional. Hence, it must be a reaction force, acting perpendicular to the surface. The only lines perpendicular to the surface of the sphere are (extended) radii, which must pass through the centre.

1027. Prove, from first principles, that, if the derivative of f is f', then, for any constant k,

$$\frac{d}{dx}(kf(x)) = kf'(x).$$

Set up $\displaystyle\lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h}$.

The factor k is constant, so it can be taken out of the limit. The limit is then equal to $f'(x)$, by definition:

$$\begin{aligned}
&\frac{d}{dx}(kf(x)) \\
&= \lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h} \\
&= k\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= kf'(x).
\end{aligned}$$

1028. Find the intersection of the diagonals of the convex quadrilateral with vertices at

$$(-4, 8), \quad (4, 9), \quad (12, 0), \quad (0, -1).$$

"Convex" means that the edges of the quadrilateral do not cross. So, find the equations of the relevant lines, and solve simultaneously.

The relevant diagonals are between $(-4, 8)$ and $(12, 0)$, and between $(4, 9)$ and $(0, -1)$. These have equations $x + 2y = 12$ and $5x - 2y = 2$. Solving simultaneously for their intersection gives $(\frac{7}{3}, \frac{29}{6})$.

1029. State, with a reason, whether these hold:

(a) $a^2 \in \{-1, 1\} \iff a \in \{-1, 1\}$,
(b) $a^3 \in \{-1, 1\} \iff a \in \{-1, 1\}$,
(c) $a^4 \in \{-1, 1\} \iff a \in \{-1, 1\}$.

Remember that curly brackets signify a list of elements, not an interval.

(a) This does not hold, as $a = -1$ is a counter-example to the leftwards implication.

(b) This holds, since $(\pm 1)^3 = \pm 1$.

(c) This does not hold, as $a = -1$ is a counter-example to the leftwards implication.

1030. Show that the y intercept of the line through the points (a, b) and $(2a, b + c)$ is independent of a.

Consider the x values $0, a, 2a$.

The gradient is $\frac{c}{a}$. Moving back from $x = a$ to $x = 0$ is a change of $-a$, which gives a y change of $-c$. Hence, the y intercept is $b - c$, which is independent of a.

1031. Events X and Y have probabilities $P(X) = \frac{2}{3}$ and $P(Y) = \frac{1}{4}$. Find all possible values of $P(X' \cap Y')$.

On a Venn diagram, consider the extreme values: minimal overlap and maximal overlap between X and Y.

The maximum possible value of $P(X' \cap Y')$ is $\frac{1}{3}$, which is attained if $Y \subset X$. The minimum value is $1 - \frac{2}{3} - \frac{1}{4} = \frac{1}{12}$, which is attained when X and Y are mutually exclusive. So, the set of possible values is $P(X' \cap Y') \in [\frac{1}{12}, \frac{1}{3}]$.

1032. Functions f and g are such that $x = a$ is a root of $f(x) = 0$, $x = b$ is a root of $g(x) = 0$, and $x = c$ is a root of $f(x) = g(x)$. State, with a reason, whether the following hold:

(a) If $a = b$, then $f(a) = g(a)$.

(b) If $a = b$, then $a = c$.

The statement "$x = a$ is a root of $f(x) = 0$" simply means that $f(a) = 0$.

(a) This is true. If $a = b$, then $x = a$ is also a root of $g(x) = 0$, so $f(a) = g(a) = 0$.

(b) This is not true. If $a = b$, then a is a root of $f(x) = g(x)$, but c could be another root of the same equation, not equal to a.

1033. A cuboid has dimensions in the ratio $1 : 2 : 5$. Its total surface area is 612 cm^2. Find its volume.

Call the lengths $x, 2x, 5x$ and set up an equation.

The edge lengths are $x, 2x, 5x$. The surface area is then $2x^2 + 5x^2 + 10x^2 = 17x^2 = 612$. Taking the positive square root gives $x = 6$. The volume is then $10x^3 = 2160$ cm^3.

1034. Solve for a in the following equation:

$$\left[x^2 - 2x\right]_0^a = \left[x^2 + 2x\right]_0^2.$$

This is a quadratic in a.

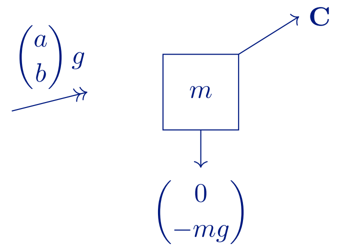
Expanding the evaluations, we have

$$\begin{aligned} & (a^2 - 2a) - (0) = (4 + 4) - (0) \\ \implies & a^2 - 2a - 8 = 0 \\ \implies & a = 4, -2. \end{aligned}$$

1035. At takeoff, an aeroplane accelerates at $(a\mathbf{i} + b\mathbf{j})g$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in horizontal and vertical directions, and $a, b \in \mathbb{R}$. Find the contact force on a passenger of mass m.

Draw a force diagram for the passenger, writing contact force as a single vector $\mathbf{C}$.

Working in column vectors, we can model the passenger as follows:



$F = ma$ is

$$\mathbf{C} + \begin{pmatrix} 0 \\ -mg \end{pmatrix} = m \begin{pmatrix} a \\ b \end{pmatrix} g.$$

This gives the contact force as

$$\mathbf{C} = \begin{pmatrix} a \\ b + 1 \end{pmatrix} mg.$$

1036. Find the mean of the interior angles of a 16-gon, giving your answer in radians.

Find the sum of the interior angles first.

The sum of the interior angles of a 16-gon is $(16 - 2)\pi = 14\pi$. Hence, the mean is $\frac{7\pi}{8}$.

1037. Explain, with reference to Newton's laws, which of the following objects would be worth grabbing if you are about to be blown (slowly and wearing a spacesuit) out of the airlock of a spacecraft:

(a) a vacuum cleaner,

(b) a machine-gun,

(c) a fan,

(d) a fire-extinguisher.

Consider Newton III.

The vacuum cleaner and the fan would be useless, as they both work by propelling air. The machine-gun and the fire-extinguisher would be useful, as they both propel their contents (bullets or foam), and, by NIII, such propulsion involves a force exerted on the propelling entity. This force could be used, according to Newton II, to navigate back to the spacecraft.

1038. Determine the value of $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$.

This is a geometric series.

This is an infinite geometric series, with first term $a = 1$ and common ratio $r = -\frac{1}{2}$. Hence, quoting the standard formula, which holds since $|r| < 1$, we have

$$S_\infty = \frac{a}{1 - r} = \frac{1}{1 - (-\frac{1}{2})} = \frac{2}{3}.$$

1039. Show that, for constants $a, b > 0$, the curve $y = a^x$ is a stretch of $y = b^x$, and give the scale factor and direction of the enlargement.

Write b as $a^{\log_a b}$.

Writing a as $b^{\log_b a}$, we have a transformation $b^x \to b^{x \log_b a}$. This is a replacement of x by $x \log_b a$, which is a stretch in the x direction. The scale factor is $\frac{1}{\log_b a}$, which is equal to $\log_a b$.

1040. A set of socks has seven pairs: two are labelled Monday, two Tuesday, etc. The socks are mixed up individually in a drawer, and I pick out two socks at random.

(a) Find the probability that they match.

(b) Given that they match, find the probability that they are the correct pair for that day.

In (b), consider the possibility space as the seven pairs.

(a) After the first is picked, the probability that the second matches is $P(\text{pair}) = \frac{1}{13}$.

(b) Given that they match, the individual socks are no longer relevant. There are seven days, and seven pairs, so the probability is simply $P(\text{correct} \mid \text{pair}) = \frac{1}{7}$.

1041. Using the binomial expansion, evaluate

$$\lim_{a \to 0} \frac{(1+a)^3 - 1}{(1+a)^4 - 1}.$$

After using the binomial expansion, cancel a factor of a before taking the limit.

Expanding and simplifying, we get

$$\begin{aligned}
&\lim_{a \to 0} \frac{(1 + 3a + 3a^2 + a^3) - 1}{(1 + 4a + 6a^2 + 4a^3 + a^4) - 1} \\
= &\lim_{a \to 0} \frac{3a + 3a^2 + a^3}{4a + 6a^2 + 4a^3 + a^4} \\
= &\lim_{a \to 0} \frac{3 + 3a + a^2}{4 + 6a + 4a^2 + a^3} \\
= &\ \tfrac{3}{4}.
\end{aligned}$$

1042. Two lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ are drawn, where $0 < m_1 < m_2$. Find a formula for the acute angle between the two lines.

Find an expression for the angle between each line and the x axis.

The two lines are at angles $\arctan m_1$ and $\arctan m_2$ above the positive x axis. The angle between the lines, therefore, is the difference between these: $\theta = \arctan m_2 - \arctan m_1$.

1043. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $\tan\theta = 1 \implies \theta = \frac{\pi}{4}$ rad,

(b) $\tan\theta = 1 \impliedby \theta = \frac{\pi}{4}$ rad.

Solve $\tan\theta = 1$.

The equation $\tan\theta = 1$ has another root $\theta = \frac{5\pi}{4}$. Hence, the first statement is false, disproved by the counterexample $\theta = \frac{5\pi}{4}$.

1044. A set of bivariate data (x, y) is being analysed. The mean $(\bar{x}, \bar{y})$ has been calculated. The correlation coefficient r is then found, and a line of best fit, which passes through $(\bar{x}, \bar{y})$, is drawn on a scatter diagram. Subsequently, it is discovered that $(\bar{x}, \bar{y})$ was itself mistakenly entered as a data point for calculation of r and the line of best fit. State, with a reason, the effect of removing it on

(a) r,

(b) the line of best fit.

The correlation coefficient r measures the extent to which bivariate data have a linear relationship.

(a) Since this extra data point lies exactly on the line of best fit, it will have contributed to an overestimation of the extent to which the data is linear. Unless correlation is perfect, removing it will reduce $|r|$ slightly.

(b) Removal will have no effect on the line of best fit, as the line of best fit always passes through $(\bar{x}, \bar{y})$.

1045. Prove that, according to the projectile model, the trajectory of a particle is symmetrical about its highest point.

Consider the equation of the trajectory.

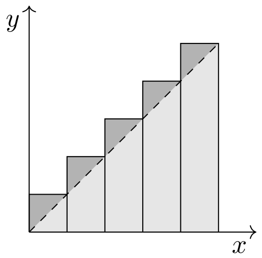
In general, $x = at$ and $y = bt - \frac{1}{2}gt^2$. Since x and t are related linearly, substituting gives $y = q(x)$, where q is a negative quadratic function. A negative parabola is symmetrical about its maximum, as may be shown explicitly by completing the square.

1046. Two functions f and g are such that $f'(x) = g''(x)$. Either prove or disprove the following statement: "If $y = g(x)$ is stationary at $x = \alpha$, then $f(x) = 0$ has a root at $x = \alpha$."

The statement is false. Integrate both side of $f'(x) = g''(x)$ to see why.

The statement is false. Integrating, we can see that $f(x) = g'(x) + c$, for some constant c. If $y = g(x)$ is stationary at $x = \alpha$, then $g'(\alpha) = 0$. However, if $c \neq 0$, then $f(\alpha) \neq 0$, which means $f(x) = 0$ doesn't have a root at $x = \alpha$.

1047. The sum of the first n integers is represented graphically below, as an area.



By formulating expressions for the areas of the lighter and darker shaded regions, prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$

The height of the right-hand rectangle is n.

The height of the right-hand rectangle is n, so the lighter, large triangle has area $\frac{1}{2}n^2$. The n darker, small triangles each have area $\frac{1}{2}$. Hence, the total area is $\frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n+1)$ as required.

1048. By quoting a standard derivative, show that

$$\int_0^{\frac{\pi}{4}} \sec^2 x \, dx = 1.$$

The relevant derivative is $\frac{d}{dx}(\tan x) = \sec^2 x$.

Since $\frac{d}{dx}(\tan x) = \sec^2 x$, we have

$$\begin{aligned} &\int_0^{\frac{\pi}{4}} \sec^2 x \, dx \\ = &\left[\tan x\right]_0^{\frac{\pi}{4}} \\ = &\ 1 - 0 \\ = &\ 1. \end{aligned}$$

1049. A six-sided die and a twelve-sided die are rolled at the same time. Find the probability that the scores on the two dice are the same.

The possibility space has $6 \times 12 = 72$ outcomes.

The possibility space has $6 \times 12 = 72$ outcomes, of which 6 are successful. So the probability is $\frac{1}{12}$.

1050. Solve $\dfrac{1}{1 - \frac{1}{x} + \frac{1}{x^2}} = x$.

Multiply the top and bottom of the large fraction by x^2.

$$\begin{aligned} &\frac{1}{1 - \frac{1}{x} + \frac{1}{x^2}} = x \\ \Longrightarrow\ &\frac{x^2}{x^2 - x + 1} = x \\ \Longrightarrow\ &x^2 = x^3 - x^2 + x \\ \Longrightarrow\ &x^3 - 2x^2 + x = 0 \\ \Longrightarrow\ &x(x-1)^2 = 0 \\ \Longrightarrow\ &x = 0, 1. \end{aligned}$$

But $x = 0$ gives division by zero in the original equation, so the solution is $x = 1$.

1051. The equation $\cos\theta = k$ has exactly one root in $[0, 360°)$. Determine the possible values of k.

Consider a unit circle, or a graph.

Considering intersections of the unit circle and the line $x = k$, the only values of k with exactly one intersection are $k = \pm 1$.

1052. It is given that $x - 2y$ is constant. Find $\dfrac{dy}{dx}$.

Express the first sentence as an equation.

If $x - 2y$ is constant, then $x - 2y = c$, for some constant c. Differentiating both sides of this equation gives $1 - 2\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = \frac{1}{2}$.

1053. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $x \in A$,
- $x \in A \cap B$.

$x \in P \implies x \in Q$ means: if x is an element of set P, then x is an element of set Q

The backwards implication $\Longleftarrow$ links them. If x is in $A \cap B$, then it must be in A, but the converse isn't true.

1054. True or false?

(a) The sum of the first n integers is odd.
(b) The sum of the first n odd integers is odd.
(c) The sum of the first n even integers is even.

Try e.g. $n = 1, 2, 3$.

(a) False. It may be even or odd.
(b) False. It may be even or odd.
(c) True.

1055. If $\dfrac{d}{dx}(x+y+1)=0$, find $\dfrac{dy}{dx}$.

Apply the differential operator $\frac{d}{dx}$ to the three terms inside the bracket.

Applying the differential operator $\frac{d}{dx}$ to the three terms inside the bracket, we get $1+\frac{dy}{dx}+0=0$. Hence, $\frac{dy}{dx}=-1$.

1056. A student has attempted to calculate the total area enclosed by the curves $y=x^3-x$ and $y=3x$ using the integral

$$I=\int_{-2}^{2} x^3-4x\,dx.$$

(a) Explain how the integrand has been obtained.

(b) Explain how the limits have been obtained.

(c) Explain what is wrong with the calculation.

(d) Show that the total area enclosed is 8.

Remember that an integral calculates signed area, not simply area.

(a) The integrand x^3-4x represents the vertical distance between the curves.

(b) The limits are the x values of intersections of the curves.

(c) There is another intersection, at $x=0$, where the curves cross and the signed area changes from positive to negative. This has not been taken into account.

(d) The correct calculation is

$$\int_{-2}^{0} x^3-4x\,dx+\int_{0}^{2} 4x-x^3\,dx=8.$$

1057. Without expanding the brackets, solve

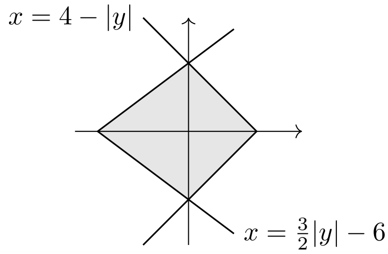
$$(3x-2)^2(x-1)+(3x-2)(x-1)^2=0.$$

Take out a common factor of $(3x-2)(x-1)$.

Taking out a factor of $(3x-2)(x-1)$, we get

$$\begin{aligned}&(3x-2)(x-1)((3x-2)+(x-1))=0\\ \Longrightarrow\ &(3x-2)(x-1)(4x-3)=0\\ \Longrightarrow\ &x=\tfrac{2}{3},\tfrac{3}{4},1.\end{aligned}$$

1058. Two graphs are represented below:



Find the area of the shaded region.

Find the coordinates of the axis intercepts, and use the area formula for triangles.

The axis intercepts are $x=-6,4$ and $y=\pm4$. Hence, the area of the shaded kite, considered as two triangles, is $2\times\frac{1}{2}\times10\times4=40$ square units.

1059. By considering the signs of the factors, solve the inequality $(x^2+1)(x-2)\geq0$, giving your answer in set notation.

The quadratic factor is always positive.

Since the quadratic factor is always positive, we can divide by it, giving $x-2\geq0$. Hence, the solution is $x\in[2,\infty)$.

1060. An irregular hexagon has sides whose lengths are in geometric progression. Its perimeter is 60 cm. Determine the possible values of l, the length of the shortest side, giving your answer in set notation.

The extreme cases here are a regular hexagon and one with a side of length 0.

In a regular hexagon of perimeter 60 cm, all sides have length 10 cm. The fact that the sides are in GP doesn't make a difference in this question: any value for the shortest side up to 10 cm can produce such an irregular hexagon. So $l\in(0,10)$ cm.

1061. Verify that the parabola $y=x^2+2$ satisfies the differential equation

$$y\frac{dy}{dx}-4x=2x^3.$$

Find $\frac{dy}{dx}$ and substitute.

Differentiating $y=x^2+2$, we get $\frac{dy}{dx}=2x$. Hence,

$$\begin{aligned}&y\frac{dy}{dx}-4x\\ =\ &(x^2+2)(2x)-4x\\ =\ &2x^3+4x-4x\\ =\ &2x^3,\text{ as required.}\end{aligned}$$

1062. Two functions f and g are such that, for all $x \in \mathbb{R}$,

$$\frac{d}{dx}\big(f(x) - g(x)\big) = 2.$$

Show that $f(x) = g(x)$ has exactly one root.

Integrate the equation given.

Integrating the equation given, we get $f(x) - g(x) = 2x + c$. Since the RHS is linear and non-constant, it is zero at exactly one x value. Hence, so is the LHS. This is equivalent to saying that $f(x) = g(x)$ has exactly one root.

1063. The curve $y = x^2$ passes through diagonally opposite corners of a rectangle, three of whose vertices are on the coordinate axes. Show that the curve divides the area of the rectangle in the ratio $1 : 2$.

Sketch the scenario, and show that one of the vertices must be the origin.

One of the vertices must be the origin. Labelling the diagonally opposite vertex (a, a^2), we see that the rectangle has area a^3. The area beneath the curve is given by the integral

$$\begin{aligned}&\int_0^a x^2\,dx\\ =\ &\left[\tfrac{1}{3}x^3\right]_0^a\\ =\ &\tfrac{1}{3}a^3.\end{aligned}$$

Hence, a third of the area is beneath the curve, which is the required result.

1064. Prove, by exhaustion, that $x^2 + y^2 = 42$ has no solutions $x, y \in \mathbb{Z}$.

A proof by exhaustion requires checking all possibilities.

Since x^2 and y^2 are both positive, we need only check that no sum of two squares from the set $\{1, 4, 9, 16, 25, 36\}$ adds to 42. This is easily verified.

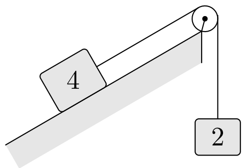
1065. Simplify $e^{3\ln a - \ln b}$.

Use log laws.

Using the laws of logarithms:

$$\begin{aligned}&e^{3\ln a - \ln b}\\ =\ &e^{\ln a^3 - \ln b}\\ =\ &e^{\ln \frac{a^3}{b}}\\ =\ &\frac{a^3}{b}.\end{aligned}$$

1066. Two masses are connected by a light, inextensible string, which is passed over a smooth, light, fixed pulley as shown in the diagram. The 4 kg mass sits on a smooth slope, in equilibrium.



Determine the angle of inclination of the slope.

Resolve parallel to the slope.

Since the string is inextensible, the hanging mass is also in equilibrium. Hence, the tension in the string is $T = 2g$. Resolving parallel to the slope, we have $T - 4g\sin\theta = 0$. Substituting gives $\sin\theta = \frac{1}{2}$. Since the angle of inclination must be acute, we get $\theta = 30°$.

1067. Find the values of p such that $\mathbf{r} = \frac{1}{7}(2\mathbf{i} + 3\mathbf{j} + p\mathbf{k})$ is a unit vector.

Use 3D Pythagoras.

Using 3D Pythagoras, we require that

$$\tfrac{1}{7}\sqrt{2^2 + 3^2 + p^2} = 1.$$

This gives $p = \pm 6$.

1068. A hand of five cards is dealt from a standard deck. Find the probability that the hand is a flush, i.e. all the cards are the same suit.

Consider the cards one by one.

Pick the first card, without loss of generality. Then the probability that the remaining cards are from the same suit is

$$1 \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} = 0.00198 \text{ (3sf)}.$$

1069. State, with a reason, whether the graph $y = |x|$ intersects the following curves:

(a) $y = 1 - x^2$,

(b) $y = 1 + x^2$.

Sketch the graphs, and consider the quadratic discriminant.

(a) These graphs must intersect, since $y = 1 - x^2$ is a negative parabola with a maximum at $y = 1$.

(b) If these graphs are to intersect, we require $1 + x^2 = x$ to have at least one root. But it has discriminant $\Delta = -3 < 0$, so the graphs do not intersect.

1070. If $z = a^x$, write $(\sqrt[3]{a})^x$ in terms of z.

Use index laws to reshuffle the indices.

We reshuffle the indices as follows:

$$(\sqrt[3]{a})^x = \left(a^{\frac{1}{3}}\right)^x = a^{\frac{1}{3}x} = (a^x)^{\frac{1}{3}} = z^{\frac{1}{3}}.$$

1071. The quadratic function $f(x) = ax^2 + 8x - a$ has discriminant 100. Find all possible values of a.

Set $b^2 - 4ac = 100$, and solve.

Setting $\Delta = 100$, we get $64 + 4a^2 = 100$, which has solution $a = \pm 3$.

1072. Solve $x - 3x^{\frac{2}{3}} + 2x^{\frac{1}{3}} = 0$.

This is a disguised cubic. Take out a factor of $x^{\frac{1}{3}}$ first.

This is a cubic in $x^{\frac{1}{3}}$. It factorises as

$$\begin{aligned}
& x - 3x^{\frac{2}{3}} + 2x^{\frac{1}{3}} = 0 \\
\Longrightarrow\ & x^{\frac{1}{3}}(x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 2) = 0 \\
\Longrightarrow\ & x^{\frac{1}{3}}(x^{\frac{1}{3}} - 1)(x^{\frac{1}{3}} - 2) = 0 \\
\Longrightarrow\ & x^{\frac{1}{3}} = 0, 1, 2 \\
\Longrightarrow\ & x = 0, 1, 8.
\end{aligned}$$

1073. An iteration is defined as $B_{n+1} = a(B_n + 1)$, where a is a constant. You are given that $B_1 = 3$ and $B_3 = 18$. Find the possible values of a.

Apply the iteration twice, and solve for a.

Applying the iteration twice, we get $B_2 = 4a$, and then $B_3 = a(4a + 1)$. So, we require

$$\begin{aligned}
& a(4a + 1) = 18 \\
\Longrightarrow\ & 4a^2 + a - 18 = 0 \\
\Longrightarrow\ & (a - 2)(4a + 9) = 0 \\
\Longrightarrow\ & a = 2, -\tfrac{9}{4}.
\end{aligned}$$

1074. Prove that, for $x, y > 0$, $\log_x y \times \log_y x = 1$.

Set $\log_x y = a$ and use index laws.

Setting $\log_x y = a$, we have $x^a = y$. Therefore, $x = y^{\frac{1}{a}}$, and, raising both sides to the power $\frac{1}{a}$, $\log_y x = \frac{1}{a}$. Multiplying the two logarithms gives the required result.

1075. Prove that, whatever the values of the constants a, b, c, d, the following function is not well-defined over $\mathbb{R}$:

$$f(x) = \frac{(x^2 - a^2 - 1)(x^2 + b^2 - 1)}{(x^2 - c^2 - 1)(x^2 + d^2 - 1)}.$$

Show that one of the factors in the denominator must have roots.

This is not well defined over $\mathbb{R}$, because the factor $(x^2 - c^2 - 1)$ is zero at $x = \pm\sqrt{1 + c^2}$. Whatever the value of c (and whatever the values of the other constants) there is a division by zero at these x values, and the fraction is undefined.

1076. A set of four lines forms a square. The first three are $y = 2x + 1$, $y = 2x + 4$ and $x + 2y = 5$. Find the two possible equations of the last line.

The fourth line must be of the form $x + 2y = k$.

The fourth line must be of the form $x + 2y = k$. Since its distance from $x + 2y = 5$ must be the same as the distance of $y = 2x + 1$ from $y = 2x + 4$, the possible equations are $x + 2y = 2$ and $x + 2y = 8$.

1077. Find the area, in the (x, y) plane, of the annulus

$$a \le (x - p)^2 + (y - q)^2 \le b.$$

An annulus is a ring-shaped region.

The fact that the annulus is centred on (p, q) is irrelevant. All we require is a disc of radius $\sqrt{b}$, with a disc of radius $\sqrt{a}$ removed from it. So, the area is $\pi(b - a)$.

1078. State that the following holds, or explain why not: "A resultant moment of zero is necessary, but not sufficient, for an object to be in equilibrium."

Consider the fact that a resultant force of zero is also necessary for equilibrium.

The statement holds. This is because, for equilibrium, both a resultant force of zero and a resultant moment of zero are required.

1079. You are given that the line $x + 3y = k$ is a normal to the curve $y = x^3$ at point P.

(a) Determine the possible coordinates of P.
(b) Find all possible values of k.

Set $\frac{dy}{dx} = 3$.

(a) We require $3x^2 = 3$, which gives $x = \pm 1$. So the possible coordinates of P are $(\pm 1, \pm 1)$.
(b) Substituting gives $k = \pm 4$.

1080. By attempting the factorisation and explaining exactly why it cannot be done, prove that $(x^2 + 1)$ is not a factor of $4x^5 + x + 1$.

Any factorisation would have to be of the form

$$(x^2 + 1)(ax^3 + bx^2 + cx + d) \equiv 4x^5 + x + 1.$$

Any factorisation would have to be of the form

$$(x^2+1)(ax^3+bx^2+cx+d) \equiv 4x^5+x+1.$$

Multiplying out and comparing coefficients of x^5, x^4, x^3, x^2 tells us that $a=4$, $b=0$, $a+c=0$, $b+d=0$. This requires $c=-4$ and $d=0$. But this gives no constant term. Hence, the factorisation is impossible.

1081. A trapeze artist of mass 55 kg swings across a stage, holding onto a light, rigid bar. At the lowest point of her swing, she experiences an acceleration of 6 ms^{-2} upwards. Find the force exerted on the bar by the trapeze artist at this instant.

Consider the Newton III pair of the force required.

The Newton pair of this force is the upwards force T exerted on the trapeze artist. Vertically, we have $T-55g=6\cdot 55$, which gives 869 Newtons upwards. So the force on the bar is 869 Newtons downwards.

1082. Solve the following simultaneous equations:

$$\begin{aligned}16x^2y^2-8xy+1&=0,\\4x+4y&=5.\end{aligned}$$

Factorise the first equation as a quadratic in xy.

The first equation is a quadratic in xy. Factorising gives $(4xy-1)^2=0$, so $4xy=1$. Substituting this into the second equation yields a quadratic. Solving produces $(x,y)=(1,\frac{1}{4}),(\frac{1}{4},1)$.

1083. An ellipse may be viewed as a transformed circle. By considering scale factors in the x and y directions, write down a formula for the area of the ellipse given by the parametric equations $x=a\cos\theta$, $y=b\sin\theta$.

Consider stretches of the unit circle, which has area π.

The unit circle $x=\cos\theta$, $y=\sin\theta$ has area π. Stretching by a factor a in x and b in y scales this area by ab, giving $A=\pi ab$.

1084. Find $f\left(\dfrac{a+b}{2}\right)$, if f is a linear function such that

$$\int_a^b f(x)\,dx=0.$$

Sketch a graph, considering the fact that an integral gives the signed area, not the area.

Since f is linear, and the signed area between the curve and the x axis is zero, the line $y=f(x)$ must be symmetrical around the midpoint of the interval $[a,b]$. Hence, the value of the function at this midpoint must be zero.

1085. Prove that the area of a rhombus is $A=\frac{1}{2}pq$, where p and q are the lengths of its diagonals.

The diagonals of a rhombus are perpendicular.

Since the diagonals of a rhombus are perpendicular, a rhombus can be considered as consisting of four right-angled triangles with sides of length $\frac{1}{2}p$ and $\frac{1}{2}q$. The area of all four is then given by

$$A=4\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}p\cdot\tfrac{1}{2}q=\tfrac{1}{2}pq.$$

1086. The equation of a straight line, gradient m, passing through the point (a,b) is

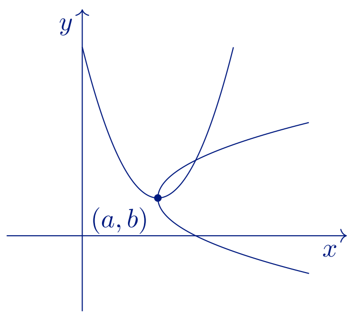
$$\frac{y-b}{x-a}=m.$$

Sketch the following graphs:

(a) $\dfrac{y-b}{(x-a)^2}=m$, (b) $\dfrac{(y-b)^2}{x-a}=m$.

Both graphs are positive parabolae.

Both graphs are parabolae, whose vertex is at (a,b), stretched by a factor m.



1087. Determine which of the points $(4,2)$ and $(3,3)$ is closer to the circle $x^2+y^2=1$.

This is equivalent to asking "Which point is closer to the origin?"

This is equivalent to asking "Which point is closer to the origin?" The distances are $\sqrt{20}$ and $\sqrt{18}$, so $(3,3)$ is closer.

1088. We define a logarithm by $a^x=b \iff \log_a b=x$. Prove, using this definition, that

(a) $\log_a 1=0$,

(b) $\log_a a=1$,

(c) $\log_a \sqrt{a} = \frac{1}{2}$.

Rewrite each logarithmic statement as an index statement.

In each case, the index statement is true as shown. Since the implication is $\iff$, the logarithmic statement is true too:

(a) $\log_a 1 = 0 \iff a^0 = 1$. True by definition.

(b) $\log_a a = 1 \iff a^1 = a$. True by definition.

(c) $\log_a \sqrt{a} = \frac{1}{2} \iff a^{\frac{1}{2}} = \sqrt{a}$. Squaring both sides of the index statement verifies that $a^{\frac{1}{2}}$ and $\sqrt{a}$ are equal.

1089. The parabola $y = x^2$ has a tangent drawn to it at a point with x coordinate a. Show that this tangent crosses the x axis at $x = \frac{1}{2}a$.

The tangent has equation $y = 2ax + c$. Find c.

Evaluating the derivative at $x = a$ produces a tangent line $y = 2ax + c$. Substituting (a, a^2) gives $c = -a^2$. So the tangent line has equation $y = 2ax - a^2$. To find the x axis intercept, we set $y = 0$, which gives $0 = 2ax - a^2$. Hence, the tangent crosses the x axis at $x = \frac{1}{2}a$.

1090. State, with a reason, which of the following shapes, in which diameter is defined vertex-to-vertex, has the larger area:

- a regular $2n$-gon of diameter d,
- a regular $(2n+2)$-gon of diameter d.

Sketch an example, e.g. a square and a hexagon.

Since the diameter is the same, both shapes may be inscribed in the same circle. The edges of the $(2n+2)$-gon form a better approximation to the circle. Hence, the $(2n+2)$-gon has the greater area.

1091. Given $Z \sim N(0, 1)$, find $P(Z^2 > Z)$.

Solve the inequality $Z^2 > Z$ first.

Solving $Z^2 > Z$ gives $Z \in (-\infty, 0) \cup (1, \infty)$. So, the probability required is

$$\begin{aligned} & P(Z < 0) + P(Z > 1) \\ = \; & 0.5 + 0.15865... \\ = \; & 0.659 \text{ (3sf)}. \end{aligned}$$

1092. A sequence is defined by

$$u_{n+1} = u_n + n, \quad u_1 = 1.$$

(a) Give the first five terms.

(b) By considering the differences, find an ordinal n^{th} term formula for the sequence.

The sequence is quadratic.

(a) $1, 2, 4, 7, 11, ...$

(b) The first differences are $1, 2, 3, 4, ...$, and the second difference is 1. Hence, the sequence is quadratic, with n^{th} term $u_n = \frac{1}{2}n^2 + pn + q$. The first two terms require $1 = \frac{1}{2} + p + q$ and $2 = 2 + 2p + q$. Solving gives $p = -\frac{1}{2}$, $q = 1$. Hence, the ordinal formula is

$$u_n = \tfrac{1}{2}n^2 - \tfrac{1}{2}n + 1.$$

1093. Find the radius of the smallest circle that can contain a $(3, 4, 5)$ triangle.

Consider the angle in a semicircle theorem.

The smallest circle which could possibly contain any triangle has diameter equal to its longest edge, in this case 5. And this circle is indeed possible, since a $(3, 4, 5)$ triangle is right-angled; hence, by the angle in a semicircle theorem, if the hypotenuse lies on the diameter, then the other vertex will lie on the circumference. So the radius is $\frac{5}{2}$.

1094. In an industrial process, a carbon nanotube in the shape of a cylinder (with no ends) grows from an initial seed. The radius remains fixed, while the rate of change of length is constant. Describe, as linear, quadratic or similar, the rate of change of the surface area of the nanotube.

This can be visualised directly, or seen algebraically by differentiating surface area $A = 2\pi rl$ with respect to time.

Differentiating surface area $A = 2\pi rl$ with respect to time, we get $\frac{dA}{dt} = 2\pi r \frac{dl}{dt}$, since r is a constant. Then, since $\frac{dl}{dt}$ is a constant, so is $\frac{dA}{dt}$. Hence, the rate of increase is linear.

1095. Solve the simultaneous equations

$$\begin{aligned} 2\sqrt{x} + 3y^2 &= 18, \\ 3\sqrt{x} - 2y^2 &= 1. \end{aligned}$$

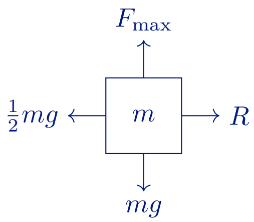
This is a pair of linear simultaneous equations in $\sqrt{x}$ and y^2.

This is a pair of linear simultaneous equations in $\sqrt{x}$ and y^2. Solving for these values by elimination or substitution gives $\sqrt{x} = 3$, $y^2 = 4$. So $x = 9$ and $y = \pm 2$.

1096. A cube of mass m is held in equilibrium against a vertical wall by a horizontal force of magnitude $\frac{1}{2}mg$. Find the least possible value of μ, the coefficient of friction between the wall and the cube.

Draw a force diagram!

The forces, with the left-hand face of the cube in contact with the wall, are as follows:



The least possible value of μ is attained when the block is in limiting equilibrium, with $\mu R - mg = 0$. By horizontal equilibrium, $R = \frac{1}{2}mg$, so $\mu_{\text{min}} = \frac{1}{2}$.

1097. The line $4x + y = 8$ can be expressed in the form $x = 1 - 2t$, $y = a + bt$. Find the constants a and b.

The point $(1, a)$ must lie on the line.

The point $(1, a)$ must lie on the line, at $t = 0$. So $4 + a = 8$, and $a = 4$. Then the gradient of the line gives $-4 = \frac{b}{-2}$, so $b = 8$.

1098. Provide a counterexample to the following:

$$p = q \implies \frac{p - q}{r - s} = 0, \text{ for all } p, q, r, s \in \mathbb{R}.$$

The implication cannot hold if the fraction is undefined.

A counterexample is $p, q, r, s = 1$. In that case, $p = q$, but the right-hand equation is not well defined.

1099. The quartic $y = x^4 - 8x^2 + 16$ has two double roots.

(a) Factorise the quartic fully, and hence find the double roots.

(b) Show explicitly, using calculus, that the gradient is zero at the double roots.

(c) Hence, sketch the curve.

The quartic is a biquadratic, that is, a quadratic in x^2.

(a) The quaratic factorises

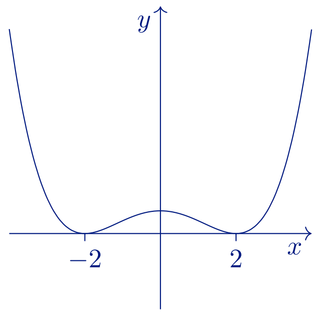
$$\begin{aligned} & x^4 - 8x^2 + 16 \\ \equiv\ & (x^2 - 4)(x^2 - 4) \\ \equiv\ & (x + 2)^2(x - 2)^2. \end{aligned}$$

So, the double roots are at $x = \pm 2$.

(b) Differentiating gives $\frac{dy}{dx} = 4x^3 - 16x$. Evaluating this at $x = \pm 2$ yields

$$4(\pm 2)^3 - 16(\pm 2) = \pm 32 \mp 32 = 0.$$

(c) The curve is a positive quartic which is tangent to the x axis at $x = \pm 2$:



1100. Integers $a < b < c$ are such that the ratio $c - b : a - c$ is $2 : 3$. Prove that, if a is a multiple of 5, then so is b.

Translate the ratio information into an algebraic equation, and manipulate it.

The ratio information is $3(c - b) = 2(a - c)$. This rearranges to $5c - 2a = 3b$. If a is a multiple of 5, then the whole LHS is. Hence the RHS is too. This implies that b must be a multiple of 5.

1101. By multiplying out and equating coefficients, or otherwise, write the expression $12x^2 + 2x - 3$ in the form $a(2x + 1)^2 + b(2x + 1) + c$.

Consider the coefficients of x^2, then x, then the constant term.

Comparing coefficients of x^2, we require $a = 3$. This gives the coefficients of x as $12 + 2b = 2$, so $b = -5$. Then, for the constant term, we need $3 - 5 + c = -3$, so $c = -1$. This gives

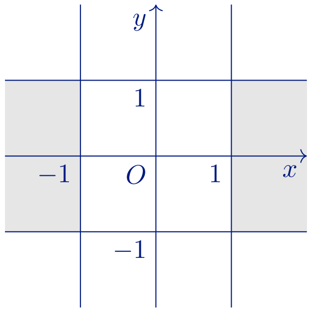
$$3(2x + 1)^2 - 5(2x + 1) - 1.$$

1102. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$|x| \geq 1, \qquad |y| \leq 1.$$

The x inequality produces two regions.

The region is



1103. A pilot sample of ten data is taken from a large population. Find the probability that all ten data lie between the quartiles of the population.

In a large population, the probability that any single datum lies between the quartiles is $\frac{1}{2}$.

When taken from a large population, the probability that any single datum lies between the quartiles is $\frac{1}{2}$. So, the probability is $\frac{1}{2}^{10} = \frac{1}{1024}$.

1104. A linear function g is defined from a function f, in terms of an unknown constant a, by

$$g(x) = f'(a)(x - a) + f(a).$$

Show that $y = g(x)$ is tangent to $y = f(x)$.

The graphs are tangent at $x = a$.

Substituting $x = a$, the y values are $y = f(a)$ and $y = g(a) = f'(a)(a - a) + f(a) = f(a)$. Hence, the curves intersect at $x = a$. Also, the gradient of the line $y = g(x)$ is $f'(a)$, which is the same as that of $y = f(x)$ at $x = a$. Hence, the graphs are tangent at $x = a$.

1105. Simplify the following, where $n \in \mathbb{Z}$, giving your answers in standard form:

(a) $2.3 \times 10^n + 1.2 \times 10^{n-1}$,

(b) $5 \times 10^n + 9.7 \times 10^{n+1}$.

Before adding, take out factors of 10 in order to match up the indices.

(a)
$$\begin{aligned} & 2.3 \times 10^n + 1.2 \times 10^{n-1} \\ = \; & 2.3 \times 10^n + 0.12 \times 10^n \\ = \; & 2.42 \times 10^n, \end{aligned}$$

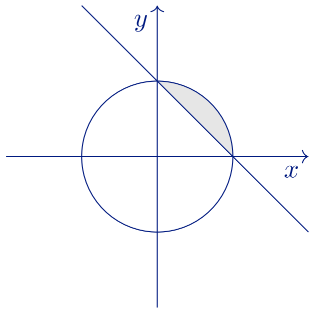
(b)
$$\begin{aligned} & 5 \times 10^n + 9.7 \times 10^{n+1} \\ = \; & 0.5 \times 10^{n+1} + 9.7 \times 10^{n+1} \\ = \; & 10.2 \times 10^{n+1} \\ = \; & 1.02 \times 10^{n+2}. \end{aligned}$$

1106. Find the exact area of the region of the (x, y) plane defined by the simultaneous inequalities

$$\begin{aligned} & x + y > 1, \\ & x^2 + y^2 < 1. \end{aligned}$$

The region is a segment of the unit circle.

Sketching, we require the area of the region below:



The area of a quarter-circle sector is $\frac{1}{4}\pi$, and the area of the subtracted triangle is $\frac{1}{2}$, which gives the area of the shaded segment as $A = \frac{1}{4}\pi - \frac{1}{2}$.

1107. For each of the following, state, with a reason, whether such an entity can exist:

(a) A function of the form $f(x) = ax^2 + bx + c$, for constants $a, b, c \in \mathbb{R}$, which is linear.

(b) A prime expressible as mn, for $mn \in \mathbb{N}$.

Both can exist.

Both can exist, as trivial cases:

(a) constants $a = 0$, $b = 1$, $c = 1$ gives a linear function.

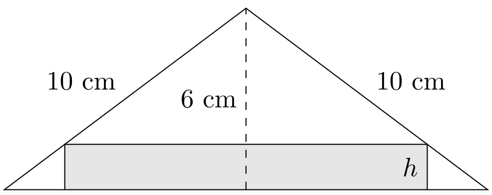
(b) natural numbers $m = 1$, $n = 3$ gives a prime product.

1108. Write down the area scale factor when $y = f(x)$ is transformed to $y = kf(kx)$.

The answer is independent of k.

The output transformation scales the area by k, but the input transformation scales it by $\frac{1}{k}$, so the overall scale factor is 1.

1109. The diagram shows an isosceles triangle of height 6 cm and slant height 10 cm. A shaded rectangle is drawn inside the triangle, of variable height h.



(a) Show that the area of the rectangle can be expressed as
$$A = 16h - \frac{8h^2}{3}.$$

(b) Hence show that maximum possible area of the rectangle is half that of the triangle.

In (a), using similar triangles. In (b), use calculus: differentiate and set $\frac{dA}{dh} = 0$.

(a) By Pythagoras, the half-base has length 8 cm. So, using similar triangles, the half-width w of the shaded rectangle is given by $\frac{6}{h} = \frac{8}{8-w}$. Solving this gives $w = 8 - \frac{4h}{3}$, so the area is
$$A = 2wh = 16h - \frac{8h^2}{3}.$$

(b) Differentiating and setting $\frac{dA}{dh} = 0$ gives $16 - \frac{16h}{3} = 0$, so $h = 3$. This gives $A = 24$ cm^2, which is half the area of the triangle.

1110. Show that the pair of circles $(x-2)^2 + y^2 = 2$ and $x^2 + y^2 = 2$ are normal to one another.

Show that the radii are perpendicular at the points of intersection.

Solving simultaneously or by inspection, the pair intersect at $(1, 1)$. The radii to this point have vectors $-\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{j}$, which are perpendicular. Hence, since tangent and radius are always perpendicular, the circles are normal. N.B. they are also normal at the other point of intersection, but the question doesn't explicitly require this.

1111. Simplify the following sets:

(a) $[0, 1] \cup [1, 2] \cup [2, 3]$,

(b) $[0, 3] \cap [1, 4] \cap [2, 5]$.

Sketch a number line.

(a) $[0, 3]$,

(b) $[2, 3]$.

1112. The function $f(x) = x^n(x + 1)$, for $n \in \mathbb{N}$, has $f''(x) = ax^3(bx + c)$, for $a, b, c \in \mathbb{N}$. Find n, a, b, c.

Find n first.

Since $f''(x)$ has degree 4, $f(x)$ must have degree 6. So $n = 5$. Differentiating $f(x) = x^6 + x^5$ twice gives $f''(x) = 30x^4 + 20x^3 = 10x^3(3x + 2)$. Hence $a = 10$, $b = 3$, $c = 2$.

1113. Prove by contradiction that every pentagon must have at least one interior angle $\theta \geq 108°$.

Begin: "Assume, for a contradiction, that a pentagon has five interior angles, each of which is smaller than 108°."

Assume, for a contradiction, that a pentagon has five interior angles, each of which is smaller than 108°. Then the sum of the interior angles must be less than $5 \times 108 = 540°$. But the interior angles of a pentagon add to 540°. This is a contradiction, so at least one angle must satisfy $\theta \geq 108°$.

1114. Electricity pylons stand every 50 metres alongside a railway track. A train accelerating constantly at a ms^{-2} covers successive gaps between pylons in 2.2 and 1.8 seconds.

(a) Show that initial speed u and a satisfy
$$50 = 2.2u + 2.42a,$$
$$100 = 4u + 8a.$$

(b) Hence, find u and a.

Consider the first gap on its own and both gaps together.

(a) Using the formula $s = ut + \frac{1}{2}at^2$, the first gap gives
$$50 = 2.2u + \tfrac{1}{2}a \cdot 2.2^2,$$
which simplifies to the first required equation, and both gaps together gives
$$100 = 4u + \tfrac{1}{2}a \cdot 4^2,$$
which simplifies to the second required equation.

(b) Solving simultaneously gives $a = \frac{250}{99}$ ms^{-2}, $u = \frac{1975}{99}$ ms^{-1}.

1115. Give the range of $g : x \mapsto (\cos x + 3)^3$.

Consider this as a pair of output transformations acting on the cosine function.

We assume the domain is $\mathbb{R}$, so the range of the cosine function is $[-1, 1]$. The range of $(\cos x + 3)$ is then $[2, 4]$, so the range of g is $[8, 64]$.

1116. A sample, with $r = -0.426$, of forty bivariate data is being tested, looking, at the 5% significance level, for evidence of negative correlation in the population.

(a) Write down suitable hypotheses.

(b) Carry out the test.

Statistical hypotheses should always be phrased in terms of the population, not the sample.

(a) With ρ defined as the population correlation coefficient, the hypotheses are

$$H_0 : \rho = 0,$$
$$H_1 : \rho < 0.$$

(b) The critical value, for a one-tailed test with $n = 40$, is $r_c = -0.264$. The sample value $-0.426 < -0.264$, so the result is (highly) significant. The sample provides (plenty of) evidence for negative correlation in the population.

1117. The graph $x = y^2$ is translated by the vector $a\mathbf{i}+b\mathbf{j}$. Write down the equation of the new graph.

This is easiest to visualise in term of replacement of x and replacement of y.

To enact this transformation, we replace x by $x-a$ and y by $y-b$. This gives the new graph as $x-a = (y-b)^2$.

1118. Write down the broadest real domains over which the following functions may be defined:

(a) $x \mapsto \sqrt{x-1}$,

(b) $x \mapsto \sqrt{1-x}$.

Consider the necessity for the quantity under the square root to be greater than or equal to zero.

(a) We require $x - 1 \geq 0$, so the broadest possible real domain is $[1, \infty)$.

(b) We require $1 - x \geq 0$, so the broadest possible real domain is $(-\infty, 1]$.

1119. Determine the exact roots of the equation

$$\sqrt{2}x^2 + \sqrt{24}x - \sqrt{8} = 0.$$

Use the quadratic formula, and simplify the resulting surds. Either that, or divide the whole equation by $\sqrt{2}$ first.

The question says "Determine...", so a full explanation is required, not just an answer. The quadratic formula gives

$$\begin{aligned} x &= \frac{-\sqrt{24} \pm \sqrt{24 + 4\sqrt{2}\cdot\sqrt{8}}}{2\sqrt{2}} \\ &= \frac{-\sqrt{24} \pm \sqrt{40}}{2\sqrt{2}} \\ &= \frac{-\sqrt{48} \pm \sqrt{80}}{4} \\ &= -\sqrt{3} \pm \sqrt{5}. \end{aligned}$$

1120. A washing line of length 4 m hangs with negligible departure from horizontal. A bag of clothes pegs weighing 0.8 kg is then hung at its centre, and the line sags by 20 vertical cm. Find the tension in the line in this state.

Resolve vertically.

Using the lengths, the angle between each section of line and the horizontal is $\theta = \arctan \frac{0.2}{2}$. Resolving vertically gives $2T \sin\theta = 0.8g$, so $T = 39.4$ N (3sf).

1121. Show that $\displaystyle\int_1^8 \frac{(x+1)^2}{\sqrt[3]{x}}\,dx = \frac{5493}{40}$.

Expand the brackets and split up the fraction.

Expanding the brackets and splitting up the fraction gives

$$\begin{aligned} &\int_1^8 x^{\frac{5}{3}} + 2x^{\frac{2}{3}} + x^{-\frac{1}{3}}\,dx \\ &= \left[\tfrac{3}{8}x^{\frac{8}{3}} + \tfrac{6}{5}x^{\frac{5}{3}} + \tfrac{3}{2}x^{\frac{2}{3}}\right]_1^8 \\ &= \frac{702}{5} - \frac{123}{40} \\ &= \frac{5493}{40}. \end{aligned}$$

1122. AB and CD are two diameters of the same circle. You are given $A : (7, 4)$, $B : (-1, 2)$, $C : (0, 5)$. Find the coordinates of D.

Find the coordinates of the centre, then use vectors.

Since AB is a diameter, the centre O is at $(3, 3)$. Then, since CD is a diameter, $\overrightarrow{CO} = \overrightarrow{OD}$. This gives the coordinates of D as $(6, 1)$.

1123. Describe all functions f for which f'' is constant.

Write this algebraically, then integrate twice.

We know that $f''(x) = a$, for some constant a. Integrating twice gives $f(x) = px^2 + qx + r$, for some constants p, q, r (where $p = \frac{1}{2}a$). This is all quadratic and linear functions.

1124. A *gradian* is a decimal measure of angle, sometimes used in surveying and geology, in which a right angle contains 100 gradians. Determine formulae, with angle θ given in gradians, for arc length l and sector area A.

Express angle as a fraction of a full circle, i.e. of 400 gradians.

Since there are 400 gradians in a full circle, arc length and sector area should be given as fractions out of 400. So the formulae, for θ is gradians, are

$$A = \frac{\theta}{400}\pi r^2 \qquad l = \frac{\theta}{400}2\pi r.$$

1125. Solve the equation $|x^2 - x| = |x - 1|$.

Use $|a| = |b| \implies a^2 = b^2$.

Using $|a| = |b| \implies a^2 = b^2$, we have

$$\begin{aligned} & (x^2 - x)^2 = (x-1)^2 \\ \implies & (x^2 - x)^2 - (x-1)^2 = 0 \\ \implies & x^2(x-1)^2 - (x-1)^2 = 0 \\ \implies & (x-1)^2(x^2-1) = 0 \\ \implies & (x-1)^3(x+1) = 0 \\ \implies & x = \pm 1. \end{aligned}$$

1126. Show that $(x-a+\sqrt{b})(x-a-\sqrt{b})$, where $a, b \in \mathbb{Z}$ and $b \geq 0$, is a quadratic with integer coefficients.

Consider the expression as a difference of two squares.

The expression is a difference of two squares:

$$\begin{aligned} & (x-a+\sqrt{b})(x-a-\sqrt{b}) \\ \equiv\ & (x-a)^2 - (\sqrt{b})^2 \\ \equiv\ & x^2 - 2ax + a^2 - b. \end{aligned}$$

Since a and b are integers, so are $-2a$ and $a^2 - b$. Hence, the expression is a quadratic with integer coefficients.

1127. A sledge is sliding down a snowy slope. Explain how you know that the reaction force exerted by the sledge on the slope and the frictional force exerted by the sledge on the slope are at right angles.

Regardless of the scenario, these forces are always at right angles.

Regardless of the scenario, these forces are always at right angles. Reaction forces act perpendicular to surfaces, frictional forces act parallel to surfaces. By definition, these are at right angles to one another.

1128. Verify that the function $g(x) = \tan x$ satisfies the nonlinear differential equation

$$g'(x) = 1 + (g(x))^2.$$

Use a Pythagoraean identity.

The derivative of $\tan x$ is $\sec^2 x$. By the second Pythagorean identity, we know that this is equal to $1+\tan^2 x$, which is $1+(g(x))^2$. This verifies that $g(x) = \tan x$ satisfies the differential equation.

1129. It is given that there are constants A, B for which the following is an identity:

$$\frac{2x^2 + 3x + c}{2x - 1} \equiv Ax + B.$$

Find the value of the constant c.

Use the factor theorem.

Multiplying up, we have

$$2x^2 + 3x + c \equiv (2x-1)(Ax+B).$$

Comparing coefficients of x^2 and x, we require that $A = 1$ and $B = 2$. This gives $c = -2$.

1130. "The coordinate axes are normal to the curve $x^2 + 2x + y^2 = 1$." True or false?

The curve is a circle. Find its centre.

The curve is a circle. Completing the square to give $(x+1)^2 + y^2 = 2$ tells us that the centre is $(-1, 0)$. Hence the x axis is normal to the curve, but the y axis is not. So the statement is false.

1131. The interior angles of a quadrilateral are in AP. Prove that the quadrilateral is convex, i.e that all of its interior angles are less than 180°.

Consider the least possible interior angle, in conjunction with the average angle.

The mean of the interior angles of a quadrilateral is 90°. Since, in this case, the interior angles are in AP, they are distributed symmetrically around their mean, so the difference between the largest and 90° is the same as the difference between the smallest and 90°. The smallest must be greater than 0°, so the largest must be less than 180°.

1132. A bank account offers compound interest at 2% per annum, with the money paid into the account as a lump sum at the end of the year. Determine the number of years after which an initial investment gives a return of over 25%.

Set up an equation and solve using logs.

The yearly scale factor is 1.02, so we require that $1.02^n \geq 1.25$. Hence, $n \geq \log_{1.02} 1.25 = 11.268$. Since the money is paid in as a lump sum, the return will exceed 25% after 12 years.

1133. If $u = 3x + 4$, write $x^2 + 6x + 1$ as a simplified quadratic in u.

Write x in terms of u.

Substituting $x = \dfrac{u-4}{3}$ gives

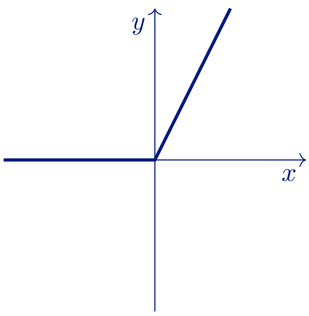
$$\begin{aligned}&\left(\frac{u-4}{3}\right)^2 + 6\left(\frac{u-4}{3}\right) + 1\\ \equiv\ &\tfrac{1}{9}(u^2 - 8u + 16) + 2(u-4) + 1\\ \equiv\ &\tfrac{1}{9}u^2 + \tfrac{10}{9}u - \tfrac{47}{9}.\end{aligned}$$

1134. A function g is defined by $g : x \mapsto x + |x|$.

(a) Show that $g(x) = 0$ for $x \leq 0$.
(b) Show that $g(x) = 2x$ for $x \geq 0$.
(c) Hence, sketch $y = g(x)$.

The graph has a single vertex, like a standard modulus graph, but is asymmetrical.

(a) For negative x, $g(x) = x + (-x) = 0$.
(b) For positive x, $g(x) = x + x = 2x$.
(c) Sketching the two straight line parts, we get



1135. Prove that, if $a, b \in \mathbb{Q}$, then $a - b \in \mathbb{Q}$.

Set up $a = \frac{p}{q}$ and $b = \frac{r}{s}$, for $p, q, r, s \in \mathbb{Z}$.

Since a and b are rational, $a = \frac{p}{q}$ and $b = \frac{r}{s}$, for $p, q, r, s \in \mathbb{Z}$, where $q, s \neq 0$. Subtracting and putting over a common denominator gives

$$a - b = \frac{ps - rq}{qs}.$$

Both $ps - rq$ and qs are integers, and $qs \neq 0$, so $a - b \in \mathbb{Q}$.

1136. A function f is such that, for some constant k,

$$\int f(x)\,dx = kf(x) + c.$$

Prove that f cannot be a polynomial function.

Prove this by contradiction, assuming that f is a polynomial of degree n.

Assume, for a contradiction, that $f(x)$ is a polynomial of degree n. Then $f(x) = ax^n + ...$, where $a \neq 0$. Integrating gives

$$\int f(x)\,dx = \frac{a}{n+1}x^{n+1} + ...$$

This is a polynomial of degree $n+1$. But $kf(x)+c$ is a polynomial of degree n. This is a contradiction. Hence, f is not a polynomial function.

1137. Two arithmetic sequences have n^{th} terms a_n and b_n. Prove that their average $\frac{1}{2}(a_n + b_n)$ forms another arithmetic progression.

Express the facts algebraically.

These APs have have general ordinal formulae $a_n = a + (n-1)d$ and $b_n = b + (n-1)e$. Their average has n^{th} term $\frac{1}{2}(a+b) + \frac{1}{2}(n-1)(d+e)$. This is an AP with first term $\frac{1}{2}(a+b)$ and common difference $\frac{1}{2}(d+e)$.

1138. The parabolae $y = (x+1)(x-a)$ and $y = (x-2)^2+b$ are mirror images in the line $x = 3$. Find a and b.

Consider the x coordinate of the vertex of the second parabola.

The x coordinate of the vertex of the second parabola is $x = 2$. Reflecting this in the line $x = 3$ gives the x coordinate of the vertex of the first parabola as $x = 4$. Since the roots of the first parabola, which are $x = -1$ and $x = a$, must be symmetrical around the vertex, we know that $a = 9$. This gives the y coordinate of both vertices as $b = -25$.

1139. An equation in y is given as

$$\Big[\log_x y\Big]_{x=2}^{x=4} = 3.$$

(a) Prove the general result $\log_{a^2} b \equiv \frac{1}{2}\log_a b$.
(b) Hence, solve for y.

Use the change of base formula in (a).

(a) Using the change of base formula, we get

$$\log_{a^2} b = \frac{\log_a b}{\log_a a^2} = \tfrac{1}{2}\log_a b.$$

(b) Expanding the notation gives

$$\begin{aligned}&\log_4 y - \log_2 y = 3\\ \Longrightarrow\ &\tfrac{1}{2}\log_2 y - \log_2 y = 3\\ \Longrightarrow\ &\log_2 y = -6\\ \Longrightarrow\ &y = \tfrac{1}{64}.\end{aligned}$$

1140. Simplify $\dfrac{x^2+2xy+y^2}{x^2-y^2}$.

Factorise top and bottom and look for common factors.

$$\frac{(x+y)^2}{(x+y)(x-y)} \equiv \frac{x+y}{x-y}.$$

This simplification is valid for $x+y \neq 0$.

1141. A car moves, for 10 seconds, with position given, in metres, by $x = t^2$. Find the time t during the motion at which the car's instantaneous speed is equal to its average speed over the ten seconds.

Use calculus.

The displacement over the 10 seconds is $10^2 - 0^2 = 100$ m. So, the average speed is 10 ms^{-1}. Differentiating gives the instantaneous speed as $v = 2t$. So, we require $2t = 10$, whence $t = 5$.

1142. Find the length of the line segment

$$x = a + bt, \quad y = c + dt, \quad t \in \left[0, \frac{1}{\sqrt{b^2+d^2}}\right].$$

Use Pythagoras.

The constants a and c are not relevant, as they translate the line segment. For unit change in t, the length is given by $\sqrt{b^2+d^2}$. Hence, the line segment has length 1.

1143. It is given that the functions $f(x) = -x^2 + kx + 4$ and $g(x) = -3x^2 - 6x + 5$ have the same range over $\mathbb{R}$. Find the value(s) of the constant k.

Find the coordinates of the vertices of the parabolae, either using calculus or by completing the square.

Differentiating and setting to zero gives $-2x+k = 0$ and $-6x - 6 = 0$. Hence, the vertices of the parabolae are $(\frac{1}{2}k, \frac{1}{4}k^2+4)$ and $(-1, 8)$. Since the ranges are the same, we require $\frac{1}{4}k^2+4 = 8$, which has roots $k = \pm 4$.

1144. Prove the following results:

(a) $\cos 30° = \dfrac{\sqrt{3}}{2}$,

(b) $\cos 45° = \dfrac{\sqrt{2}}{2}$.

Use an equilateral triangle and a square.

(a) Splitting an equilateral triangle of side length 2 in half gives two right-angled triangles with sides $(2, 1\sqrt{3})$. By definition, $\cos 30° = \frac{\sqrt{3}}{2}$.

(b) The diagonal of a unit square is $\sqrt{2}$, by Pythagoras, which gives the required result.

1145. By considering normals, find the shortest distance between the circles $(x-3)^2 + (x+2)^2 = 16$ and $(x-11)^2 + (y+8)^2 = 9$.

Find the distance between the centres.

The shortest distance between circles is normal to both, so it lies along the line of centres. These are $(3, -2)$ and $(11, -8)$, and the distance between them is $\sqrt{8^2+6^2} = 10$. Subtracting both radii gives the distance as $10 - 4 - 3 = 3$ units.

1146. Using integration, show that the average value of the function $f(x) = x^2$ on the domain $[0, 12]$ is 48.

The relevant integral is $\displaystyle\int_0^{12} x^2\, dx$.

A definite integral sums the integrand over the given domain. In this case, we get

$$\int_0^{12} x^2\, dx = \left[\tfrac{1}{3}x^3\right]_0^{12} = 576.$$

Dividing by the width of the domain gives $\frac{576}{12} = 48$ as required.

1147. Verify, by calculating derivatives, that the curve $y = \sqrt{x}$ satisfies the differential equation

$$4y^3\frac{d^2y}{dx^2} + 1 = 0.$$

Find $\dfrac{d^2y}{dx^2}$ and substitute.

Differentiating twice gives $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ and then $\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}}$. Substituting y and $\frac{d^2y}{dx^2}$, we get

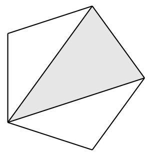
$$\begin{aligned} & 4y^3\frac{d^2y}{dx^2} + 1 \\ = \; & 4(\sqrt{x})^3 \cdot -\tfrac{1}{4}x^{-\frac{3}{2}} + 1 \\ = \; & -x^{\frac{3}{2}} \cdot x^{-\frac{3}{2}} + 1 \\ = \; & -1 + 1 \\ = \; & 0, \text{ as required.} \end{aligned}$$

1148. If $a = 2^x$ and $b = 3^x$, write a in terms of b.

Use logarithms.

Rewriting $a = 2^x$ as an exponential base 3, we have $a = (3^{\log_3 2})^x = (3^x)^{\log_3 2} = b^{\log_3 2}$.

1149. The regular pentagon below has side length 2.

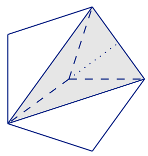


Show that the area of the shaded triangle is

$$A = \operatorname{cosec} \tfrac{\pi}{5} + \cot \tfrac{\pi}{5}.$$

Draw in three radii.

Adding three radii, we have



One sector subtends $\frac{2\pi}{5}$ radians at the centre, so one half-sector subtends $\frac{\pi}{5}$. Hence, the dashed radii have length $\operatorname{cosec} \frac{\pi}{5}$, and the dotted apothem has length $\cot \frac{\pi}{5}$. Hence, the perpendicular height of the triangle is $h = \operatorname{cosec} \frac{\pi}{5} + \cot \frac{\pi}{5}$. Since the base is 2, this is also the area.

1150. Complete the square on $\sqrt{3}x^2 - \sqrt{48}x - \sqrt{27}$.

Take out a factor of $\sqrt{3}$ first.

Taking out $\sqrt{3}$ and simplifying $\sqrt{27}$, we get

$$\begin{aligned} & \sqrt{3}(x-2)^2 - 4\sqrt{3} - 3\sqrt{3} \\ = \; & \sqrt{3}(x-2)^2 - 7\sqrt{3}. \end{aligned}$$

1151. State, with a reason, whether, in a game of cards, being dealt a straight (consecutive numbers) is more probable if the cards are picked

(a) with replacement,

(b) without replacement.

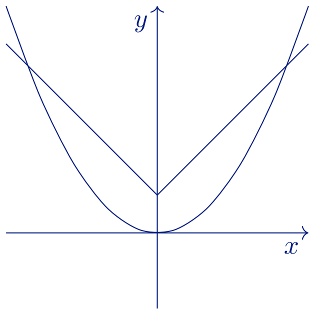
Consider the number of successful outcomes in the possibility space.

A straight is more likely without replacement, since removing a card from the possibility space makes it likelier that the next card is something different.

1152. Show that the area enclosed by $y = 3x^2$ and $y = |x| + 2$ is 3.

Sketch the graphs first.

The graphs are



Solving $3x^2 = x + 2$ gives intersections at $x = \pm 1$. We can then calculate the area with

$$\begin{aligned} A &= 2 \int_0^1 x + 2 - 3x^2 \, dx \\ &= 2 \Big[\tfrac{1}{2}x^2 + 2x - x^3 \Big]_0^1 \\ &= 2 \cdot \tfrac{3}{2} \\ &= 3. \end{aligned}$$

1153. A function is defined, over the domain $\mathbb{R}$, by the instruction $f : x \mapsto x^4 - x^2$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find all x values such that $f''(x) = 0$.

(c) Show, by considering the sign of $f''(x)$ either side of these x values, that these are points of inflection.

At a point of inflection, $f''(x) = 0$ and $f''(x)$ changes sign.

(a) $f'(x) = 4x^3 - 2x$, $f''(x) = 12x^2 - 2$.

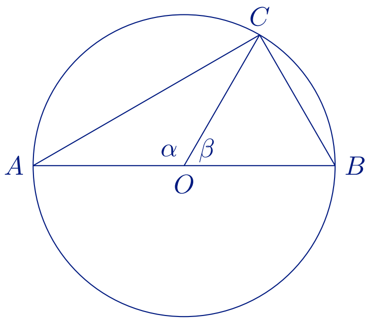
(b) Solving $12x^2 - 2 = 0$ gives $x = \pm \frac{1}{\sqrt{6}}$.

(c) Since each of these values is a single root of the quadratic $12x^2 - 2 = 0$, $f''(x)$ is both zero and changes sign. Hence, there are points of inflection at $x = \pm \frac{1}{\sqrt{6}}$.

1154. Prove the "angle in a semicircle" theorem, which states that the angle subtended by a diameter at the circumference is a right angle.

Draw in the radius to the third vertex of the triangle and label the angles subtended at the centre α and β.

Drawing in the radius to the proposed right angle and labelling the subtended angles α and β, we have



Since triangles AOC and BOC are isosceles, their other angles are $90 - \frac{1}{2}\alpha$ and $90 - \frac{1}{2}\beta$. The sum of these is $180 - \frac{1}{2}(\alpha + \beta)$, which, since α and β lie on a straight line, is $90°$. Q.E.D.

1155. In each case, state whether the given events are independent:

(a) "Coin A shows heads"; "Coin A shows tails".

(b) "Coin A shows heads"; "Coin B shows tails".

Mutual exclusivity implies dependence.

(a) These are not independent. They are mutually exclusive, which is as dependent as you can get: one rules out the other.

(b) These are independent.

1156. Solve ${}^nC_3 - {}^nC_2 = 0$, for $3 \leq n \in \mathbb{N}$.

Use the definition ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

Using ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, we have

$$\begin{aligned}
&\frac{n!}{3!(n-3)!} + \frac{n!}{2!(n-2)!} = 0 \\
\implies\; &\frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2} = 0 \\
\implies\; &n(n-1)\left(\frac{n-2}{6} - \frac{1}{2}\right) = 0 \\
\implies\; &n(n-1)\,(n-5) = 0 \\
\implies\; &n = 0, 1, 5.
\end{aligned}$$

Since the equation is only defined for $n \geq 3$, the solution is $x = 5$.

1157. If $y = 2\sin(3x + \frac{\pi}{6})$, find $\dfrac{dy}{dx}$.

Use the chain rule.

Using the chain rule, with x in radians, we get

$$\frac{dy}{dx} = 2\cos(3x + \tfrac{\pi}{6}) \cdot 3 = 6\cos(3x + \tfrac{\pi}{6}).$$

1158. Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, relative to an origin O.

(a) Show that any point P on the line AB has position vector $\mathbf{p} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$, for $\lambda \in \mathbb{R}$.

(b) Point C is now defined with position vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$. Find an expression, in terms of a new parameter μ, for the position vector $\mathbf{q}$ of any point Q along the line OC.

(c) Hence, or otherwise, prove that the diagonals of a parallelogram bisect each other.

In (a), consider $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$.

(a) The position vector of P, situated a fraction λ of the way from A to B, may be calculated

$$\begin{aligned}
\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} \\
&= \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) \\
&= (1-\lambda)\mathbf{a} + \lambda\mathbf{b}.
\end{aligned}$$

(b) $\overrightarrow{OQ} = \mu\mathbf{c} = \mu(\mathbf{a} + \mathbf{b})$.

(c) Setting $\lambda = \mu = \frac{1}{2}$, the position vectors of the midpoints of the diagonals of parallelogram $OACB$ are $\mathbf{p} = \mathbf{q} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$. Since the midpoints coincide, the diagonals must bisect each other. $\square$

1159. Two triangles have side lengths $(11, 60, 61)$ and $(11, 100, 109)$. Show that they have the same area.

The first is a Pythagorean triple.

The first is a Pythagorean triple. So its area is $\frac{1}{2} \cdot 11 \cdot 60 = 330$. Using this area to determine the perpendicular height of the other triangle, we get $\frac{1}{2} \cdot 109 \cdot h = 330$, which gives $h = \frac{660}{109}$. This height splits the side of length 109 into two parts:

$$\sqrt{11^2 - \frac{660}{109}^2} \quad \text{and} \quad 109 - \sqrt{11^2 - \frac{660}{109}^2}.$$

Using the latter to calculate the third side gives 100, as required.

1160. The following graph is defined, in which a and b are distinct constants:

$$y = \frac{a^2 + 2ax + x^2}{b^2 + 2bx + x^2}.$$

Give the equations of the two straight lines which are asymptotes of the graph.

One asymptote is horizontal; the other is vertical.

Factorising the denominator as $(b+x)^2$ gives a (double) vertical asymptote at $x=-b$. Also, rewriting as

$$y=\frac{\frac{a^2}{x^2}+\frac{2a}{x}+1}{\frac{b^2}{x^2}+\frac{2b}{x}+1},$$

we can see that, as $x\to\pm\infty$, $y\to 1$. Hence, $y=1$ is a horizontal asymptote.

1161. Show that, when two dice are rolled, a sum of seven is expected to appear twice as often as nine.

Visualise the possibility space.

The expected number of appearances is proportional to probability. In the possibility space, there are 36 outcomes, of which 6 sum to seven and 3 sum to nine. Hence, a sum of seven is expected twice as often as nine.

1162. A polynomial graph $y=f(x)$, of odd degree, has exactly two distinct roots. Prove that, at at least one of these roots, the gradient must be zero.

A graph of odd degree must have a shape broadly like that of a cubic: overall, it crosses the x axis.

The graph has odd degree, so the curve must cross (not merely intersect) the x axis an odd number of times. This graph has exactly two roots, so these cannot both single roots: at least one must be a repeated root. At a repeated root, the gradient is zero, as required.

1163. There are n routes from A to B. Counting a trip and its reverse as distinct, find the number of different return trips, if the routes out and back

(a) may be the same,

(b) must be different.

Multiply the possibilities for the way out by the possibilities for the way back.

(a) For every one of n routes out, there are n routes back, so the total is n^2.

(b) For every one of n routes out, there are $n-1$ routes back, so the total is $n(n-1)$.

1164. Solve $(\sqrt{x}+x)^3=1$.

Cube both sides and solve as a quadratic in $\sqrt{x}$.

Cubing both sides gives $\sqrt{x}+x=1$, which is a quadratic in $\sqrt{x}$. It has solution

$$\sqrt{x}=\frac{-1\pm\sqrt{5}}{2}.$$

The negative root doesn't produce a real value for x. With the positive root, squaring gives

$$x=\frac{3-\sqrt{5}}{2}.$$

1165. The lines $y=x$ and $L_1:y=-(2+\sqrt{3})x$ meet at 60° at the origin. Show that the line $L_2:y=(\sqrt{3}-2)x$ also meets $y=x$ at 60°.

Show that L_1 and L_2 are are reflections in $y=x$.

Consider the reciprocal of the gradient of L_1:

$$\begin{aligned}&\frac{1}{-(2+\sqrt{3})}\\ =\;&\frac{2-\sqrt{3}}{-(2+\sqrt{3})(2-\sqrt{3})}\\ =\;&\frac{2-\sqrt{3}}{-1}\\ =\;&\sqrt{3}-2.\end{aligned}$$

This is the gradient of L_2, which tells us that L_2 is the reflection, in $y=x$, of L_1. Hence, the angle between L_2 and $y=x$ is the same as the angle between L_1 and $y=x$, as required.

1166. Either prove or disprove the following statement: "If two rectangles have the same perimeter and area, then they must be congruent."

The statement is true.

This statement is true. A rectangle is defined by two lengths; giving the perimeter and area is two pieces of information, which can be used to solve simultaneously and find the lengths. The quadratic involved will have one or two roots, but, if it has two, these will correspond to the two lengths, and $x=2$, $y=4$ is the same rectangle as $x=4$, $y=2$. Hence, two rectangles with the same perimeter and area must be congruent.

1167. An aeroplane is climbing slowly, after take-off, at constant velocity. Explain, using Newton's laws, how you know that the magnitudes of the following pairs of forces are equal:

(a) the lift experienced by the plane; the weight of the plane,

(b) the gravitational force of the Earth on the plane; the gravitational force of the plane on the Earth,

(c) the thrust on the plane; the drag on the plane.

In each, quote either NII or NIII.

(a) Velocity is constant, so there is no vertical acceleration. Therefore, by NII, these two vertical forces must be equal in magnitude.

(b) These are equal in magnitude by definition: they are a Newton III pair.

(c) Velocity is constant, so there is no horizontal acceleration. Therefore, by NII, these two horizontal forces must be equal in magnitude.

1168. The cubic $y = x^3 - x + 2$ has two stationary points and one point of inflection. Show that these three points are collinear.

Set the first and second derivatives to zero.

To find stationary points, we set

$$\frac{dy}{dx} = 3x^2 - 1 = 0.$$

This gives the stationary points as $(\pm\frac{1}{\sqrt{3}}, 2 \mp \frac{2}{\sqrt{3}})$. To locate the point of inflection, we set

$$\frac{d^2y}{dx^2} = 6x = 0,$$

which gives $(0, 2)$. The stationary points are symmetrically located around the point $(0, 2)$, which means the three are collinear.

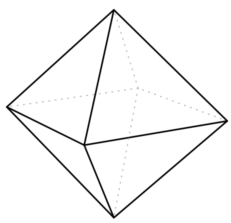
1169. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $(x - a)(x - b)(x - c) = 0$,
- $(x - a)(x - b) = 0$.

Consider the truth of each statement if $x = c$.

The implication is backwards (or upwards on the page): $\Longleftarrow$. The forwards implication doesn't hold: $x = c$ is the counterexample.

1170. A regular octahedron is shown below.



Two of the vertices are selected at random. Find the probability that they are joined by an edge.

Place one vertex, without loss of generality, and consider the possibility space as those remaining.

We can place one vertex without loss of generality. Of the remaining five vertices, four are connected to the original one by an edge. Hence, the required probability is $\frac{4}{5}$.

1171. Prove that, if two distinct parabolae of the form $y = f(x)$ are tangent at a point, then they cannot intersect elsewhere.

Consider the degree of the equation satisfied by the intersections.

Solving for intersections, $f(x) = g(x)$ can have degree at most two. It must have a double root, since we are told that the parabolae are tangent at a point. Since it has a double root, it cannot have any further roots. Hence, the parabolae cannot intersect elsewhere.

1172. A function has instruction

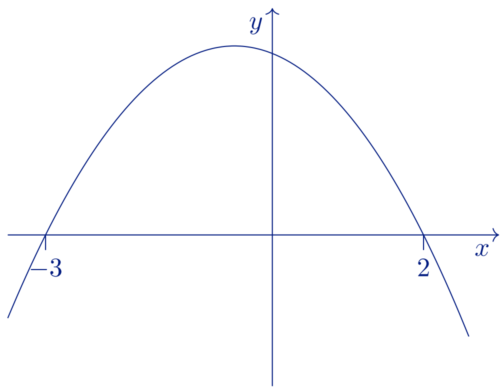
$$f : x \longmapsto \frac{1}{\sqrt{6 - x - x^2}}.$$

(a) Sketch the curve $y = 6 - x - x^2$.

(b) Hence, or otherwise, show that f is well defined over the domain $[-2, 1]$.

Show that the roots of the quadratic are not in the domain of the function.

(a) Sketch of $y = 6 - x - x^2$:



(b) The domain $[-2, 1]$ lies between the roots, which means that $6 - x - x^2 > 0$ for all input values. Hence, both square rooting and division are well defined.

1173. Solve $x - (\sqrt{x} - 1)^3 = 1$.

Multiply out with the binomial expansion, and simplify.

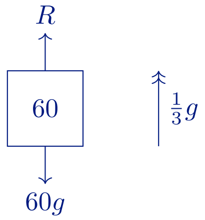
Multiplying out with the binomial expansion,

$$\begin{aligned}
& x - (x^{\frac{3}{2}} - 3x + 3x^{\frac{1}{2}} - 1) = 1 \\
\Longrightarrow\ & -x^{\frac{3}{2}} + 4x - 3x^{\frac{1}{2}} = 0 \\
\Longrightarrow\ & x^{\frac{1}{2}}(x - 4x^{\frac{1}{2}} + 3) = 0 \\
\Longrightarrow\ & x^{\frac{1}{2}}(x^{\frac{1}{2}} - 3)(x^{\frac{1}{2}} - 1) = 0 \\
\Longrightarrow\ & x^{\frac{1}{2}} = 0, 1, 3 \\
\Longrightarrow\ & x = 0, 1, 9.
\end{aligned}$$

1174. A strange individual of mass 60 kg is standing on a set of weighing scales in a lift which is accelerating upwards at $\frac{1}{3}g$ ms^{-2}. Determine the reading, in kg, on the scales.

The reading on a set of scales is a measurement of reaction force, not mass.

The forces on the strange individual are



The reading on the scales is a measurement of the reaction force exerted, converted to mass. $F = ma$ gives $R - 60g = 60 \cdot \frac{1}{3}g$, therefore $R = 80g$. Hence, the scales read 80 kg.

1175. The indefinite integral of a function g is cubic. Show that $y = g(x)$ has a stationary point.

The function g must be quadratic.

Since the indefinite integral of g is cubic, g, as the derivative of a cubic function, must be quadratic. Hence, $y = g(x)$ is a parabola. Every parabola has precisely one stationary point, the vertex.

1176. One of the following statements is true; the other is not. State, with a reason, which is which.

(a) $xyz^{-1} = 0 \implies x = 0$,

(b) $xy^{-1}z^{-1} = 0 \implies x = 0$.

The first statement is false.

(a) This is false: $y = 0$ is a counterexample.

(b) This is true, since neither y^{-1} nor z^{-1}, being reciprocals, can equal zero.

1177. Explain why one of the following expressions is well-defined and the other is not:

$$\lim_{h\to 0} \tfrac{1}{h}\big((x+h)^2 - x^2\big), \qquad \tfrac{1}{h}\big((x+h)^2 - x^2\big)\,\big|_{h=0}$$

The limit is well defined, but the evaluation is not.

The left-hand expression is well defined, as, before taking the limit, we can divide by (non-zero) h. Following this step, we can safely take the limit. The right-hand expression is ill defined, however, as evaluating at $h = 0$ gives $\frac{1}{0} \cdot 0$.

1178. Two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ are perpendicular, and both of unit magnitude. Show that $a_1b_1 + a_2b_2 = 0$.

Use gradients.

The gradients of these perpendicular vectors must multiply to give negative 1:

$$\begin{aligned} &\frac{a_2}{a_1} \cdot \frac{b_2}{b_1} = -1 \\ \implies &\frac{a_2}{a_1} = -\frac{b_1}{b_2} \\ \implies &a_2b_2 = -a_1b_1 \\ \implies &a_1b_1 + a_2b_2 = 0, \text{ as required.} \end{aligned}$$

1179. State, with a reason, whether the following claims are true in the Newtonian system:

(a) "If an object has negligible mass, then the resultant force on it is negligible."

(b) "If an object has negligible weight, then the resultant force on it is negligible."

Remember that mass is fixed in the Newtonian system, but weight is not.

(a) This is true. If an object has negligible mass, then $F = ma \approx 0 \cdot a = 0$ guarantees that resultant force will be negligible, barring astronomical accelerations (for which the system would break down anyway).

(b) This is false. A massive object can be weightless in space, yet still have a resultant force applied (by thrusters, or suchlike).

1180. Disprove the following statement:

$$\text{If } f^2(x) \equiv g^2(x), \text{ then } f(x) \equiv g(x).$$

The notation $f^2(x)$ means $f(f(x))$.

This is false. A counter example is $f(x) = x$ and $g(x) = -x$. In both cases, the composition gives $f^2(x) = g^2(x) = x$, but the functions are distinct.

1181. A regular hexagon has two adjacent vertices at $(0, 0)$ and $(6, 8)$.

(a) Find the side length.

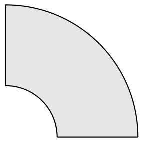
(b) Show that the area of the hexagon is $150\sqrt{3}$.

Consider the hexagon as six equilateral triangles.

(a) By Pythagoras, the side length is 10.

(b) The area of an equilateral triangle of side length l is $A = \frac{\sqrt{3}l^2}{4}$. The hexagon consists of six such triangles with side length 10, so it has area $6 \cdot \frac{100\sqrt{3}}{4} = 150\sqrt{3}$.

1182. The shape below consists of two concentric arcs, and of two perpendicular part-radii. The shaded region has perimeter $P = 4 + \frac{\pi}{2}$.



Show that the area shaded is $\frac{1}{4}\pi + 1$.

Call the shorter arc length l, set up an equation, and solve.

Call the shorter arc length l. Its radius length is then $r = \frac{2l}{\pi}$. Hence, the radius of the longer arc is $R = (1 + \frac{2l}{\pi})$, and its arc length is therefore $\frac{1}{2}\pi(1 + \frac{2l}{\pi})$, which simplifies to $\frac{1}{2}\pi + l$. The perimeter gives us the following equation:

$$3l + \tfrac{1}{2}\pi + l = 4 + \tfrac{1}{2}\pi.$$

So, $l = 1$. Hence, the area of the shaded region is

$$\begin{aligned} A &= \tfrac{1}{4}\pi(R^2 - r^2) \\ &= \tfrac{1}{4}\pi\left(\left(1 + \tfrac{2}{\pi}\right)^2 - \tfrac{2}{\pi}^2\right) \\ &= \tfrac{1}{4}\pi\left(1 + \tfrac{4}{\pi}\right) \\ &= \tfrac{1}{4}\pi + 1 \end{aligned}$$

1183. A function f is such that $0 \le f(x) \le 1$ for all x in the domain $\mathbb{R}$. State, with a reason, whether the following are necessarily true, with k as a positive constant,

(a) $kf(x) \in [0, k]$ for all $x \in \mathbb{R}$,

(b) $x \mapsto kf(x)$ has range $[0, k]$ over $\mathbb{R}$.

The second statement is not true.

(a) This is true. We know that $0 \le f(x) \le 1$; scaling this by positive k gives $0 \le kf(x) \le k$.

(b) This is not true. We don't that $f(x)$ takes *every* value in the interval $[0, 1]$, only that its outputs lie in that interval. So, we don't know that $kf(x)$ takes every value in $[0, k]$. The range will be a subset of $[0, k]$.

1184. Differentiate $y = \sqrt{x}$ from first principles.

Use the same technique as for rationalising the denominator of surds.

We set up the limit as follows:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

1185. Write the following in simplified interval notation

(a) $\{x \in \mathbb{R} : |x| < 2\} \cap [1, 3]$,

(b) $\{x \in \mathbb{R} : |x| > 2\} \cap [1, 3]$,

(c) $\{x \in \mathbb{R} : |x| \le 2\} \cap [1, 3]$.

Visualise/draw the sets on a number line.

(a) $[1, 2)$,

(b) $(2, 3]$,

(c) $[1, 2]$.

1186. Describe the transformation that takes the graph $y = x^2 + 3x + 1$ onto the graph $y = x^2 - 3x + 1$.

Complete the square on both.

Completing the square, we have $y = (x + \frac{3}{2})^2 - \frac{5}{4}$ and $y = (x - \frac{3}{2})^2 - \frac{5}{4}$. These are related by either reflection in the y axis, or equivalently translation by vector $3\mathbf{i}$.

1187. A firework rocket weighing 2 kg blasts off from rest at ground level. A vertical driving force of 31.6 Newtons acts on the rocket for 4.0 seconds, until the fuel is exhausted. At that point, the rocket becomes a projectile. Assume the mass is constant throughout.

(a) Find the speed at which the rocket is travelling when the fuel runs out.

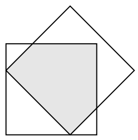
(b) Determine the greatest height achieved by the rocket during the motion.

Draw a force diagram to calculate a, and then use *suvat* to find the height and speed at which the fuel runs out. Use the final velocity of the first stage as the initial velocity of the second stage.

(a) Vertical $F = ma$ is $31.6 - 2g = 2a$, so $a = 6$ ms^{-2}. Over 4 seconds, starting from rest, this gives 24 ms^{-1}.

(b) The height attained during the first phase is $s = \frac{1}{2} \cdot 6 \cdot 4^2 = 48$. In the subsequent projectile motion, the vertical displacement is given by $0 = 24^2 - 2gh$, so $h = 29.3877...$. The greatest height is $48 + 29.3877... = 77.4$ m (3sf).

1188. Two unit squares are placed as depicted below. All acute angles in the diagram are 45°.



Determine the exact area of the shaded region.

Calculate the lengths of the sides of the small isosceles triangles.

The large unshaded isosceles triangle has perpendicular sides of length $\frac{\sqrt{2}}{2}$, hence area

$$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{4}.$$

The small unshaded isosceles triangles have perpendicular sides of length $1 - \frac{\sqrt{2}}{2}$, hence area

$$\frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)^2 = \frac{3 - 2\sqrt{2}}{4}.$$

The shaded area is then calculated by subtracting these areas from 1, giving

$$\begin{aligned} & 1 - \frac{1}{4} - 2 \cdot \frac{3 - 2\sqrt{2}}{4} \\ = \ & \sqrt{2} - \frac{3}{4}. \end{aligned}$$

1189. The equation $f(x) = 0$ has exactly one root, at $x = a$. Solve the following equations, giving your answers in terms of a:

(a) $f(2x) = 0$,

(b) $f(x - 1)f(x + 1) = 0$.

(a) Consider the necessary value of $2x$.

(b) If two numbers multiply to give zero, then...

(a) To get zero, we require $2x = a$, so $x = \frac{1}{2}a$.

(b) We require either $x-1 = a$ of $x+1 = a$. Hence, $x = a \pm 1$.

1190. Disprove the following statement: "The difference of any two irrational numbers is irrational."

Engineer a counterexample in the form $a + b$ and $a + c$.

A counterexample is π and $\pi + 1$, both of which are irrational, but whose difference is 1, which is rational.

1191. A student has tried to solve the simultaneous equations $3x = 13 + 4y$ and $6x - 8y = 7$, and has found that something goes wrong in the standard technique. Explain, using a graph or otherwise, what the correct interpretation of this fact is.

Sketch the two lines carefully.

Doubling and rearranging the first equation, we get $6x - 8y = 26$. Considered as a straight line graph, this is parallel to the first equation. It is correct, therefore, that the standard technique should break down, as there are no intersections.

1192. Show that, if $y = x^2$, then

$$\int_0^k y\, dx + \int_0^{k^2} y\, dy = k^3.$$

Calculate the two integrals explicitly, noting that the second is an integral with respect to y.

Evaluating the integrals:

$$\begin{aligned} & \int_0^k y\, dx + \int_0^{k^2} x\, dy \\ = & \int_0^k x^2\, dx + \int_0^{k^2} y^{\frac{1}{2}}\, dy \\ = & \left[\tfrac{1}{3}x^3\right]_0^k + \left[\tfrac{2}{3}y^{\frac{3}{2}}\right]_0^{k^2} \\ = & \tfrac{1}{3}k^3 + \tfrac{2}{3}k^3 \\ = & k^3, \text{ as required.} \end{aligned}$$

1193. The two roots of the quadratic $x^2 + px + q = 0$ differ by 4 and their sum is 2. Find p and q.

Consider the difference and sum of the roots, as given by the quadratic formula.

The quadratic formula gives the difference between the roots as $\sqrt{p^2 - 4q} = 4$ and the sum as $-p = 2$. Hence, $p = -2$, and $q = -3$.

1194. Solve $\dfrac{\sqrt{x} - 1}{\sqrt{x} + 1} = 2 - \sqrt{x}$.

Multiply both sides by $\sqrt{x} + 1$.

Multiplying up:

$$\begin{aligned}
&\frac{\sqrt{x}-1}{\sqrt{x}+1} = 2-\sqrt{x}\\
\Longrightarrow\ &\sqrt{x}-1 = (2-\sqrt{x})(\sqrt{x}+1)\\
\Longrightarrow\ &\sqrt{x}-1 = 2+\sqrt{x}-x\\
\Longrightarrow\ &-1 = 2-x\\
\Longrightarrow\ &x = 3
\end{aligned}$$

1195. In this question, the vectors $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and non-zero.

(a) Determine the values of p and q to make the following an identity:

$$p(\mathbf{a}+\mathbf{b})+q(\mathbf{a}-\mathbf{b}) = 4\mathbf{a}+6\mathbf{b}.$$

(b) Explain how you used the facts that $\mathbf{a}$ and $\mathbf{b}$

i. are non-parallel,

ii. are non-zero.

Compare the coefficients of the two vectors.

(a) Comparing coefficients of $\mathbf{a}$ and $\mathbf{b}$, we get $p+q=4$ and $p-q=6$. Solving simultaneously gives $p=5$, $q=-1$.

(b) i. If the vectors were parallel, we wouldn't have been able to split the equation up into an $\mathbf{a}$ equation and a $\mathbf{b}$ equation:

$$\begin{aligned}
p\mathbf{a}+q\mathbf{a} &= 4\mathbf{a}\\
p\mathbf{b}-q\mathbf{b} &= 6\mathbf{b}
\end{aligned}$$

ii. If either of the vectors had been equal to zero, we wouldn't have been able to extract the coefficients from the relevant equation. For example:

$$\begin{aligned}
&p\mathbf{a}+q\mathbf{a} = 4\mathbf{a}\\
\not\Longrightarrow\ &p+q = 4
\end{aligned}$$

because $\mathbf{a}$ could be zero.

1196. An AP has first term a and common difference d. Prove that, for such a sequence:

(a) the n^{th} term is given by $u_n = a+(n-1)d$,

(b) the sum of the first n terms is given by $S_n = \frac{1}{2}n\left(2a+(n-1)d\right)$.

(a) Consider the number of steps required to get to the n^{th} term.

(b) Consider the mean of the sequence.

(a) The number of steps required to get to the n^{th} term is $(n-1)$. Each of these steps adds the common difference d. Hence, the n^{th} term is $u_n = a+(n-1)d$.

(b) The mean of an AP is the mean of the first and last terms. Using the n^{th} term formula, this is given by $\frac{1}{2}(a+(a+(n-1)d))$, which simplifies to $\frac{1}{2}(2a+(n-1)d)$. To get the required formula, we multiply this mean by the number of terms in the sequence, n.

1197. Show that, if h is a linear function defined over $\mathbb{R}$, then, for any $a \neq b$, the following is constant:

$$\frac{h(b)-h(a)}{b-a}.$$

Consider the expression as a gradient.

The expression is a gradient formula for $y=h(x)$: the numerator is Δy, and the denominator is Δx. Since h is a linear function, $y=h(x)$ is a straight line, which means this gradient is constant.

1198. Eliminate a from the following equations, to find a relation of the form $f(x)=g(y)$.

$$\begin{aligned}
2ax+y &= 4,\\
3x-ay &= 1.
\end{aligned}$$

Use either elimination or substitution.

Multiplying up gives

$$\begin{aligned}
2axy+y^2 &= 4y,\\
6x^2-2axy &= 2x
\end{aligned}$$

Adding the two equations, we get

$$\begin{aligned}
&6x^2+y^2 = 2x+4y\\
\Longrightarrow\ &6x^2-2x = 4y-y^2.
\end{aligned}$$

1199. In a gliding competition, the challenge is to travel around the triangle of greatest area. The winner performs three legs, of distances 10, 17 and 21 km. Find the area of the triangle.

Use the cosine rule to find an angle, then the sine area formula. (Or use Heron's formula, if you happen to know it.)

The cosine rule produces an angle

$$\theta = \arccos\frac{10^2+17^2-21^2}{2\cdot 10\cdot 17}.$$

The sine area formula then gives

$$A = \tfrac{1}{2}\cdot 10\cdot 17\sin\theta = 84.$$

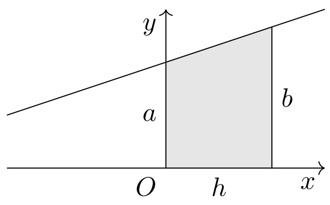
1200. If $f(x) = \dfrac{1}{1+x}$, prove that $f^2(x) = \dfrac{x+1}{x+2}$. Insert the output $f(x)$ in as the input of the function, and simplify the inlaid fractions. We need to simplify

$$f^2(x) = \frac{1}{1+\frac{1}{1+x}}.$$

Using the standard technique, we multiply top and bottom of the large fraction by the denominator of the small fraction, which gives

$$f^2(x) = \frac{1+x}{1+x+1} = \frac{1+x}{2+x}.$$

1201. By finding the equation of the line and setting up an integral, verify the area formula $A = \frac{1}{2}(a+b)h$ for the right-angled trapezium depicted below.



The line is $y = a + \frac{b-a}{h}x$.

The line is $y = a + \dfrac{b-a}{h}x$. So, the area is

$$\begin{aligned} A &= \int_0^h a + \frac{b-a}{h}x\,dx \\ &= \left[ax + \frac{b-a}{2h}x^2\right]_0^h \\ &= ah + \frac{b-a}{2h}h^2 \\ &= \tfrac{1}{2}(2a+b-a)h \\ &= \tfrac{1}{2}(a+b)h. \end{aligned}$$

1202. True or false?

(a) $\dfrac{x^2-1}{x^2+1} = 0$ has two roots,

(b) $\dfrac{x^2-1}{x^2+x} = 0$ has two roots,

(c) $\dfrac{x^2-1}{x^2-2} = 0$ has two roots.

Consider common factors.

Each numerator has roots $x = \pm 1$. So, the overall equation will have exactly these roots, unless the denominator also does.

(a) True, as the denominator is never zero.

(b) False, as the denominator is zero at $x = 0, -1$.

(c) True, as the denominator is zero at $x = \pm\sqrt{2}$.

1203. Find an irrational number which is a member of the set $[3.6, 3.61]$.

Construct one by dividing e.g. π by a large number, and adding it to 3.6.

There are infinitely many we could construct. For example, $\frac{\pi}{1000} = 0.003141...$ is less than 0.01, and is irrational. Hence, $3.6 + \frac{\pi}{1000}$ satisfies the conditions of the question.

1204. A graph has equation $y = \dfrac{2x+7}{x+3}$.

(a) Express the graph as $y = a + \dfrac{b}{x+3}$.

(b) Hence, sketch the curve.

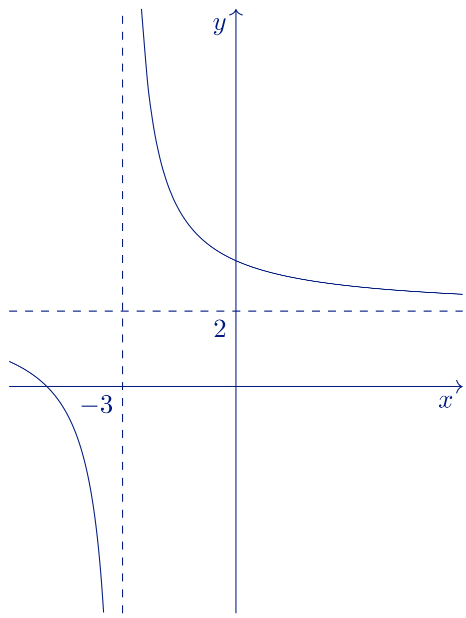
(a) Write the numerator in terms of $(x+3)$.

(b) Consider the graph as $y = \frac{1}{x}$ transformed.

(a) We write $2x+7 = 2(x+3)+1$. Hence, the graph is

$$y = 2 + \frac{1}{x+3}.$$

(b) The curve is a transformed version of the standard reciprocal graph $y = \frac{1}{x}$. Replacing x by $x+3$ translates the curve by $-3\mathbf{i}$, and adding two to the outputs translates it by $2\mathbf{j}$. Hence, the graph has asymptotes at $x = -3$ and $y = 2$:



1205. A set of 100 straight lines is drawn in a plane, such that no three lines are concurrent and no two lines are parallel. Find the number of intersections.

Consider the fact that, since no two lines are parallel, every pair of lines must cross exactly once.

Since no two lines are parallel, every pair crosses exactly once. There are ${}^{100}C_2$ such pairs. And, since no three lines are concurrent, these points are all distinct. Hence, the total number is ${}^{100}C_2 = 4950$ intersections.

1206. A differential equation is given as

$$\frac{dy}{dx} = 2x - 3y.$$

A linear solution $y = mx + c$ is suggested.

(a) Write down $\dfrac{dy}{dx}$.

(b) Substitute to show that m and c must satisfy the identity $3mx + m \equiv 2x - 3c$.

(c) Hence, show that the only linear solution to the differential equation is $9y - 6x + 2 = 0$.

In (c), use the fact that the formula in (b) is an identity to equate coefficients.

(a) $\dfrac{dy}{dx} = m$.

(b) Substituting, we get

$$\begin{aligned} m &= 2x - 3(mx + c) \\ \implies 3mx + m &\equiv 2x - 3c. \end{aligned}$$

(c) Since the above is an identity, we can equate coefficients of x^1 and x^0. This gives

$$\begin{aligned} x^1 &: 3m = 2 \\ x^0 &: m = -3c. \end{aligned}$$

Hence, $m = \frac{2}{3}$, $c = -\frac{2}{9}$ is the only solution. Multiplying $y = mx + c$ by 9 and rearranging gives the required result.

1207. Solve the equation $x^5 = 8x^2$.

Rearrange to the form $f(x) = 0$ and factorise.

Rearranging and factorising produces $x^2(x^3 - 8) = 0$. This gives the solution as $x = 0$ or $x = 2$.

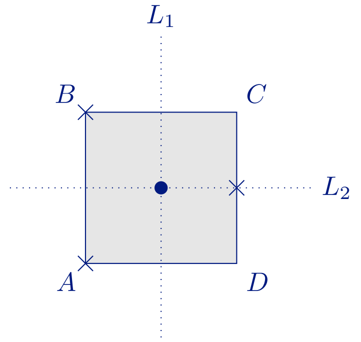
1208. A uniform solid cube of mass m has vertices $ABCDEFGH$, where $ABCD$ is the horizontal lower face. The cube is supported by vertical forces at A, B and the midpoint of CD.

(a) Explain, using a plan sketch, why the forces at A and B must have the same magnitude.

(b) Find the magnitudes of the forces.

Draw a plan view of the square $ABCD$, marking the forces on as points. (They are perpendicular to the square.)

(a) Drawing the three reaction forces and the weight as points (all forces are perpendicular to the diagram), we have



Since the diagram has L_2 as a line of symmetry, the forces R_A and R_B must be equal.

(b) Taking moments about line L_1, we get $2R_A = R_{CD}$. Then $F = ma$ perpendicular to $ABCD$ gives $2R_A + R_{CD} = mg$. Solving these, we have $R_A = R_B = \frac{1}{4}mg$, $R_{CD} = \frac{1}{2}mg$.

1209. Write down the value of $\displaystyle\int_{-1}^{1} \sqrt{1 - x^2}\, dx$.

Consider the equation of a circle.

The integrand is a semicircular function. Hence, the value of the integral is the area of a unit semicircle, $\frac{1}{2}\pi$.

1210. State, with a reason, whether $x = k$, where $k \in \mathbb{R}$ is a constant, intersects the following curves:

(a) $y = x^2 + k + 1$,

(b) $x = y^2 + k + 1$.

(a) Consider the fact that $x = k$ is a vertical line, parallel to the y axis.

(b) Solve simultaneously, and use the discriminant.

(a) Since $x = k$ is a line parallel to the y axis, it must intersect a curve which is defined for all real values of x.

(b) Solving simultaneously, we have $y^2 + 1 = 0$, which has no real roots. Hence, there are no intersections.

1211. A square is undergoing an enlargement. The rate of change of the lengths of its sides is 2 cm/s. Find the rate of change of the lengths of its diagonals.

Express the relationship between a, the side length, and b, the diagonal length, as an equation, and differentiate it with respect to time.

The relationship between a, the side length, and b, the diagonal length, is $b = \sqrt{2}a$. Differentiating this with respect to time tells us that the rates of change are related as

$$\frac{db}{dt} = \sqrt{2}\frac{da}{dt}.$$

We are told that $\dfrac{da}{dt} = 2$, so $\dfrac{db}{dt} = 2\sqrt{2}$ cm/s.

1212. This question is about the use of the Newton-Raphson iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to factorise $f(x) = x^3 - 5x^2 - 2x - 24$.

(a) Set up the iteration for f.

(b) Run the iteration with $x_0 = 0$, and verify that it converges to a root α of f.

(c) Explain why $(x - \alpha)$ must be a factor of $f(x)$.

(d) Factorise $f(x)$ into a linear and a quadratic factor.

(e) Use the discriminant $\Delta = b^2 - 4ac$ to show that no further factorisation is possible.

Part (c) is the factor theorem.

(a) $x_{n+1} = x_n - \dfrac{x^3 - 5x^2 - 2x - 24}{3x^2 - 10x - 2}$,

(b) Running the iteration with $x_0 = 0$ gives $x_n \to 6$. Substituting, we get $f(6) = 6^3 - 5 \cdot 6^2 - 2 \cdot 6 - 24 = 0$, which verifies that $x = 6$ is a root.

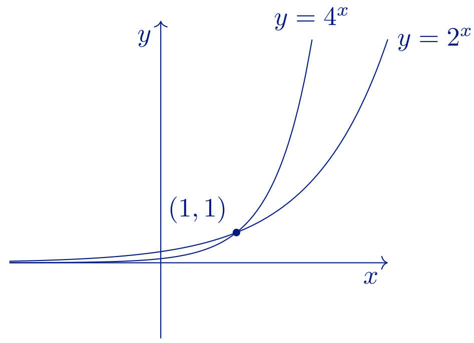
(c) The factor theorem states that, if $x = \alpha$ is a root of a polynomial, then $(x - \alpha)$ is a factor.

(d) $f(x) = (x - 6)(x^2 + x + 4)$.

(e) The discriminant of the quadratic is $\Delta = -15 < 0$. Hence, the quadratic has no real roots. Thus, over the reals $\mathbb{R}$, no further factorisation is possible.

1213. On one set of axes, sketch the graphs $y = 2^{x-1}$ and $y = 4^{x-1}$, marking any points of intersection.

Consider $y = 2^x$ and $y = 4^x$ first, then translate them. The exponential graphs $y = 2^x$ and $y = 4^x$ both pass through $(0, 1)$. Since the latter is $y = 4^x = (2^2)^x = 2^{2x}$, it is the former after a stretch, factor $\frac{1}{2}$, in the x direction. The graphs in the question are then translated by $\mathbf{i}$, since x has been replaced by $x - 1$. This gives



1214. If $\dfrac{d}{dx}(x^2 + 2y) = 10$, find $\dfrac{dy}{dx}$ in terms of x.

Enact the operation "differentiate with respect to x", as encoded in the differential operator $\frac{d}{dx}(\ )$, and rearrange.

Enacting the operation "differentiate with respect to x", as encoded in the differential operator $\frac{d}{dx}(\ )$, we get $2x + 2\frac{dy}{dx} = 10$. Thus $\frac{dy}{dx} = 5 - x$.

1215. A statistician is setting up a hypothesis test for the mean μ of a normal distribution $X \sim N(\mu, \sigma^2)$. Under the assumption of the null hypothesis, an acceptance region (a, b) is calculated, such that $P(a < \bar{X} < b) = 0.99$.

(a) Is the test one-tail or two-tail?

(b) What is the significance level of the test?

The acceptance region is the complement of the critical region.

(a) The test is two tail, as the acceptance region is bounded at both ends.

(b) The significance level is 1%.

1216. A geometric sequence has n^{th} term u_n. Show that, for any constants $p, q \in \mathbb{R}$, $w_n = pu_n + qu_{n+1}$ is also geometric.

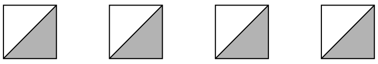
Simplify the ratio $\dfrac{w_{n+1}}{w_n}$.

We know hat $u_n = ar^{n-1}$, for some $a, r \in \mathbb{R}$. To prove that w_n is also geometric, we need to prove that the ratio of consecutive terms is constant. So, we consider

$$\begin{aligned} &\frac{w_{n+1}}{w_n} \\ &= \frac{pu_{n+1} + qu_{n+2}}{pu_n + qu_{n+1}} \\ &= \frac{par^n + qar^{n+1}}{par^{n-1} + qar^n} \\ &= \frac{pr + qr^2}{p + qr}. \end{aligned}$$

Since p, q, r are all constant, this is constant.

1217. The four tiles below are placed together, in random orientations, to form a two-by-two square.



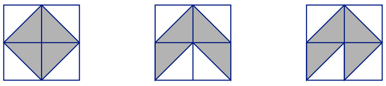
(a) Show that there are 13 configuration in which the shading forms one region.

(b) Hence, find the probability that the shading forms one region.

Count up the cases by exhaustion, giving $1+4+8=13$.

(a) Counting up the cases:

1 case | 4 cases | 8 cases



(b) Each small square has 4 possible orientations, so there are $4^4 = 256$ possible configurations. Of these, 13 are successful. So, the required probability is $\frac{13}{256}$.

1218. Either prove or disprove the following statement: "The product of irrational numbers is irrational."

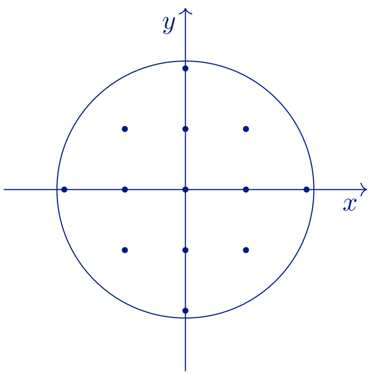
The statement is false.

The statement is false. We need only provide a counterexample, such as $\sqrt{2}\times\sqrt{2}=2$.

1219. Determine the number of points (x, y) for which $x, y \in \mathbb{Z}$ and $2x^2 + 2y^2 < 9$.

Count the points by exhaustion.

The inequality is the interior of a circle centred at the origin. The relevant points then form a square, with vertices at $(0, \pm 2)$ and $(\pm 2, 0)$. There are 13 points in the square.



1220. The equations $f(x) = 0$ and $g(x) = 0$, where f and g are linear functions defined over $\mathbb{R}$, have the same solution set S. The equation $f(x) = g(x)$ is denoted E. State, with a reason, whether the following claims hold:

(a) "E has solution set S",

(b) "the solution set of E contains S",

(c) "the solution set of E is a subset of S".

Since the equations are linear and have non-zero derivatives, S must consist of a single element s.

(a) This is not true. If f and g are the same function, then E has solution set $\mathbb{R}$.

(b) This is true. If $s \in S$, then $f(s) = g(s) = 0$, so s must also be in the solution set of E.

(c) This is not true. If f and g are the same function, then E has solution set $\mathbb{R}$.

1221. Show that $\displaystyle\int_1^2 \frac{1}{x^2} + \frac{2}{x^3}\,dx = \frac{5}{4}$.

Write the integral as $\displaystyle\int_1^2 x^{-2} + 2x^{-3}\,dx$.

$$\begin{aligned}
&\int_1^2 x^{-2} + 2x^{-3}\,dx \\
=\ &\Big[-x^{-1} - x^{-2}\Big]_1^2 \\
=\ &\left(-2^{-1} - 2^{-2}\right) - \left(-1^{-1} - 1^{-2}\right) \\
=\ &\left(-\tfrac{3}{4}\right) - (-2) \\
=\ &\tfrac{5}{4}.
\end{aligned}$$

1222. A reaction force is so called because it occurs in "reaction" to contact. Explain whether or not it is possible for an object to experience a reaction force while weightless.

The modern use of the word *reaction* is not per Newton's "action and reaction". We now use reaction force to refer to any contact force acting perpendicular to the surfaces in contact.

Yes, this is entirely possible. A "reaction" force is any contact force which is perpendicular to the surfaces in contact. Two floating astronauts, even if they are weightless, can exert reaction forces on one another by pushing.

1223. Find the sum of the first 1000 odd numbers.

Consider the numbers as a sequence with first term 1 and common difference 2.

The odd numbers are a sequence with first term 1 and common difference 2. Using the standard formula, the sum is

$$S_{1000} = \tfrac{1000}{2}\big(2\cdot 1 + (1000-1)2\big) = 1000000.$$

1224. The quadratic functions f, g, h have discriminants which are negative, zero, and positive respectively. The functions share no roots. Write down the numbers of roots of the following equations:

(a) $f(x)g(x) = 0$,

(b) $f(x)g(x)h(x) = 0$,

(c) $g(x)(h(x))^{-1} = 0$.

Since the functions share no roots, the factors can be treated separately. The solution sets for the individual factors combine.

The functions f, g, h have $0, 1, 2$ roots respectively.

(a) 1 root.

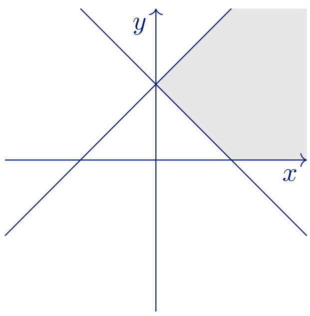
(b) 3 roots.

(c) 1 root, since $h(x)^{-1}$ can never be zero.

1225. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$y > x + 1, \qquad y < 1 - x.$$

The region is a right-angled quadrant bounded by two straight lines.



1226. A sample has $\bar{x} = 1.17$, $\sum x^2 = 227$, and variance 0.9011. Find the number of data in the sample.

Use $s^2 = \dfrac{\sum x^2 - n\bar{x}^2}{n}$

Using $s^2 = \dfrac{\sum x^2 - n\bar{x}^2}{n}$, we get

$$0.9011 = \frac{227 - n\cdot 1.17^2}{n}.$$

This gives $0.9011n = 227 - 1.17^2 n$, hence

$$n = \frac{227}{0.9011 + 1.17^2} = 100.$$

1227. The scores a and b on two exams, out of A and B marks respectively, are to be combined into one score X, given out of a hundred. Each mark is to have equal weighting. Find a formula for X.

Consider scale factor required to act on $a + b$.

The total score is $a+b$, which is out of $A+B$. The scale factor required to take the number of total marks available to 100, therefore, is $\frac{100}{A+B}$. So, the formula is

$$X = \frac{100(a+b)}{A+B}.$$

1228. Prove that a pair of distinct circles cannot have more than two intersections.

Prove this by contradiction. Show that the centres and radii of the circles must be the same.

Assume, for a contradiction, that a pair of distinct circles has (at least) three distinct intersections. By constructing perpendicular bisectors on these three points, we can show that the two circles have the same centre. If they intersect, they must also have the same radius. Hence, they are the same circle, i.e. not distinct. This is a contradiction.

Hence, a pair of distinct circles cannot have more than two intersections.

1229. A_n is a quadratic sequence and B_n is arithmetic. Describe the following sequences as quadratic, arithmetic or neither:

(a) $u_n = (A_n)^2$,

(b) $u_n = (B_n)^2$.

Propose generic ordinal formulae for A_n and B_n.

(a) A_n has ordinal formula $A_n = pn^2 + qn + r$, for some $p, q, r \in \mathbb{R}$. Hence, $(A_n)^2$ is quartic, which is neither arithmetic nor quadratic.

(b) B_n has ordinal formula $B_n = p + qn$, for some $p, q \in \mathbb{R}$. Hence, $(B_n)^2$ is quadratic.

1230. A jar contains 6 blue and 4 red counters. From the jar, two counters are taken out at random. Find the probability that these are different colours.

Consider the two possibilities BR and RB.

There are two possibilities, BR and RB. Each has probability $\frac{6\times 4}{10\times 9}$. Hence, the total probability is

$$P(\text{different colours}) = 2 \times \frac{6\times 4}{10\times 9} = \frac{8}{15}.$$

1231. Prove that, for any function f and constant c, the curves $y = f(x)$ and $y = f(x) + c$ have the same set of x values at which they are stationary.

Consider the transformation between the two curves.

The transformation between the two curves is a translation in the y direction by c. This does not affect the shape of the curve. Hence, the stationary points will remain at the same x values.

1232. Show that, as $x \longrightarrow 0$, the gradient of the tangent to $y = \sqrt{x}(x-1)$ grows without bound.

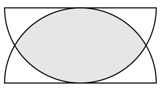
Expand the brackets before differentiating.

Expanding the brackets and differentiating, we get

$$\lim_{x\to 0} \frac{dy}{dx} = \lim_{x\to 0} \left(\tfrac{3}{2}x^{\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}}\right).$$

Since $\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, this grows without bound (asymptotically) as $x \to 0$.

1233. Two semicircles, with parallel diameters of length 2, overlap symmetrically as follows:



Show that the unshaded region has area $\sqrt{3} - \frac{1}{3}\pi$.

Draw a line joining the two points of intersection. Calculate the area of one segment with sector minus triangle, then double it.

Half of the shaded area is a segment which subtends an angle of 120° at the centre. The associated sector has area $\frac{1}{3}\pi$. The associated triangle has area $\frac{1}{2}\sin 120° = \frac{\sqrt{3}}{4}$. Therefore, the shaded area is $\frac{2}{3}\pi - \frac{\sqrt{3}}{2}$. The unshaded area is then

$$\pi - 2\left(\tfrac{2}{3}\pi - \tfrac{\sqrt{3}}{2}\right) = \sqrt{3} - \tfrac{1}{3}\pi \text{ as required.}$$

1234. Verify that the quadratic $2x^2 + 7x - 4$ is a factor of the cubic $2x^3 - 7x^2 - 53x + 28$.

Do the factorisation explicitly.

Factorising, we have

$$2x^3 - 7x^2 - 53x + 28 = (2x^2 + 7x - 4)(x - 7).$$

1235. A number is given in standard form as $z = a \times 10^b$.

(a) Write down the set of possible values of a.

(b) Show that $\log_p z$ is linear in b, for any $p > 0$.

"Linear in b" means expressible as $mb+c$, for some constants m and c.

(a) $a \in [1, 10)$.

(b) Using log rules, we can express $\log_p z$ as

$$\log_p \left(a \times 10^b\right) = \log_p a + b \log_p 10.$$

This is of the form $mb + c$, with $m = \log_p 10$ and $c = \log_p a$, so $\log_p z$ is linear in b.

1236. Take $g = 10$ ms^{-2} in this question, and ignore air resistance.

A rocket of mass 120 kg is fired vertically into the air from rest. Upon ignition, due to its boosters, it feels a constant force of 1500 N vertically.

(a) Find the time taken for the rocket to reach an altitude of 500 m.

(b) At this time, the fuel is exhausted. Determine the speed at which the rocket hits the ground.

Draw a force diagram, find the acceleration, and then use $s = ut + \frac{1}{2}at^2$.

(a) $F = ma$ gives $1500 - 120 \times 10 = 120a$. Hence, $a = \frac{300}{120} = 2.5$ ms^{-2}. Then, $s = ut + \frac{1}{2}at^2$ gives $500 = \frac{1}{2} \times 2.5 \times t^2$. Taking the positive square root, we get $t = 20$ seconds.

(b) When the fuel runs out, the vertical speed is $u = 20 \times 2.5 = 50$ ms^{-1}, and the height is 50 ms^{-1}. So, for hitting the ground, we have

s	-50
u	50
v	v
a	-10
t	not relevant

$v^2 = u^2 + 2as$ gives $v^2 = 50^2 + 2 \cdot -10 \cdot -50$. The speed at which the rockets hits the ground is $v = 59.160... = 59.2$ ms^{-1} (3sf).

1237. The expression $k - 7x + 6x^2$ has discriminant 121. Find the value of the constant k.

The discriminant is $\Delta = b^2 - 4ac$. Setting $\Delta = 121$, we have $49 - 24k = 121$. Hence, $k = 3$.

1238. Evaluate $\displaystyle\lim_{x\to 2} \frac{2x^2 + 6x - 20}{3x^2 - 7x + 2}$.

Look for common factors on the top and bottom.

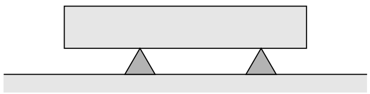
Factorising the top and bottom, we have

$$\begin{aligned} &\lim_{x\to 2} \frac{2x^2 + 6x - 20}{3x^2 - 7x + 2} \\ =& \lim_{x\to 2} \frac{(2x-4)(x+5)}{(3x-1)(x-2)} \\ =& \lim_{x\to 2} \frac{2(x+5)}{3x-1} \end{aligned}$$

At this point, having cancelled factors of $(x-2)$, we can take the limit. This gives

$$\lim_{x\to 2} \frac{2(x+5)}{3x-1} = \frac{14}{5}.$$

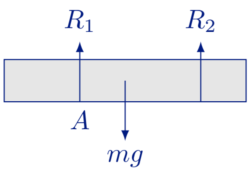
1239. A uniform block of mass m kg rests on supports, as depicted. The supports divide the length of the block into sections in the ratio $2:3:1$.



Find the reaction forces at the supports.

Draw a force diagram of the block, and set up two equations: vertical $F = 0$ and moments around a sensible point (one of the supports, say).

Drawing a force diagram, we have



$$\begin{aligned} \updownarrow &: \quad R_1 + R_2 - mg = 0 \\ \overset{\frown}{A} &: \quad mg - 3R_2 = 0. \end{aligned}$$

Solving gives $R_2 = \frac{1}{3}mg$ and $R_1 = \frac{2}{3}mg$.

1240. Prove that, if a parallelogram is also a kite, then it is a rhombus.

A rhombus has four sides of equal length.

Suppose the shape is $ABCD$, with AC taken to be the line of symmetry of the kite. Since $ABCD$ is a parallelogram, $|AB| = |CD|$ and $|AD| = |BC|$. Furthermore, since $ABCD$ is a kite, $|AB| = |AD|$. Hence, $|AB| = |BC| = |CD| = |DA|$, and the shape is a rhombus, as required.

1241. Determine the values of the constants a and b for the following to be an identity in x:

$$x(x-2)(x-a) \equiv x^3 + bx^2 + 6x.$$

Set $x = 2$ to find b, then set $x = a$.

Setting $x = 2$, we get $0 = 20 + 4b$. This gives $b = -5$. Then factorising the RHS, we get $a = 3$.

1242. Two statements about hexagon H are as follows. P is "H is regular." Q is "H has sides described, top-to-tail around the perimeter, by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, -\mathbf{a}, -\mathbf{b}, -\mathbf{c}$."

State, with a reason, which, if any, of the following implications hold:

(a) $P \implies Q$,

(b) $P \impliedby Q$,

(c) $P \iff Q$.

Draw a sketch with

$$\mathbf{a} = \mathbf{i}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j}, \quad \mathbf{c} = \mathbf{j}$$

Implication (a) holds, because, if the hexagon is regular, then every pair of opposite sides must be parallel and the same length. This gives Q. The other implications don't hold; a counterexample is

$$\mathbf{a} = \mathbf{i}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j}, \quad \mathbf{c} = \mathbf{j}$$

1243. Solve the equation $2\log_x 2 = 1 - 2\log_x 3$.

Put the logarithms on one side, and use log rules.

$$\begin{aligned} & 2\log_x 2 = 1 - 2\log_x 3 \\ \implies & \log_x 4 = 1 - \log_x 9 \\ \implies & \log_x 4 + \log_x 9 = 1 \\ \implies & \log_x 36 = 1 \\ \implies & x = 36. \end{aligned}$$

1244. A function F is set up as follows: given a monic quadratic graph $y = f(x)$ as an input, the output of F is the vertex, as a pair of coordinates. For example, $F[y = (x-2)^2 + 5] = (2, 5)$.

(a) Find $F[y = x^2 - 2x + 7]$.

(b) Find the function g in $F[y = g(x)] = (3, 0)$.

You can translate the function F as answering the question "What is the vertex of this parabola?".

(a) Completing the square, we have

$$F[y = (x-1)^2 + 6] = (1, 6).$$

(b) $y = g(x)$ is a monic quadratic with vertex at $(3, 0)$. Hence, $g(x) = (x-3)^2$.

1245. Show that an equilateral triangle of side length 2 will fit exactly inside a circle of radius $\frac{2\sqrt{3}}{3}$.

Find the distance from the centre of the equilateral triangle to one of its vertices, using trigonometry.

If A is a vertex, M the midpoint of side AB, and O the centre of the triangle ABC, then triangle AMO is a right-angled triangle. Trigonometry gives

$$\cos 30° = \frac{\sqrt{3}}{2} = \frac{1}{AO}$$

Therefore

$$AO = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

So, the equilateral triangle will fit exactly inside a circle of radius $\frac{2\sqrt{3}}{3}$.

1246. Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{3}{10^k}$ as a simplified fraction.

This is the sum to infinity S_∞ of a geometric sequence. Equivalently, it may be thought of as a recurring decimal.

This is the sum to infinity S_∞ of a GP with first term $a = \frac{3}{10}$ and common ratio $r = \frac{1}{10}$. Hence, using the standard formula $S_\infty = \frac{a}{1-r}$,

$$S_\infty = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{1}{3}.$$

Alternatively, the series is the recurring decimal 0.33333..., giving the same result.

1247. Use the Newton-Raphson iteration to determine, to 3sf, the input value for which the reciprocal function and the natural logarithm function give the same output.

Translate to the equation $\frac{1}{x} = \ln x$, and solve using Newton-Raphson.

Translating into algebra, we want to find x such that $\frac{1}{x} = \ln x$, so we set $f(x) = \frac{1}{x} - \ln x = 0$. The Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Hence, in this case, the iteration is

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - \ln x_n}{-\frac{1}{x_n^2} - \frac{1}{x_n}}.$$

This simplifies to

$$x_{n+1} = x_n + \frac{\frac{1}{x_n} - \ln x_n}{\frac{1}{x_n^2} + \frac{1}{x_n}}.$$

Running the iteration with $x_1 = 1$, we get $x_2 = 1.5$ and $x_n \to 1.76322228....$ So, the root is $x = 1.76$ to 3sf. We can verify this with error bounds:

$$f(1.755) = 0.0733... > 0$$
$$f(1.765) = -0.00157... < 0.$$

1248. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $-\sqrt{x} = 3$,
- $x = 9$.

Consider squaring both sides of the first equation.

The implication goes forwards. By squaring both sides, we get $-\sqrt{x} = 3 \implies x^2 = 9$. But the reverse implication doesn't hold, as 9 has two square roots.

1249. "The x axis is tangent to the curve $y = x^2 - x - 6$." True or false?

Consider the number of roots of the quadratic $x^2 - x - 6 = 0$.

This is false. Factorising gives $y = (x-3)(x+2)$, i.e. distinct roots. For the x axis to be a tangent, we would require one, repeated root.

Alternatively, $\Delta = b^2 - 4ac = 25 \neq 0$.

1250. Simultaneous equations are given as:

$$a + 4b + c = 10,$$
$$a + b + 2c = 12,$$
$$3a + b + c = 9.$$

(a) From the first two, show that $3a + 7c = 38$.

(b) Find another equation linking a and c.

(c) Hence, solve for a, b, c.

In (b), use the second and third equations to show that $2a - c = -3$.

(a) $4 \times (1) - (2)$ gives the required result.

(b) $(3) - (2)$ gives $2a - c = -3$.

(c) Solving these two equations simultaneously, we get $a = 1, c = 5$. Substituting this back in gives $b = 1$.

1251. State, with a reason, whether or not the following are valid identities:

(a) $\sin|x| \equiv \sin x$,

(b) $\cos|x| \equiv \cos x$,

(c) $\tan|x| \equiv \tan x$.

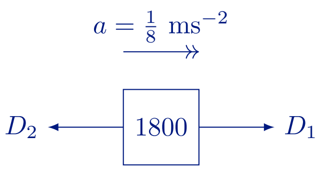
For such equations to be true, the relevant graph, e.g. $y = \sin x$, has to have the y axis as a line of symmetry.

This is true iff the relevant graph has the y axis as a line of symmetry, because, e.g. substituting $-30°$ into (b), we require $\cos 30° = \cos -30°$. This is only true for the cosine graph. Hence, (b) is the only valid identity.

1252. In a rugby scrum, two teams are pushing against one another. The two teams have total mass 850 kg and 950 kg respectively. The scrum accelerates constantly from rest, moving 1 metre in 4 seconds. Determine the difference in driving force exerted by the two teams.

Draw a force diagram of the entire scrum, with D_1 and D_2 as the driving forces exerted on the scrum. Calculate the acceleration using suvat, then use $F = ma$.

The acceleration is given by $s = ut + \frac{1}{2}at^2$, which is $1 = \frac{1}{2}a \cdot 16$. So, $a = \frac{1}{8}$ ms^{-2}. Modelling the entire scrum, then, considering only horizontal forces,



Newton II gives $D_1 - D_2 = 1800 \times \frac{1}{8} = 225$ N.

1253. A quadratic sequence starts $7, 15, 25, \ldots$. Find the value of the hundredth term.

With $u_n = an^2 + bn + c$, the second difference is $2a$.

With ordinal formula $u_n = an^2 + bn + c$, the second difference is $2a$. The first differences are $8, 10$, so $a = 1$. The first term gives $7 = 1 + b + c$, and the second term gives $15 = 4 + 2b + c$. Solving these simultaneously yields $b = 5, c = 1$. Hence, the hundredth term is

$$u_{100} = 100^2 + 5 \cdot 100 + 1 = 10501.$$

1254. Prove by exhaustion that no square number ends in any of the digits $2, 3, 7, 8$.

Proof by exhaustion means checking all the possibilities. But you only need to check the single-digit numbers.

Since the units digit of n^2 depends only on the units digit of n, we need only check the squares of $0, 1, \ldots, 9$. These are $0, 1, 4, 16, 25, 36, 49, 64, 81$, none of which ends in $2, 3, 7$, or 8. Q.E.D.

1255. A stepladder has the shape of two sides of an equilateral triangle of side length 240 cm. Without the use of a calculator, show that the ladder will not fit underneath a shelf 2 metres off the ground.

Find the height of the ladder as a surd, then square the surd in order to compare it.

The vertical height of the ladder, in metres, is

$$2.4 \sin 60° = \frac{\sqrt{3}}{2} \times 2.4 = \frac{6\sqrt{3}}{5}.$$

We need to show that this is greater than 2, hence that its square is greater than 4. Squaring the height gives

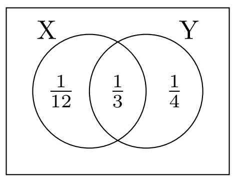
$$\left(\frac{6\sqrt{3}}{5}\right)^2 = \frac{36 \times 3}{25} = \frac{108}{25} = 4\tfrac{8}{25} > 4.$$

1256. Make x the subject in $y = \dfrac{a\sqrt{x} + b}{c\sqrt{x} + d}$.

Multiply up by the denominator of the fraction, and gather the $\sqrt{x}$ terms.

$$\begin{aligned}
& y = \frac{a\sqrt{x} + b}{c\sqrt{x} + d} \\
\Longrightarrow\ & cy\sqrt{x} + dy = a\sqrt{x} + b \\
\Longrightarrow\ & cy\sqrt{x} - a\sqrt{x} = b - dy \\
\Longrightarrow\ & \sqrt{x}(cy - a) = b - dy \\
\Longrightarrow\ & \sqrt{x} = \frac{b - dy}{cy - a} \\
\Longrightarrow\ & x = \left(\frac{b - dy}{cy - a}\right)^2.
\end{aligned}$$

1257. The probabilities of events X and Y are given in the following Venn diagram,



Represent the same information on a tree diagram, conditioned on X, finding and labelling all six branch probabilities.

"Conditioned on X" means that the primary branches should be X and X'; the secondary branches should then be Y and Y'. Calculate probabilities using

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

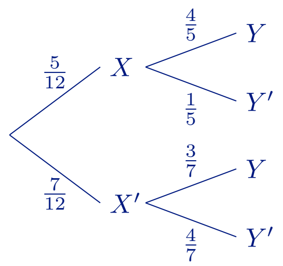
We begin with $P(X) = \frac{5}{12}$ and $P(X') = \frac{7}{12}$. Then, in the circle representing X, we have

$$P(Y \mid X) = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{12}} = \frac{4}{5}$$
$$P(Y' \mid X) = \frac{\frac{1}{12}}{\frac{1}{3} + \frac{1}{12}} = \frac{1}{5}.$$

Equivalent calculations in X' give

$$P(Y \mid X') = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{3}} = \frac{3}{7},$$
$$P(Y' \mid X') = \frac{\frac{1}{3}}{\frac{1}{4} + \frac{1}{3}} = \frac{4}{7}.$$

So, the tree diagram is



1258. Part of the *Diophantus identity* states that

$$(a^2 + b^2)(c^2 + d^2) \equiv (ac - bd)^2 + (ad + bc)^2.$$

Prove this result.

Multiply out the LHS and RHS separately, and prove that they simplify to the same expression.

LHS: $(a^2 + b^2)(c^2 + d^2)$
$\equiv a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2.$

RHS: $(ac - bd)^2 + (ad + bc)^2$
$\equiv a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2$
$\equiv a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2.$

LHS $\equiv$ RHS, proving the identity.

1259. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $x \in A$,
- $x \in A \cup B$.

Draw a Venn diagram to help visualisation.

The implication is $\Longrightarrow$. If x is an element of A, then it must be in A's union with any set. The reverse is not true, however: a counterexample is any $x \in B \setminus A$.

1260. Write 4^{2x+3} in terms of 2^x.

Express 4 as 2^2 and use index laws.

$4^{2x+3} = (2^2)^{2x+3} = 2^{4x+6} = 2^{4x} \cdot 2^6 = 64(2^x)^4.$

1261. A circle is given as $x^2 + ax + y^2 + by = 0$, for $a, b \in \mathbb{R}$. A square is inscribed, with all four of its vertices on the circumference. Show that the square has area $2(a^2 + b^2)$.

Complete the square for x and for y. This will allow you to find the radius of the circle.

Completing the square for both x and y gives $(x + a)^2 + (y + b)^2 = a^2 + b^2$. So, the radius of the circle is $\sqrt{a^2 + b^2}$. Inscribing a square, the right-angled triangle of two radii and one edge has sides in the ratio $1 : 1 : \sqrt{2}$. So, the area of the square is $\sqrt{2(a^2 + b^2)}^2 = 2(a^2 + b^2)$ as required.

1262. Three dice are rolled. State which, if either, of the following events has the greater probability:

- all scores are odd,
- no scores are odd.

"No scores are odd" is "all scores are even".

Since "no scores are odd" is the same as "all scores are even", the two events have equal probability.

1263. Simplify $\dfrac{1 + \sqrt{\frac{1}{c}}}{1 - \sqrt{\frac{1}{c}}} + \dfrac{1 - \sqrt{\frac{1}{c}}}{1 + \sqrt{\frac{1}{c}}}$.

When you have inlaid fractions, begin by multiplying top and bottom of the main fraction by the denominator(s) of the inlaid fraction(s).

Multiplying top and bottom of the large fractions by $\sqrt{c}$ gives

$$\frac{\sqrt{c} + 1}{\sqrt{c} - 1} + \frac{\sqrt{c} - 1}{\sqrt{c} + 1}.$$

Putting these over common denominator, we have

$$\frac{(\sqrt{c} + 1)^2 + (\sqrt{c} - 1)^2}{(\sqrt{c} + 1)(\sqrt{c} - 1)} = \frac{2c^2 + 2}{c - 1}.$$

1264. A particle is projected horizontally, at 5 ms^{-1}, from a point 1.6 metres above flat ground.

(a) Find the time of flight.

(b) Determine the angle below the horizontal at which the particle is travelling as it lands.

Resolve into horizontal and vertical components, with $a_x = 0$ and $a_y = -g$.

(a) Vertically, $s = -1.6$, $u = 0$, $a = -g$, and we want to find t. So, $-1.6 = \frac{1}{2} \cdot -gt^2$. Hence, $t^2 = \frac{16}{49}$. Taking the positive square root, the time of flight is $t = \frac{4}{7}$ seconds.

(b) At $t = \frac{4}{7}$, the components of the velocity are $v_x = 5$, $v_y = -g \times \frac{4}{7} = -5.6$. Hence, the angle below the horizontal is given by $\arctan \frac{5.6}{5} = 48.239... = 48.2°$ (1dp).

1265. Solve the equation $4\sqrt{x}(1+x) - (\sqrt{x}+1)^4 = 1$.

Use the binomial expansion to multiply out the brackets.

Using the binomial expansion, we have

$$\begin{aligned} & 4\sqrt{x} + 4x\sqrt{x} - \left(x^2 + 4x\sqrt{x} + 6x + 4\sqrt{x} + 1\right) = 1 \\ \implies & x^2 + 6x + 2 = 0 \\ \implies & x = \frac{-6 \pm \sqrt{36 - 4 \cdot 2}}{2} = -3 \pm \sqrt{7}. \end{aligned}$$

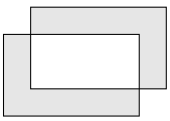
However, both of these roots are negative, and are not in the domain of the square root function. Hence, the equation has no roots.

1266. Prove that the sum of the interior angles of an n-sided polygon is given, in radians, by $S = (n-2)\pi$.

Split the n-gon up into $n-2$ triangles.

By drawing diagonals from a vertex to every other vertex, any convex n-gon can be split up into $n-2$ triangles. The interior angles of these triangles all lie at the vertices of the n-gon. Hence, the sum of the interior angles of the $n-2$ triangles is equal to the sum of the interior angles of the n-gon. Each triangle has π radians, giving $(n-2)\pi$. □

1267. Two identical rectangles are overlaid as depicted. Each has dimensions 12 cm × 8 cm, and the shaded area is 100 cm^2.



In each case, find the given quantity, or state that there is not enough information to find it:

(a) the area of the unshaded rectangle,

(b) the perimeter of the unshaded rectangle.

The area can be calculated, but the perimeter cannot.

(a) Each rectangle has area 96 cm^2. Therefore, calling the area of the unshaded rectangle A, $2 \times 96 - 2A = 100$. Solving, we get $A = 46$ cm^2.

(b) The perimeter cannot be calculated. The unshaded rectangle, while maintaining an area of 100 cm^2, could have different dimensions, and hence perimeters.

1268. Show that the following algebraic fraction cannot be simplified to a polynomial:

$$\frac{16x^4 + 5x^2 - 6x + 1}{x - 3}.$$

Show that $(x-3)$ is not a factor of the numerator, by using the factor theorem.

This algebraic fraction simplifies to a polynomial iff $(x-3)$ is a factor of the numerator. We can check this using the factor theorem. Substituting $x = 3$ into the numerator gives

$$16 \cdot 3^4 + 5 \cdot 3^2 - 6 \cdot 3 + 1 = 1324.$$

Since this is non-zero, $(x-3)$ is not a factor.

1269. Determine the distance from the centre of a cube of side length l to one of its vertices.

Use 3D Pythagoras.

Using 3D Pythagoras, the distance is

$$d = \sqrt{\left(\tfrac{l}{2}\right)^2 + \left(\tfrac{l}{2}\right)^2 + \left(\tfrac{l}{2}\right)^2} = \frac{\sqrt{3}}{2}l.$$

1270. Simplify $\{z \in \mathbb{R} : |z| \leq 5\} \setminus [2, \infty)$.

The slash is "set minus". So, picture a number line with all z between -5 and $+5$, with $[2, \infty)$ removed.

$\{z \in \mathbb{R} : |z| \leq 5\} \setminus [2, \infty)$ can be written as $[-5, 5] \setminus [2, \infty)$. Subtracting, we are left with $[-5, 2)$.

1271. The numbers p, $2p+1$, $2p^2-3$ are in arithmetic progression. Find all possible values of p.

Translate into algebra: "The difference between the first two terms is equal to the difference between the second and third terms."

Setting the differences equal to each other, we have

$$\begin{aligned} & 2p + 1 - p = 2p^2 - 3 - (2p+1) \\ \implies & 2p^2 - 3p - 5 = 0 \\ \implies & (2p-5)(p+1) = 0 \\ \implies & p = \tfrac{5}{2}, -1. \end{aligned}$$

1272. Consider the graph $\log_2 x + \log_2 y = 1$.

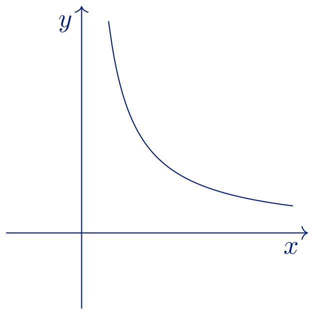
(a) Show that the relationship between x and y is one of inverse proportion.

(b) Hence, sketch the graph.

In (a), use log rules.

(a) Using log rules, we can write $\log_2(xy) = 1$. Then, exponentiating both sides, base 2, we get $xy = 2$, which is inverse proportion.

(b) The graph is a standard reciprocal $xy = 2$, apart from the fact that both x and y, as inputs to a logarithm function, must be positive. Hence, the graph is in the positive quadrant:



1273. Show that $\displaystyle\lim_{h\to 0} \frac{(x+h)^4 - (x-h)^4}{2h} = 4x^3$.

Expand the two brackets in the numerator using the binomial expansion. Simplify, divide top and bottom by h, then take the limit.

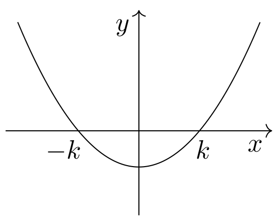
Using the binomial expansion, we have

$$(x \pm h)^4 = x^4 \pm 4x^3h + 6x^2h^2 \pm 4xh^3 + 1.$$

Hence, we can simplify to

$$\begin{aligned} &\lim_{h\to 0} \frac{8x^3h + 8xh^3}{2h} \\ = &\lim_{h\to 0} 4x^3 + 4xh^2 \\ = &\; 4x^3 \quad \text{as required.} \end{aligned}$$

1274. A quadratic graph $y = ax^2 + bx + c$ is shown below.



State, with a reason, whether the following facts are necessarily true:

(a) "a is positive",

(b) "b is zero",

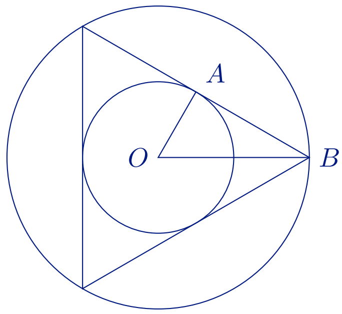
(c) "c is negative".

Consider the graph as $y = a(x-k)(x+k)$

Since it has roots at $x = \pm k$, the graph can be expressed as $y = a(x+k)(x-k) = ax^2 - ak^2$. Therefore, since the graph is a positive parabola, all three facts are necessarily true: $a > 0, b = 0$ and so $c = -ak^2 < 0$.

1275. On an equilateral triangle, circles are maximally inscribed and minimally circumscribed. Show that the ratio of the areas of the circles is 1 : 4.

Sketch the scenario, and use trig to show that the ratio of radii is 1 : 2.



In the above, triangle OAB has angles $30°, 60°, 90°$. Therefore, $OA = \frac{1}{2}OB$. Since the ratio of lengths is 1 : 2, the ratio of areas is therefore 1 : 4.

1276. Two dice are rolled. Given that at least one shows a four, find the probability that both do.

Draw a possibility space diagram (6 by 6 table), and use the information "at least one shows a four" to restrict the possibility space.

The information given restricts the possibility space to the shaded outcomes, of which only one is successful.

	1	2	3	4	5	6
1						
2						
3						
4				✓		
5						
6						

Therefore $P(\text{both 4's} \mid \text{at least one 4}) = \frac{1}{11}$.

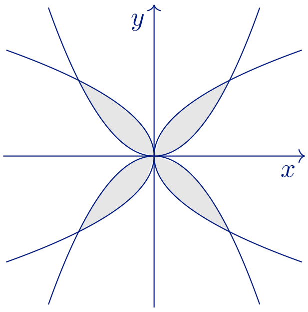
1277. A symmetrical design is created by taking the curve $y = x^2$ and rotating it through $90°$, $180°$, and $270°$, with centre of rotation at the origin. Points with either $|x|$ or $|y|$ greater than 2 are not drawn. Areas enclosed by the curves are then shaded.

(a) Sketch the design.

(b) Show that the design has area $\frac{4}{3}$.

In (b), use integration.

(a) The design is



(b) One of the leaves of the design is given by the following integral

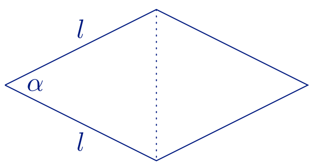
$$\begin{aligned}&\int_0^1 \sqrt{x} - x^2 \, dx \\ =&\left[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{3}x^3\right]_0^1 \\ =&\left(\tfrac{2}{3} - \tfrac{1}{3}\right) - (0) \\ =&\tfrac{1}{3}.\end{aligned}$$

Hence, the total shaded area is $\frac{4}{3}$.

1278. Prove that the area A of a rhombus is given by the formula $A = l^2 \sin\alpha$, where l is side length and α is any interior angle.

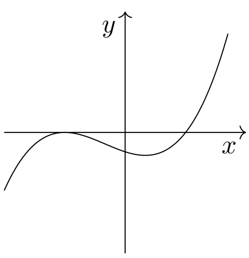
Use the sine area formula $A = \frac{1}{2}ab\sin C$.

The rhombus looks as follows:



Using the formula $A = \frac{1}{2}ab\sin C$, the area to the left of the dotted line is $\frac{1}{2}l^2\sin\alpha$. Hence, the full area is given by $A = l^2\sin\alpha$.

1279. Determine whether $y = (x+1)(x-1)^2$ could be the equation generating the following graph:



Consider whether $x = 1$ and $x = -1$ are single or double roots.

The given equation could not generate the graph. The equation $(x+1)(x-1)^2 = 0$ has a single root at $x = -1$ and a double root at $x = +1$, but the graph shows the opposite: a single root (crossing) at $x = +1$ and a double root (tangency) at $x = -1$.

1280. By attempting the factorisation explicitly, prove that $2x^2 - x + 1$ is not a factor of $2x^4 - 3x^3 + 2$.

Assume that $(2x^2 - x + 1)(ax^2 + bx + c) = 2x^4 - 3x^3 + 2$, and find a contradiction.

Assume, for a contradiction, that

$$(2x^2 - x + 1)(ax^2 + bx + c) = 2x^4 - 3x^3 + 2.$$

Comparing coefficients of x^4, we know that $a = 1$. Then, comparing coeffs of x^3 gives $2b - a = -3$, so $b = -1$. Then, comparing coeffs of x^2 gives $2c - b + a = 0$, so $c = -1$. But then the constant term is -1 on the LHS and $+2$ on the RHS. Hence, such a factorisation isn't possible.

1281. A triangle is drawn in a unit circle $x^2+y^2 = 1$, with its vertices on the circumference. Two of its sides are defined by the lines $x = 0$ and $y = \frac{3}{5}x - 1$. Show that the other side is defined by the line $y = 1 - \frac{5}{3}x$.

Use the angle in a semicircle theorem.

Two of the vertices are on the line $x = 0$; they have coordinates $(0, \pm 1)$. This side is a diameter of the circle. Therefore, by the angle in a semicircle theorem, the other two sides are perpendicular. The second side has gradient $\frac{3}{5}$ and y intercept -1. So, the third side has gradient $-\frac{5}{3}$ and y intercept 1; its equation is $y = 1 - \frac{5}{3}x$ as required.

1282. Two distributions are given by $X \sim B(100, \frac{1}{2})$ and $Y \sim N(50, 25)$. Show that the values of $P(X = 40)$ and $P(39.5 < Y < 40.5)$ are approximately equal.

X is a binomial distribution with $n = 100$ and $p = \frac{1}{2}$; Y is a normal distribution with $\mu = 50$ and $s = 5$.

Using the binomial distribution

$$P(X = 40) = {}^{100}C_{40} \times 0.5^{100} = 0.0108439$$

Using the normal distribution

$$P(39.5 < Y < 40.5) = 0.0108521$$

The values are approximately equal.

1283. Find b in simplified terms of a, if the quadratic $x^2 + 4x + a$ has a factor of $x - b$.

Use the factor theorem: if a polynomial $f(x)$ has a factor of $x - b$, then $f(b) = 0$.

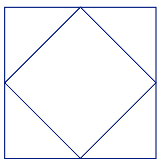
The factor theorem tells us that, if a polynomial $f(x)$ has a factor of $x - b$, then $f(b) = 0$. Hence $b^2 + 4b + a = 0$. This is a quadratic in b. Using the formula, we get

$$b = \frac{-4 \pm \sqrt{16 - 4a}}{2} = -2 \pm \sqrt{4 - a}.$$

1284. Find the area of the largest square which, rotating about its centre, remains inside a unit square.

Rotate the inner square so that its sides are at 45° to those of the outer square, then calculate its area.

The largest such square has its vertices at the midpoints of the sides of the unit square:



By Pythagoras, the length scale factor between the two squares is $1 : \sqrt{2}$, so the area scale factor is $1 : 2$. Hence, the area is $\frac{1}{2}$.

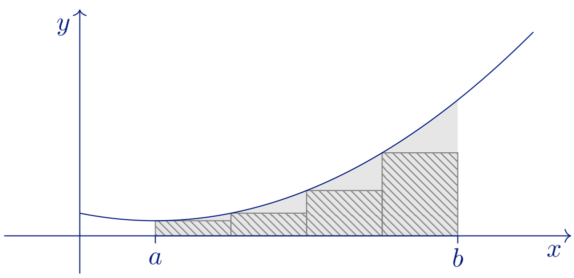
1285. It is a fundamental result of calculus that

$$\lim_{\delta x \to 0} \sum_{a}^{b} f(x)\,\delta x = \int_a^b f(x)\,dx.$$

With reference to a sketch of a function $y = f(x)$, give a graphical interpretation of the two sides of the equation, and hence explain their equality.

The LHS can be seen as (a limit of) the sum of the areas of a set of thin rectangular strips, while the the right hand side can be seen as the area under a curve.

On a graph of $y = f(x)$, we can approximate the area under the curve with a set of rectangular strips, as follows:



Each rectangle has width δx and height $f(x)$, hence area $f(x)\delta x$. Summing these gives the area of the hatched region. The integral, on the other hand, gives the area shaded. If one takes thinner and thinner rectangles (letting $\delta x \to 0$), then the total area of the rectangles approaches the area under the curve. In the limit, therefore, the given equation holds.

1286. Find the length of the line segment

$$x = p + t\cos\theta, \quad y = q + t\sin\theta, \quad t \in [-1, 1].$$

One end of the line segment, at $t = -1$, has coordinates $(p - \cos\theta, q - \sin\theta)$. Find the coordinates at $t = 1$, and then use Pythagoras.

The coordinates of the endpoints, at $t = \pm 1$, are

$$(p \pm \cos\theta, q \pm \sin\theta).$$

So, the vector between them is $2\cos\theta\mathbf{i} + 2\sin\theta\mathbf{j}$. Finding its length l, we get $l^2 = 4(\cos^2\theta + \sin^2\theta)$. The identity $\sin^2\theta + \cos^2\theta \equiv 1$ gives $l = 2$.

1287. A diameter, joining opposite vertices, is drawn on a regular 16-gon. A pair of distinct vertices A and B are then chosen from among the remaining 14. Find the probability that the chord AB crosses the diameter.

It doesn't matter where vertex A is, so place it anywhere. Then consider the placement of vertex B.

Vertex A can be placed anywhere, without loss of generality. The probability, then, that chord AB crosses the diameter is the probability that B is on the opposite side to A. There remain 13 vertices to be chosen from, of which 7 are on the opposite side, so the probability is $\frac{7}{13}$.

1288. Two functions f and g are related by $f'(x) = g'(x)$.

(a) Show that $f(x) - g(x)$ is constant.

(b) A student claims "If $f'(x) = g'(x)$, then, for $a > 0$, the ratio $\log_a f(x) : \log_a g(x)$ doesn't depend on x." His proof is as follows:

"Taking logs of both sides of $f(x) - g(x) = c$, we get $\log_a(f(x) - g(x)) = \log_a c$. The RHS is a constant which doesn't depend on x; let's call it k. Using a log rule on the LHS, we get

$$\frac{\log_a f(x)}{\log_a g(x)} = k.$$

Hence, the ratio $\log_a f(x) : \log_a g(x)$ doesn't depend on x."

Identify the error in the proof.

In (a), integrate both sides. In (b), take logs of part (a).

(a) Integrating, and combining both $+c$'s, we get $f(x) = g(x) + c$. So, $f(x) - g(x) = c$.

(b) A log rule has been misused. The true rule is

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y,$$

using which the logarithm of a fraction may be expressed as a difference of logarithms. The student has instead expressed the logarithm of a difference $\log_a(f(x) - g(x))$ as a fraction of logarithms $\log_a f(x)/\log_a(g(x))$. The result proposed is not, in fact, true.

1289. The interior angles of a quadrilateral form an AP. Give, in radians, the sum of the smallest and largest angles.

Consider the mean of the four angles of a quadrilateral.

The mean interior angle in a quadrilateral is $\frac{\pi}{2}$ radians. Since these interior angles form an AP, we know that the mean of the smallest and largest angles is therefore $\frac{\pi}{2}$. So, their sum is π radians.

1290. Find c such that the curve $y = x^2 + c$ satisfies the differential equation

$$\frac{dy}{dx} = \sqrt{4y - 20}.$$

Differentiate to find $\frac{dy}{dx}$, substitute in and solve for c.

Differentiating, $\dfrac{dy}{dx} = 2x$. Substituting gives

$$2x = \sqrt{4(x^2 + c) - 20}.$$

We require that $4x^2 + 4c - 20 \equiv 4x^2$ for all x. Setting $x = 0$ yields $c = 5$.

1291. If $x = \sin u$, show that $\left(\dfrac{du}{dx}\right)^2 = \tan^2 u + 1$.

Differentiate to find $\frac{dx}{du}$, reciprocate and square both sides then use a Pythagorean trig identity.

Differentiating, we get $\frac{dx}{du} = \cos u$. Reciprocating and squaring both sides then produces

$$\left(\frac{du}{dx}\right)^2 = \sec^2 u.$$

The Pythagorean trig identity $\tan^2 u + 1 \equiv \sec^2 u$ gives the required result.

1292. The velocity of an object, for $t \geq 0$, is given by $v = \sqrt{t} - t$. At $t = 0$, the object is at the origin.

(a) Find the maximum velocity attained.

(b) Verify that the object returns to O at $t = \frac{16}{9}$.

In (a), differentiate and set $a = 0$. In (b), integrate and set $s = 0$.

(a) The acceleration is given by $a = \frac{1}{2}t^{-\frac{1}{2}} - 1$. So, maximum velocity is attained where $\frac{1}{2}t^{-\frac{1}{2}} - 1 = 0$. This is at $t = \frac{1}{4}$ seconds. Substituting, we get $v_{\text{max}} = \frac{1}{4}$ ms^{-1}.

(b) Integrating v with limits of $t = 0$ and $t = \frac{16}{9}$ gives the displacement over that interval.

$$\begin{aligned}
&\int_0^{\frac{16}{9}} \sqrt{t} - t \, dt \\
&= \left[\tfrac{2}{3}t^{\frac{3}{2}} - \tfrac{1}{2}t^2\right]_0^{\frac{16}{9}} \\
&= \tfrac{2}{3} \cdot \tfrac{64}{27} - \tfrac{1}{2} \cdot \tfrac{256}{81} \\
&= 0.
\end{aligned}$$

Hence, the object returns to O at $t = \frac{16}{9}$.

1293. Show that points $(0, -2)$ and $(3, 0)$ are equidistant from the circle $x^2 + x + y^2 - 4y = 0$.

"Equidistant from a circle" is the same statement as "equidistant from the centre of the circle."

Completing the square, the circle is

$$(x + \tfrac{1}{2})^2 + (y - 2)^2 = \tfrac{9}{4}.$$

So, the centre of the circle is $C : (-\frac{1}{2}, 2)$. The squared distances of the given points from C are $\frac{1}{2}^2 + 4^2 = \frac{65}{4}$ and $\frac{7}{2}^2 + 2^2 = \frac{65}{4}$. Hence, the points are equidistant from the circle.

1294. Point A is on the line $y = 1 - 2x$, and $\overrightarrow{OA}$ is a unit vector. By considering the equation of a circle, or otherwise, determine the possible coordinates of A.

Use the unit circle $x^2 + y^2 = 1$.

Since $\overrightarrow{OA}$ is a unit vector, point A must lie on the unit circle $x^2 + y^2 = 1$. Hence, we can solve simultaneously. Substituting for y gives $x^2 + (1 - 2x)^2 = 1$. Simplifying, $5x^2 - 4x = 0$, so $x = 0$ or $x = 0.8$. The possible coordinates, then, are $(0, 1)$ and $(0.8, -0.6)$.

1295. State whether the following functions have a sign change at $x = 1$.

(a) $g(x) = x(x - 1)$,

(b) $g(x) = x(x-1)^{-1}$,

(c) $g(x) = x(x-1)^{-2}$.

Consider the parity (oddness/evenness) of the indices.

In each case, it is the parity (oddness/evenness) of the index k in $(x-1)^k$ that determines whether a sign change takes place.

(a) Yes, $k = 1$, which is odd.

(b) Yes, $k = -1$, which is odd.

(c) No, $k = -2$, which is even.

1296. You are given that $\ln y = 3\ln x + 4$. Describe, without reference to logarithms, the relationship between y and x.

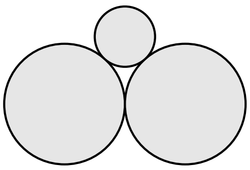
Exponentiate both sides, and use index laws.

Exponentiating both sides, we have

$$\begin{aligned} e^{\ln y} &= e^{3\ln x + 4} \\ = y &= e^{\ln x^3 + 4} \\ = y &= e^4 x^3. \end{aligned}$$

Since e^4 is a constant, the relationship is cubic.

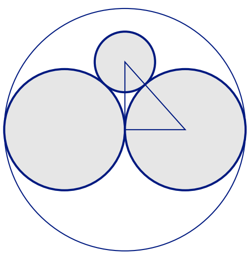
1297. Three circles of radius 5, 10 and 10 are all tangent to each other.



Show that all three can be contained within a circle of radius 20.

Place the circle of radius 20 such that its diameter passes through the centres of the two large circles.

To contain the two large circles, the surrounding circle must have its centre at their point of tangency.



The result is shown visually above. To prove it geometrically, consider the vertical height of the triangle, which is

$$\sqrt{15^2 - 10^2} = \sqrt{125} < 12.$$

Hence, the highest point is at a vertical height of $\sqrt{125} + 5 < 17$ above the centre, so will fit inside a circle of radius 20.

1298. The first four terms of an arithmetic sequence are given by $a, b, a+b, 8$. Find a and b.

Equate the differences to produce two equations: $u_2 - u_1 = u_3 - u_2$ and $u_3 - u_2 = u_4 - u_3$. Solve these simultaneously.

Equating differences gives $b - a = (a+b) - b$ and $(a+b) - b = 8 - (a+b)$. These simplify to $b = 2a$, $2a = 8 - b$. Solving simultaneously, $a = 2, b = 4$.

1299. Prove that the sum of three consecutive cubes is divisible by 3.

Begin with $n^3 + (n+1)^3 + (n+2)^3$ and simplify.

The sum of three consecutive cubes is given by

$$n^3 + (n+1)^3 + (n+2)^3,$$

where $n \in \mathbb{Z}$. Using the binomial expansion, this simplifies to

$$\begin{aligned} & n^3 + (n^3 + 3n^2 + 3n + 1) + (n^3 + 6n^2 + 12n + 8) \\ &= 3n^3 + 9n^2 + 15n + 9 \\ &= 3(n^3 + 3n^2 + 5n + 3) \end{aligned}$$

This is divisible by 3, as required.

1300. The following identity holds for all $A, B \in \mathbb{R}$:

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B.$$

Use this compound-angle formula to solve the equation $\cos(x + 30°) = \sin x$, for $x \in [0, 360°)$.

Use the exact values $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$. Then use $\tan x \equiv \frac{\sin x}{\cos x}$.

Using the compound-angle formula,

$$\begin{aligned} & \cos x \cos 30° - \sin x \sin 30° = \sin x \\ \Longrightarrow\ & \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \sin x \\ \Longrightarrow\ & \sqrt{3}\cos x = 3\sin x. \end{aligned}$$

Equations of this form are solved by writing in terms of $\tan x$:

$$\begin{aligned} & \sqrt{3}\cos x = 3\sin x \\ \Longrightarrow\ & \frac{\sqrt{3}}{3} = \frac{\sin x}{\cos x} \\ \Longrightarrow\ & \frac{\sqrt{3}}{3} = \tan x. \end{aligned}$$

Therefore $x = \arctan \frac{\sqrt{3}}{3} = 30°$ or, diametrically on a unit circle, $210°$.

1301. A quantum physicist is measuring the energy of electrons emerging from a scattering experiment. The speeds involved in the experiment are being increased linearly over time. The average energy, in Joules, of the scattered electrons is modelled as changing, over t measured in seconds, according to

$$\frac{dE}{dt} = 2.15616t - 0.088210t^2.$$

(a) Give the initial rate of change of E.

(b) Find, in each case, the times $t \geq 0$ for which the relevant quantity is increasing. Give your answers in set notation.

i. energy,

ii. rate of change of energy.

A quantity q is increasing over time when $\frac{dq}{dt} > 0$. So, for (b) i. solve the relevant inequalities, and for (b) ii. differentiate and solve the relevant inequality.

(a) $\left.\frac{dE}{dt}\right|_{t=0} = 0$.

(b) i. We require $2.15616t - 0.088210t^2 > 0$. Solving the associated equation gives $t = 0$ or $t = 24.443487...$. We have a negative quadratic, so the energy is increasing for $t \in (0, 24.443)$ to 5sf.

ii. For the rate of change of energy to be increasing, we require the second derivative to be positive, that is to say,

$$\frac{d^2E}{dt^2} = 2.15616 - 0.17642t > 0.$$

This is a linear inequality, with solution $t < 12.2217435...$. But t must be non-negative, so $t \in [0, 12.222)$ to 5sf.

1302. Explain why no events A and B (with non-zero probabilities) that are mutually exclusive can be independent.

"Mutually exclusive" means never happening simultaneously; "independent" means that the occurrence of one does not affect the probability of the other.

If events A and B are mutually exclusive, then knowledge of A's occurrence rules B out entirely, i.e. $P(B \mid A) = 0$. But since $P(B) \neq 0$, this means that $P(B \mid A) \neq P(B)$, which is the definition of dependence.

1303. Simplify the following:

(a) $2^{\log_2 x + \log_2 y}$,

(b) $e^{\ln x + \ln y}$.

The two questions are the same: $\ln x$ means "logarithm with base $e \approx 2.718$".

(a) $2^{\log_2 x + \log_2 y} = 2^{\log_2 x} \times 2^{\log_2 y} = xy$,

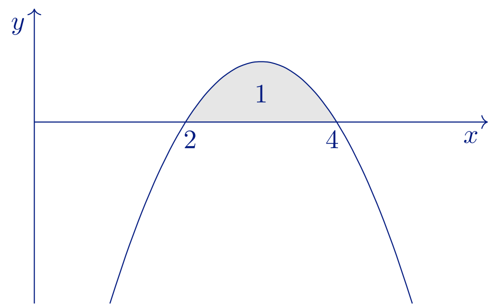
(b) $e^{\ln x + \ln y} = e^{\ln x} \times e^{\ln y} = xy$.

1304. Sketch a quadratic graph $y = h(x)$ for which

$$\int_2^4 h(x)\, dx = 1, \quad h(2) = h(4) = 0.$$

The quadratic has roots at $x = 2$ and $x = 4$, and the area beneath the graph between those limits is positive.

The quadratic crosses the x axis at 2 and 4, and, since the integral is positive, must be above the x axis between those point. So, it's a negative parabola:



1305. A water pump ejects water from a tank. Water, whose density is 1 kg per litre, accelerates from rest along a cylindrical pipe of length 0.5 m and cross-sectional area 0.002 m^2, and emerges with a velocity of 2 ms^{-1}.

(a) Find the total mass of water in the pipe.

(b) Find the acceleration (assumed constant) of the water.

(c) Hence, find the force applied by the pump.

In (a), use the volume of a prism $V = lA$ and *suvat*. In (b), use $F = ma$ with the object modelled as "the water in the pipe".

(a) The volume of water in the pipe is $0.5 \times 0.002 = 0.001$ m^3, which is 1000 litres, mass 1000 kg.

(b) The water accelerates from $u = 0$ to $v = 2$ with displacement $s = 0.5$. So, $v^2 = u^2 + 2as$ gives $4 = 2a \times 0.5 \implies a = 4$ ms^{-2}.

(c) Since the water has a consistent acceleration, it can be modelled as a single object of mass 1000 kg. $F = ma$ gives $F = 4000$ N.

1306. A sample $\{x_i\}$ has mean $\bar{x}$ and variance s_x^2. Write down the mean value of

(a) $x_i - \bar{x}$,

(b) $(x_i - \bar{x})^2 - s_x^2$.

(a) is asking "What is the average deviation from the mean?"

(a) By definition, the mean deviation from the mean is 0.

(b) The mean of the first term is the definition of the variance

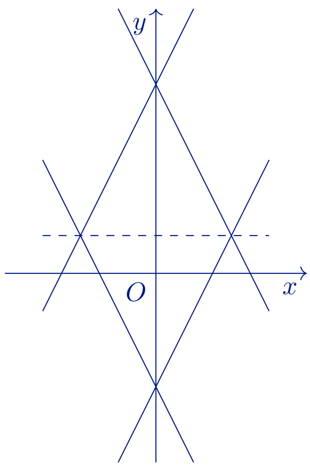
$$s_x^2 := \frac{(x_i - \bar{x})^2}{n}.$$

Hence, the mean value 0.

1307. Four straight lines are defined by the equations $y = 2x-3$, $y = 2x+5$, $y = -2x-3$, $y = -2x+5$. Show that the four lines enclose a rhombus.

Sketch the lines carefully, and consider lines of symmetry.

Sketching the lines:



The shape has the lines $x = 0$ and $y = 1$ as lines of symmetry. Hence, it is a rhombus.

1308. Functions f and g are such that $x = a$ is a root of $f(x) = 0$, $x = b$ is a root of $g(x) = 0$, and $x = c$ is a root of $f(x) = g(x)$. State, with a reason, whether the following hold:

(a) If $a = b$, then $a = b = c$.

(b) If $a = c$, then $a = b = c$.

Neither is true; look for specific counterexamples.

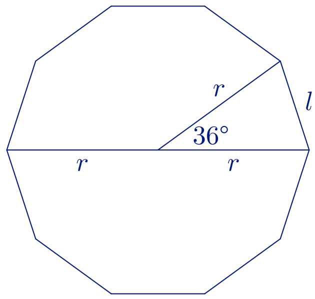
(a) No. A counterexample is $f(x) = x(x-1)$ and $g(x) = x(x-1)$, with $a = 0, b = 0, c = 1$.

(b) No. A counterexample is $f(x) = x(x-1)$ and $g(x) = x(x-1)$, with $a = 0, b = 1, c = 0$.

1309. Show that the diameter, from vertex to vertex, of a regular decagon of side length l is $l \operatorname{cosec} 18°$.

$\operatorname{cosec} 18° = \frac{1}{\sin 18°}$. Each of the reciprocal trig functions can be identified by its third letter, e.g. cosec is the reciprocal of sin.

The decagon is as follows:



Splitting the isosceles triangle shown in two, we have $\frac{l}{2} = r \sin 18°$. This gives

$$d = 2r = \frac{l}{\sin 18°} = l \operatorname{cosec} 18°.$$

1310. Solve for k in $\displaystyle\int_0^1 12x^2\,dx = \int_0^k x^3\,dx$.

Calculate the definite integrals on each side, forming an equation in k.

$$\begin{aligned} &\int_0^1 12x^2\,dx = \int_0^k x^3\,dx \\ \Longrightarrow\ &\Big[4x^3\Big]_0^1 = \Big[\tfrac{1}{4}x^4\Big]_0^k \\ \Longrightarrow\ &4 - 0 = \tfrac{1}{4}k^4 - 0 \\ \Longrightarrow\ &16 = k^4 \\ \Longrightarrow\ &k = \pm 2. \end{aligned}$$

1311. Write down the ranges of the following functions, when defined over the real numbers:

(a) $x \mapsto 2^x$,

(b) $x \mapsto 2^{-x}$,

(c) $x \mapsto -2^x$.

The function $f(x) = 2^x$ has range $(0, \infty)$. The other two functions can be thought of as transformations of this function, with either the inputs or outputs negated.

(a) The range is $(0, \infty)$.

(b) Negating the inputs leaves the range as $(0, \infty)$.

(c) Negating the outputs changes the range to $(-\infty, 0)$.

1312. The parabola $y = x^2 + px + q$ crosses the x axis at $x = a, b$. Write down the equation of the monic parabola which has roots at $x = -a, -b$.

In a monic parabola, the coefficient of x^2 is 1. Hence, this new parabola must a reflection of the old one in the y axis. So, replace x by $-x$.

The new parabola is a reflection of the old in the y axis. To enact this reflection, we replace x with $-x$, giving

$$\begin{aligned} & y = (-x)^2 + p(-x) + q \\ \implies & y = x^2 - px + q. \end{aligned}$$

1313. State true or false for the following, in which $\text{lcm}(a, b)$ denotes the lowest common multiple of distinct natural numbers a, b.

(a) $\text{lcm}(a, b) = ab$.

(b) If a and b are primes, then $\text{lcm}(a, b) = ab$.

(c) If $\text{lcm}(a, b) = ab$, then a and b are primes.

In (a), consider $a = 4, b = 6$. In (c), consider $a = 1$.

(a) This is false: $\text{lcm}(4, 6) = 12$.

(b) This is true by definition.

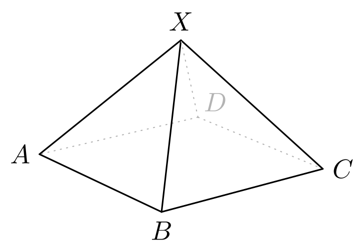
(c) This is false, $\text{lcm}(1, 4) = 4$, but 1 and 4 are not primes.

1314. Show that the normal to the curve $y = x^4 - x^2$ at $x = 1$ intersects the curve again.

You can solve this problem without doing any calculations. Consider the orders of the equations involved, i.e. quartics and cubics. Every cubic equation must have a root.

The equation for intersections of curve and normal $x^4 - x^2 = mx + c$ is quartic. It has a single root at $x = 1$, because the normal crosses the curve there. Hence, by the factor theorem, the quartic has exactly one factor of $(x - 1)$. This leaves a cubic factor. Every cubic has a root, so there must be at least one other intersection.

1315. The square-based pyramid shown below is formed of eight edges of unit length.



Find the total surface area.

Divide the equilateral sides into right-angled triangles.

Each equilateral face has area $\frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2}$, and the square base has area 1, so the total surface area is $\sqrt{3} + 1$.

1316. State, with a reason, which of the implications $\implies$, $\impliedby$, $\iff$ links the following statements concerning a real number x:

- $x \in [a, b]$,
- $x \in (a, b)$.

The implication goes backwards.

$x \in [a, b] \impliedby x \in (a, b)$, because, if $a < x < b$, then $a \leq x \leq b$. The boundary values $x = a$ and $x = b$ are the counterexamples to the forwards implication.

1317. You are given that the two lines $2x + py = 0$ and $x - (p + 1)y = 4$ are perpendicular to one another. Determine all possible values of p.

Rearrange each equation to the form $y = mx + c$, and then set up an equation $m_1 m_2 = -1$.

The gradients of the lines are $-\frac{2}{p}$ and $\frac{1}{p+1}$. Since the lines are perpendicular, these multiply to -1:

$$\begin{aligned} & -\frac{2}{p} \times \frac{1}{p+1} = -1 \\ \implies & 2 = p(p + 1) \\ \implies & p^2 + p - 2 = 0 \\ \implies & p = -2, 1. \end{aligned}$$

1318. A student writes: "Friction acts to oppose motion, so it can only go in one direction. Therefore, since Newton III has equal and opposite forces, friction can't obey Newton's third law." Explain carefully why this is incorrect.

In e.g. the friction experienced by a car braking to a halt, consider which objects the friction acts on.

Friction does act to oppose (any possible) motion, but such motion is the relative motion between two objects, not the absolute motion of one object. If a car brakes to a halt on a road, the NIII pair is

- friction acting backwards on the car, and
- friction acting forwards on the road.

These both act to oppose the motion of the car relative to the road.

1319. Write the following sets as lists of elements, in the form $\{a_1, a_2, ..., a_n\}$.

(a) $\{x \in \mathbb{Z} : x > 0\} \cap [-2, 2]$,

(b) $\{x \in \mathbb{Z} : x \leq 0\} \cap (-3, 3]$,

(c) $\{x \in \mathbb{Z} : |x| \geq 2\} \cap (-4, 4)$.

The left-hand sets are (a) positive integers, (b) non-positive integers, (c) integers whose magnitude is 2 or more.

(a) $\{1, 2\}$,

(b) $\{-2, -1\}$,

(c) $\{-3, -2, 2, 3\}$.

1320. Prove that, if u_n is an arithmetic progression and f is a function with constant first derivative, then $f(u_n)$ is also an arithmetic progression.

Express the AP generically as $u_n = a + (n-1)d$, and integrate $f'(x) = k$.

Any AP can be expressed as $u_n = a + (n-1)d$. Then, if f has constant first derivative, $f'(x) = k$, so $f(x) = kx + c$. Substituting, we have

$$\begin{aligned} f(u_n) &= ku_n + c \\ &= k(a + (n-1)d) + c \\ &= ka + c + (n-1)kd. \end{aligned}$$

This is the ordinal formula for an AP with first term $ka + c$ and common difference kd.

1321. Rearrange $pt^6 + qt^3 + r = 0$ to make t the subject.

Consider the equation as a quadratic in t^3.

This is a quadratic in t^3. So, quoting the quadratic formula, we get

$$\begin{aligned} t^3 &= \frac{-q \pm \sqrt{q^2 - 4pr}}{2p} \\ \Longrightarrow t &= \sqrt[3]{\frac{-q \pm \sqrt{q^2 - 4pr}}{2p}}. \end{aligned}$$

1322. A hypothesis test produces a p-value of 0.0421. A student writes: "Since the p-value is less than the significance level 5%, the null hypothesis is false." Explain what is wrong with this sentence.

The error is in "..., the null hypothesis **is** false."

The conclusion is too strong. A hypothesis test never tells you that a null hypothesis **is** definitely false, only that you have sufficient evidence (here at the 5% level) to reject it.

1323. Show that $2\sin^2 x + \cos x = 3$ has no real roots.

Use $\sin^2 x + \cos^2 x \equiv 1$, and then the discriminant $\Delta = b^2 - 4ac$.

Using $\sin^2 x + \cos^2 x \equiv 1$, we have

$$\begin{aligned} & 2 - 2\cos^2 x + \cos x = 3 \\ \Longrightarrow\ & 2\cos^2 x - \cos x + 1 = 0. \end{aligned}$$

This is a quadratic in $\cos x$. Its discriminant is $\Delta = -7 < 0$. So, the equation has no real roots.

1324. The vertices V_i of a regular n-gon are at points (x_i, y_i), for $i = 1, 2, ..., n$, given, in radians, by

$$V_i : \left(2 + \sin \tfrac{\pi i}{20}, 4 - \cos \tfrac{\pi i}{20}\right).$$

Find the centre and number of sides of the n-gon.

Rewrite $\frac{\pi i}{20}$ as $\frac{2\pi i}{40}$.

Rewriting $\frac{\pi i}{20}$ as $\frac{2\pi i}{40}$, we can see that each i step involves a rotation around the point $(2, 4)$ by $\frac{2\pi i}{40}$ radians. Since there are 2π radians at the centre, the polygon, centred at $(2, 4)$ has 40 sides.

1325. Show that the curves $2y = x^2 + 1$ and $2x = y^2 + 1$ are tangent.

Sketch the curves first to get your bearings, considering the fact that the curves are reflections in the line $y = x$.

As the roles of x and y switch, the two curves are reflections in $y = x$. Hence, they intersect where $y = x$. Substituting into the first equation, this gives $2x = x^2 + 1$, which is $(x-1)^2 = 0$. Since $x = 1$ is a double root, the line $y = x$ is tangent to the first curve. Therefore, the curves must be tangent to each other at $(1, 1)$.

1326. A quadratic function h has $h(3) = -1$, $h'(3) = 0$, $h''(3) = -1$. Show that $h(x) = 0$ has no real roots.

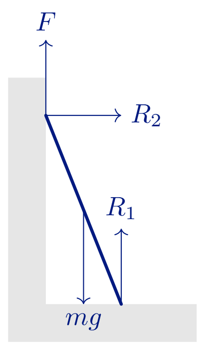
For a graph $y = h(x)$, consider the nature and location of the stationary point at $x = 3$.

The graph $y = h(x)$ is stationary at $(3, -1)$, since $h'(3) = 0$ and $h(3) = -1$. Furthermore, this is a local maximum, since $h''(3) = -1 < 0$. But, for a quadratic, a local maximum is a global maximum, so $h(x) \leq -1$. The equation $h(x) = 0$, therefore, has no real roots.

1327. Prove that no ladder can remain in equilibrium when placed on smooth horizontal ground, even if leaning against a rough vertical wall.

Draw a force diagram, and consider the horizontal equilibrium.

The forces are as follows:



The reaction R_2 must be non-zero, or there would be a resultant moment anticlockwise around the foot of the ladder. Hence, since R_2 is the only horizontal force, there is a resultant force to the right, and the ladder cannot be in equilibrium.

1328. Solve $\ln x + \ln(4 + x) = \ln 2$.

Use log laws to put the equation in the form $\ln p = \ln q$, then exponentiate both sides.

Using a log law, we have $\ln x(4 + x) = \ln 2$. Then, exponentiating both sides gives

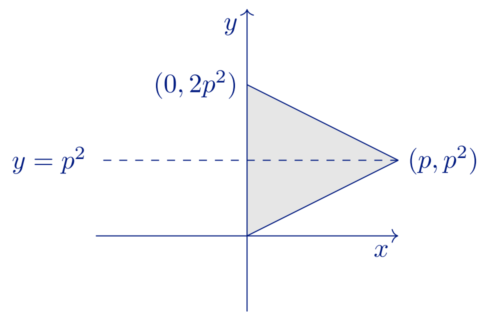
$$\begin{aligned} & x(4+x) = 2 \\ \Longrightarrow\ & x^2 + 4x - 2 = 0 \\ \Longrightarrow\ & x = \pm\sqrt{6} - 2. \end{aligned}$$

But the lower root $-\sqrt{6} - 2 < 0$ cannot be input into the natural logarithm function. Hence, the solution is $x = \sqrt{6} - 2$.

1329. A line segment is drawn from the origin to a point (p, p^2). This line segment is then reflected in the line $y = p^2$, to form, with the y axis, an isosceles triangle. Show that this triangle has area p^3.

Sketch the scenario.

The scenario is

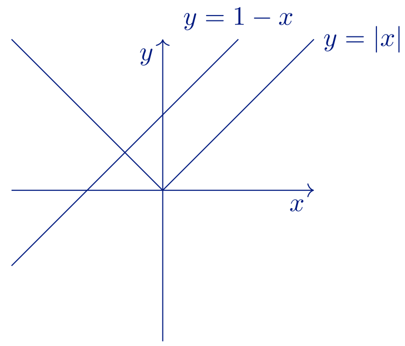


Treating the y axis as the base, the area is given by $\frac{1}{2}bh = \frac{1}{2} \times 2p^2 \times p = p^3$.

1330. Solve $x + |x| > 1$, answering in set notation.

Rearrange to $|x| > 1 - x$, and sketch $y = |x|$ and $y = 1 - x$.

Rearranging to $|x| > 1 - x$, we compare $y = |x|$ and $y = 1 - x$. These are



The intersection is at $x = -\frac{1}{2}$, and we need the mod graph to be above the straight line. Hence, $x \in (-\infty, -\frac{1}{2})$.

1331. Separate the variables in the following differential equation, writing it in the form $f(y)\frac{dy}{dx} = g(x)$ for some functions f and g:

$$\operatorname{cosec} x \frac{dy}{dx} - \tan y = 1.$$

Begin by moving the $\tan y$ term to the RHS.

$$\begin{aligned} & \operatorname{cosec} x \frac{dy}{dx} - \tan y = 1 \\ \Longrightarrow\ & \operatorname{cosec} x \frac{dy}{dx} = 1 + \tan y \\ \Longrightarrow\ & \frac{dy}{dx} = \sin x(1 + \tan y) \\ \Longrightarrow\ & \frac{1}{1 + \tan y}\frac{dy}{dx} = \sin x. \end{aligned}$$

1332. Determine the area of the smallest circle which completely encloses the curve $\frac{1}{9}x^2 + \frac{1}{16}y^2 = 25$.

The curve is an ellipse.

The given curve is an ellipse centred on the origin, with axis intercepts at $(\pm 15, 0)$ and $(0, \pm 20)$. So, the long radius is 20. The smallest circle enclosing the ellipse must have radius 20, and area 400π.

1333. Find and correct the error in the following:

$$\int (2x+1)^2\, dx = \tfrac{1}{3}(2x+1)^3 + c.$$

The effect of the chain rule has been neglected.

The (reverse) chain rule dictates that, if we are integrating such a function, a scale factor of $\frac{1}{2}$ emerges. This is because the 2 coefficient of x enacts a stretch, scale factor $\frac{1}{2}$, in the x direction. This reduces areas by a factor of $\frac{1}{2}$. The corrected version is

$$\int (2x+1)^2\, dx = \tfrac{1}{6}(2x+1)^3 + c.$$

1334. Find the horizontal range of a projectile thrown from ground level, at an angle of $15°$ above the horizontal, at initial speed 70 ms^{-1}.

Split the velocity into $u_x = 70\cos 15$ and $u_y = 70\sin 15$, then set up a vertical *suvat* with $s_y = 0$. Calculate the time taken and use it to find s_x.

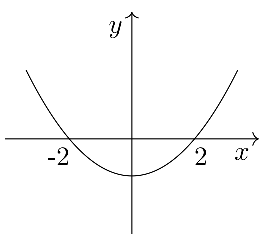
Splitting into vertical and horizontal components:

s_y	0	s_x	s_x
u_y	$70\sin 15$	u_x	$70\cos 15$
v_y	n/a	v_x	n/a
a_y	$-g$	a_x	0
t	t	t	t

Vertically, we have $0 = 70\cos 15t - \frac{1}{2}gt^2$. Solving gives $t = 0$ (launch) or $t = 140\cos 15/g$ (landing). Hence, the range is

$$s_x = \frac{140\cos 15}{g} \times 70\sin 15 = 250 \text{ m}.$$

1335. The graph below is of $y = g'(x)$, for some cubic function g defined over the real numbers.



Find all possible functions g.

Give all possible functions $g'(x)$, using the fact that $g'(x)$ has roots at $x = \pm 2$, then integrate.

Since $g(x)$ is a cubic, $g'(x)$ must be a quadratic. It has roots at ± 2, and is a positive parabola, so its equation is $g'(x) = a(x+2)(x-2) = ax^2 - 4a$. Integrating this, we get $g(x) = \frac{1}{3}ax^3 - 4ax + b$, where $a > 0$ and b are real numbers.

1336. Solve $\dfrac{1}{1+x} + \dfrac{1}{(1+x)^2} = 2$.

Multiply both sides of the equation by $(1+x)^2$, then simplify.

Multiplying both sides by $(1+x)^2$, we get

$$\begin{aligned} & 1 + x + 1 = 2(1+x)^2 \\ \Longrightarrow\; & 2x^2 + 3x = 0 \\ \Longrightarrow\; & x(2x+3) = 0 \\ \Longrightarrow\; & x = 0 \text{ or } x = -\tfrac{3}{2}. \end{aligned}$$

1337. Prove that, if a, b, c are positive numbers in GP with common ratio $r \neq 1$, then $a + c > 2b$. You may wish to consider the value of $a(1-r)^2$.

You know that $a(1-r)^2 > 0$. Begin with this inequality and rearrange to the required result.

We know that, since $r \neq 1$, $a(1-r)^2 > 0$. Multiplying out, this is $a - 2ar + ar^2 > 0$, which rearranges to $a + ar^2 > 2ar$. If a, b, c are in GP, then $ar = b$ and $ar^2 = c$, giving us the result.

1338. In a biological study, the masses of 15 adult giant Pacific octopi are measured as having $\bar{m} = 13.6$ kg. Afterwards, it is discovered that the masses of four of the octopi were undermeasured by 16%, due to a calibration error.

(a) Show that, with the information above, the best estimate of the true mean of the sample, to 3sf, is 14.3 kg.

(b) Why is your answer only an estimate?

Scale $\frac{4}{15}$ of the sample up, leaving the other $\frac{11}{15}$ unscaled.

(a) Scaling $\frac{4}{15}$ of the sample back up, while leaving the rest unchanged, we get

$$13.6\left(\frac{4}{15} \cdot \frac{1}{0.84} + \frac{11}{15}\right) = 14.29... = 14.3 \text{ (3sf)}.$$

(b) This is only an estimate because it is possible that e.g. only the heaviest or only the lightest octopi were undermeasured.

1339. Simplify $\ln \dfrac{1}{e^x} + \ln(2e^x)$.

Use log rules.

$$\begin{aligned}&\ln \frac{1}{e^x} + \ln(2e^x)\\ =\,&\ln e^{-x} + \ln 2 + \ln e^x\\ =\,&-x + \ln 2 + x\\ =\,&\ln 2.\end{aligned}$$

1340. Show that the x intercept of the line through (a, b) and (c, d) is at

$$x = \frac{bc - ad}{b - d}.$$

Find the equation of the line, and solve for the x intercept.

The gradient of the line is $\frac{d-b}{c-a}$, hence, its equation, using $y - y_0 = m(x - x_0)$, is

$$y - b = \frac{d - b}{c - a}(x - a).$$

Setting y to zero for the x intercept:

$$\begin{aligned}&-b = \frac{d-b}{c-a}(x-a)\\ \Longrightarrow\;&\frac{-bc+ab}{d-b} = x - a\\ \Longrightarrow\;&a + \frac{bc-ab}{b-d} = x\\ \Longrightarrow\;&\frac{ab-ad}{b-d} + \frac{bc-ab}{b-d} = x\\ \Longrightarrow\;&x = \frac{bc-ad}{b-d}.\end{aligned}$$

1341. Write down the functions whose derivatives are given by the following limits:

(a) $f'(x) = \displaystyle\lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x-h+1}}{2h}$.

(b) $g'(x) = \displaystyle\lim_{a,b\to x} \frac{a^2 - b^2 + \frac{1}{a} - \frac{1}{b}}{a - b}$.

Sketch the relevant gradient triangles.

(a) This is a gradient calculated using values $x+h$ and $x-h$. The function is $f(x) = \sqrt{x+1}$.

(b) This is a gradient calculated using points at a and b. The function is $g(x) = x^2 + \frac{1}{x}$.

1342. An arithmetic progression begins s, s^2, s^2+6. Find all possible values for the hundredth term.

Equate the differences $u_2 - u_1 = u_3 - u_2$.

Equating the differences, we get

$$\begin{aligned}&s^2 - s = 6\\ \Longrightarrow\;&s^2 - s - 6 = 0\\ \Longrightarrow\;&(s-3)(s+2) = 0\\ \Longrightarrow\;&s = -2, 3.\end{aligned}$$

In each case, the common difference is 6. So, $u_{100} = -2 + 99\cdot 6 = 592$ or $u_{100} = 3 + 99\cdot 6 = 597$.

1343. Ignoring leap years etc., find the probability that, in any given week, at least one of a class of 25 students will have a birthday.

Assume that every pupil has a $\frac{7}{365}$ probability of having a birthday in a given week.

Assume that each pupil has (independently) a $\frac{7}{365}$ probability of having a birthday in a given week.

$$\begin{aligned}&P(\text{at least one birthday})\\ =\,&1 - P(\text{no birthdays})\\ =\,&1 - \left(\tfrac{358}{365}\right)^{25}\\ =\,&0.38375... = 0.384 \text{ (3sf)}.\end{aligned}$$

1344. If $\dfrac{d}{dt}(4x + 2t^2) = 6t$, find $\dfrac{dx}{dt}$ in terms of t.

The differential operator $\frac{d}{dt}$ distributes over addition, so you can apply it to each term in the bracket. The first term gives $4\frac{dx}{dt}$.

Applying the differential operator, we get

$$\begin{aligned}&4\frac{dx}{dt} + 4t = 6t\\ \Longrightarrow\;&4\frac{dx}{dt} = 2t\\ \Longrightarrow\;&\frac{dx}{dt} = \frac{1}{2}t.\end{aligned}$$

1345. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a polynomial function f:

- $f(x)$ has a factor of $(x - a)$,
- $f(x)$ has a root at $x = a$.

This is the factor theorem.

These are the statements in the factor theorem, whose implication goes both ways, so $\Longleftrightarrow$.

1346. Solve $\dfrac{1}{(1+\sqrt{x})^2} + \dfrac{1}{(1-\sqrt{x})^2} = 1$.

Multiply up by the denominators.

Multiplying up gives a difference of two squares on the RHS. This simplifies to

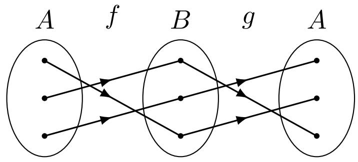
$$\begin{aligned}
&(1-\sqrt{x})^2+(1+\sqrt{x})^2=(1-x)^2\\
\Longrightarrow\ &2+2x=1-2x+x^2\\
\Longrightarrow\ &x^2-4x-1=0\\
\Longrightarrow\ &x=\frac{4\pm\sqrt{16+4}}{2}\\
\Longrightarrow\ &x=2\pm\sqrt{5}.
\end{aligned}$$

1347. Using any numerical methods, factorise fully the cubic expression $2x^3-5x^2-21x+36$.

Solve $2x^3-5x^2-21x+36=0$ on your calculator, then reverse engineer the factorisation.

Using a calculator, $2x^3-5x^2-21x+36=0$ has solution $x=-3,4,\frac{3}{2}$. So, by the factor theorem, the factorisation is $(x+3)(x-4)(2x-3)$.

1348. Functions f and g map two sets A and B to each other, according to the following diagram. The two copies of A are ordered in the same way.



(a) Give the domains of fg and gf.

(b) Describe the behaviour of $x_{n+1}=gf(x_n)$.

In (a), fg means "perform g, then f". In (b), the sequence is periodic; find the period.

(a) The right-hand function is performed first, so the domain of fg is B and of gf is A.

(b) The function $gf : A \mapsto A$ cycles the elements of A, mapping each to the one below it. Hence, the sequence given is periodic, with period 3.

1349. Prove that $\tan^2\theta \equiv \sec^2\theta-1$.

Begin with $\sin^2\theta+\cos^2\theta\equiv 1$.

Start with the standard Pythagorean trig identity, and then divide through by $\cos^2\theta$.

$$\begin{aligned}
&\sin^2\theta+\cos^2\theta\equiv 1\\
\Longrightarrow\ &\tan^2\theta+1\equiv\sec^2\theta\\
\Longrightarrow\ &\tan^2\theta\equiv\sec^2\theta-1,\ \text{as required.}
\end{aligned}$$

1350. A car of mass 400 kg is pulling a caravan of mass 800 kg up a slope of inclination 5°. Resistances of 250 N and 350 N act on car and caravan.

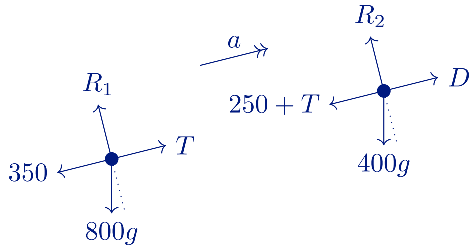
(a) Draw force diagrams for the vehicles.

(b) Determine the minimum driving force D such that the car can pull the caravan up a slope of unlimited length.

(c) Find the tension in the tow-bar with this D.

In (b), the minimum driving force is when the acceleration is zero.

(a) Force diagrams:



(b) The minimal acceleration to move indefinitely is $a=0$. Resolving up the slope for the entire system, we get $D-1200g\sin 5°-600=0$. So, the minimal driving force is $D=1625$ N (4sf).

(c) Considering now only the caravan, with $a=0$, we have $T-800g\sin 5°-350=0$, which gives the tension in the towbar as $T=1033$ N (4sf).

1351. Solve exactly the equation $x^{-\frac{1}{2}}+1=x^{\frac{1}{2}}$.

This is a disguised quadratic, in $\sqrt{x}$.

Multiplying up by $x^{\frac{1}{2}}$ to eliminate the negative power, $1+x^{\frac{1}{2}}=x$ is a quadratic in $x^{\frac{1}{2}}$.

$$\begin{aligned}
&x-x^{\frac{1}{2}}-1=0\\
\Longrightarrow\ &x^{\frac{1}{2}}=\frac{1\pm\sqrt{5}}{2}.
\end{aligned}$$

Since $x^{\frac{1}{2}}$ is always positive, we take the positive root, giving

$$x=\left(\frac{1+\sqrt{5}}{2}\right)^2=\frac{3+\sqrt{5}}{2}.$$

1352. The equation of a straight line, gradient m, passing through the point (p,q) is

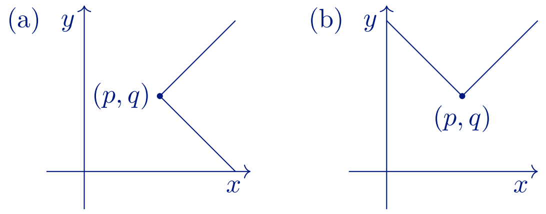
$$\frac{y-q}{x-p}=m.$$

Sketch the following graphs:

(a) $\dfrac{|y-q|}{x-p}=m$, (b) $\dfrac{y-q}{|x-p|}=m$.

In (a) the behaviours are different above and below the line $y=q$. In (b) the behaviours are different to the left and right of the line $x=p$.

The sketches are



	1	2	3	4	5	6
1						
2	✓					
3	✓	✓				
4	✓	✓	✓			

The probability is $\frac{6}{24} = \frac{1}{4}$.

1353. True or false?

(a) $\frac{d}{dx}(1 + 2x) = 2$,

(b) $\frac{d}{dy}(1 + 2y) = 2$,

(c) $\frac{d}{dx}(1 + 2y) = 2$.

Substitute $y = x^2$ into (c) and see if it holds.

(a) True.

(b) True. It's the same statement as in (a).

(c) Not true (unless $y = x$ happens to be true). For example, if y is a constant then the LHS is zero.

1354. The *triangular numbers* are defined by the ordinal formula $T_n = \frac{1}{2}n(n + 1)$, for $n \in \mathbb{N}$. Prove that

$$T_{n+1} + T_n \equiv (T_{n+1} - T_n)^2 .$$

Simplify each side separately as an expression, to reach $(n + 1)^2$.

Starting with the LHS,

$$\begin{aligned}
&T_{n+1} + T_n \\
\equiv&\tfrac{1}{2}(n + 1)(n + 2) + \tfrac{1}{2}n(n + 1) \\
\equiv&\tfrac{1}{2}(n + 1)[(n + 2) + n] \\
\equiv&\tfrac{1}{2}(n + 1)[2n + 2] \\
\equiv&(n + 1)^2.
\end{aligned}$$

Starting with the RHS,

$$\begin{aligned}
&(T_{n+1} - T_n)^2 \\
\equiv&\left(\tfrac{1}{2}(n + 1)(n + 2) - \tfrac{1}{2}n(n + 1)\right)^2 \\
\equiv&\left(\tfrac{1}{2}n^2 + \tfrac{3}{2}n + 1 - \tfrac{1}{2}n^2 - \tfrac{1}{2}n)\right)^2 \\
\equiv&(n + 1)^2.
\end{aligned}$$

Therefore, the identity holds.

1355. A four-sided die and a six-sided die are rolled at the same time. Find the probability that the score on the four-sided die is the larger of the two.

Draw a possibility space as a 4×6 grid.

Marking outcomes in the possibility space:

1356. Show that $\displaystyle\int_0^{1000} \frac{(1 + \sqrt[3]{x})^2}{\sqrt[3]{x}}\,dx = 9650.$

Multiply out the top of the integrand, then split the fraction up.

By multiplying out and splitting the fraction up, we can simplify the integrand to $x^{-\frac{1}{3}} + 2 + x^{\frac{1}{3}}$. Then, integrating, we get

$$\begin{aligned}
&\int_0^{1000} x^{-\frac{1}{3}} + 2 + x^{\frac{1}{3}}\,dx \\
=&\left[\tfrac{3}{2}x^{\frac{2}{3}} + 2x + \tfrac{3}{4}x^{\frac{4}{3}}\right]_0^{1000} \\
=&(150 + 2000 + 7500) - (0) \\
=&9650.
\end{aligned}$$

1357. By finding perpendicular bisectors, or otherwise, find the equation of the circle which passes through the points $A : (1, 0)$, $B : (7, -2)$ and $C : (17, 8)$.

Find the equation of the perpendicular bisector of AB and of AC, and solve to find their intersection. This is the centre of the circle. Then calculate the radius using Pythagoras.

The gradient $m_{AB} = -\frac{1}{3}$, so the perp. bisector is $y + 1 = 3(x - 4)$. Then, $m_{AC} = \frac{1}{2}$, so the bisector is $y - 4 = -2(x - 9)$. Solving simultaneously gives $O : (7, 8)$ as the centre. The squared distance OA^2 is then $36 + 64 = 100$. So, the equation of the circle is $(x - 7)^2 + (y - 8)^2 = 100$.

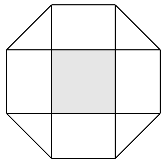
1358. Simplify $\dfrac{1 - 9x^{\frac{5}{2}}}{1 - 3x^{\frac{5}{4}}}$.

Consider the numerator as a difference of two squares.

The numerator is a difference of two squares. So,

$$\frac{1 - 9x^{\frac{5}{2}}}{1 - 3x^{\frac{5}{4}}} = \frac{(1 - 3x^{\frac{5}{4}})(1 + 3x^{\frac{5}{4}})}{1 - 3x^{\frac{5}{4}}} = 1 + 3x^{\frac{5}{4}}.$$

1359. A point is chosen at random on the interior of a regular octagon. Find the probability that it lies within the shaded square depicted.



Call the side length 1, and calculate the area of the octagon. This area is then the possibility space.

The possibility space is the interior of the octagon. Calling the side length 1, we can calculate the area of the octagon as

$$\underbrace{1}_{\text{square}} + \underbrace{4\times\frac{\sqrt{2}}{2}}_{\text{rectangles}} + \underbrace{4\times\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^2}_{\text{triangles}} = 2+2\sqrt{2}.$$

Hence, the probability is $\dfrac{1}{2+2\sqrt{2}}$.

1360. Find $\dfrac{dy}{dx}$ in terms of x, if $\dfrac{d}{dx}(4x^2+y)=0$.

Distribute the differential operator over the bracket, and rearrange to make $\frac{dy}{dx}$ the subject.

Performing the operation $\frac{d}{dx}$, we get $8x+\frac{dy}{dx}=0$. Hence, $\frac{dy}{dx}=-8x$.

1361. Two competitors are pulling against each other in a tug-of-war. Each pulls on one end of the same rope, and, when the rope has been displaced by 1 metre in either direction, the tug-of-war is won. The combined mass of both competitors and the rope is 200 kg. For $t\geq 0$, each competitor exerts a variable driving force horizontally on the ground: competitor 1 exerts $F_1=1480-20t$; competitor 2 exerts $F_2=1460-10t$.

(a) Show that competitor 1 gains initially, but fails to achieve victory.

(b) Using a numerical method, show that, after approximately 8 seconds, competitor 2 wins.

Consider the two competitors and rope as a single object, and use integration on $F=ma$.

(a) Initially, competitor 1 exerts more force, so gains. To calculate the displacement, we define a in competitor 1's direction, and NII gives

$$1480-20t-(1460-10t)=200a.$$

This simplifies to $a=\frac{1}{10}-\frac{1}{20}t$. Integrating, we get $v=\frac{1}{10}t-\frac{1}{40}t^2+c$. The $+c$ is zero, since the game starts from rest. Integrating again gives $s=\frac{1}{20}t^2-\frac{1}{120}t^3+d$. Again, since the initial displacement is zero, $d=0$.

The maximum positive displacement occurs when $v=0$. This is at $t=0$ or $t=4$. At that point, the displacement is $s=\frac{4}{15}<1$, so competitor 1 does not win.

(b) Competitor 2 wins when $\frac{1}{20}t^2-\frac{1}{120}t^3=-1$. Using a cubic solver, $x=7.91532...$. Therefore competitor 2 wins after approx. 8 seconds.

1362. Solve the equation $|x^2-3x|=|3x-1|$.

Use $|a|=|b| \implies a=\pm b$.

We get two quadratics: $x^2-3x=\pm(3x-1)$. These simplify to $x^2-1=0$ and $x^2-6x+1=0$. So, the full solution is $x=\pm 1$ or $x=3\pm 2\sqrt{2}$.

1363. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $\sin(\theta-\phi)=0 \implies \theta-\phi=0$,

(b) $\sin(\theta-\phi)=0 \impliedby \theta-\phi=0$.

The first is untrue. Find a counterexample.

The first statement is not true. Any pair (θ,ϕ) such that $\theta-\phi=\pi$ radians is a counterexample.

1364. Find the probability that, when two dice are rolled, the scores differ by more than 2.

Draw a possibility space.

	1	2	3	4	5	6
1				✓	✓	✓
2					✓	✓
3						✓
4	✓					
5	✓	✓				
6	✓	✓	✓			

So, the probability is $\frac{12}{36}=\frac{1}{3}$.

1365. It is given that $x=\frac{4}{15}$ is a root of the expression $45x^3-27x^2-26x+8$. Factorise it fully.

Using the factor theorem, $(15x-4)$ is a factor.

The factor theorem tells us $(15x-4)$ is a factor. Extracting it, we reach $(15x-4)(3x^2-x-2)$. The quadratic factorises: $(15x-4)(3x+2)(x-1)$.

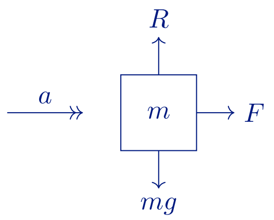
1366. A packing box of mass m is sitting on the floor of a delivery van, which is driving along a straight road. The coefficient of friction between the box and the floor of the van is μ. If the box is not to slide, determine, for the van,

(a) the maximum acceleration,

(b) the maximum magnitude of deceleration.

Give the box mass m, and draw a force diagram for it.

(a) A force diagram for the box is



Vertical $F = ma$ gives $R = mg$, which means that $F_{\max} = \mu mg$. Maximal acceleration without slipping means maximal friction, so NII horizontally is $\mu mg = ma$, giving $a_{\max} = \mu g$.

(b) The direction doesn't matter, so $a_{\max} = \mu g$.

1367. A line segment is defined by $\mathbf{r} = t\mathbf{i}$ for $t \in [-1, 1]$. Determine whether there are any points on this line segment which lie a distance $\sqrt{10}$ away from the point $(2, 3)$.

This can be done algebraically, using Pythagoras and solving for the parameter t, or geometrically, by finding the perpendicular.

Algebraically, we need the squared distance from $(t, 0)$ to $(2, 3)$ to be 10. Hence, $(t-2)^2 + 3^2 = 10$, so $t = 1, 3$. The latter value is not a valid t value for the line segment, but the former is. So, there is exactly one point, the endpoint $(1, 0)$.

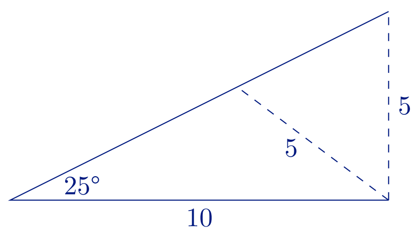
1368. Sketch, on one diagram, both triangles which have information $a = 5$, $b = 10$, $A = 25°$, and describe how the two solutions emerge in the algebra of

(a) the sine rule, finding angle B,

(b) the cosine rule, finding length c.

Either consider, in each case, the particular line of algebra at which two answers appear, or else simply do the calculations!

The triangles are as follows:



(a) In the sine rule for B, there are two angles B in $(0, 180°)$ satisfying $\sin B = 2 \sin 25°$.

(b) In the cosine rule for length c, the quadratic $25 = 100 + c^2 - 20c \cos 25°$ has two roots.

1369. Calculate $9 \displaystyle\sum_{i=1}^{\infty} \frac{1}{10^i}$.

This can be viewed either as a geometric series, or, equivalently, as a recurring decimal.

This can be viewed either as a geometric series, or, equivalently, as the recurring decimal 0.99999.... Hence, the sum to infinity is 1.

1370. Show that the binomial distribution $X \sim B(5, \frac{1}{3})$ has two modal values.

Using $E(X) = np$, the expectation is $1.\dot{6}$. So, check 1 and 2.

Since the expectation is $\frac{5}{3}$, the modal values must be 1 and 2. The probabilities are

$$P(X = 1) = {}^5C_1 \tfrac{1}{3}^1 \tfrac{2}{3}^4 = \tfrac{80}{243}$$
$$P(X = 2) = {}^5C_2 \tfrac{1}{3}^2 \tfrac{2}{3}^3 = \tfrac{80}{243}.$$

1371. It is given that a polynomial graph $y = h(x)$, with domain $\mathbb{R}$, is concave everywhere.

(a) Explain why the graph can have a maximum of one stationary point.

(b) Hence, explain why the equation $h(x) = 0$ can have a maximum of two roots.

In (a), consider the *nature* of any stationary points.

(a) The graph is concave everywhere, so any stationary point must be a maximum. But, if the graph is continuous, it is not possible to have two local maxima without a local minimum.

(b) If the graph has one stationary point, a local maximum at $x = \alpha$, then it is divided into two regions: the graph is increasing for $x < \alpha$ and decreasing for $x > \alpha$. In each of these regions, there can be at most one crossing of $y = 0$. Hence, $h(x) = 0$ has, at most, two roots.

1372. A segment subtending an angle of $k\pi$ radians at the centre is marked on a circle with radius 2 cm. The area of the segment is $\frac{5}{3}\pi - 1$. Determine k, given that it is a rational number.

Find the area of the segment by finding (algebraically) the area of the sector and subtracting the area of the triangle.

The area of the relevant sector is $\frac{1}{2}2^2 k\pi = 2k\pi$. The area of the triangle is $\frac{1}{2}2^2 \sin k\pi = 2 \sin k\pi$. So, we need to solve

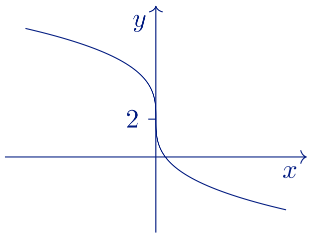
$$2k\pi - 2\sin k\pi = \tfrac{5}{3}\pi - 1.$$

Since k is a rational number, we can equate the coefficients of π, giving $2k = \frac{5}{3} \implies k = \frac{5}{6}$. Checking the rational terms, $2\sin\frac{5}{6}\pi = 2 \times \frac{1}{2} = 1$, and our solution works.

1373. Sketch $y = 2 - \sqrt[3]{x}$.

Sketch $y = \sqrt[3]{x}$ first, considering it as a reflection of $y = x^3$ in the line $y = x$. Then perform two output transformations.

The curve $y = \sqrt[3]{x}$ is a reflection of $y = x^3$ in the line $y = x$. It is then reflected in the x axis to $y = -\sqrt[3]{x}$ and translated by 2 units in the y direction to $y = 2 - \sqrt[3]{x}$:



1374. Simultaneous equations are given as follows:

$$
\begin{aligned}
x + y - 2z &= -8, \\
4x - 3y + z &= 6, \\
2x + 5y - 2z &= -3.
\end{aligned}
$$

Solve to find x, y, z.

Eliminate z from equations 1 and 2, and then from equations 1 and 3. Solve the resulting two equations in x and y.

Eliminating z from the first two equations gives $9x - 5y = 4$, and from equations 1 and 3 gives $x + 4y = 5$. Solving these simultaneously, we get $x = 1, y = 1$. Substituting back in, $z = 5$.

1375. Show that (x^2+1) is not a factor of $4x^3 - 12x^2 + 18$.

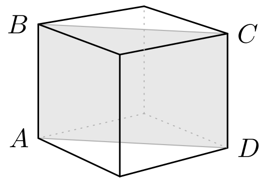
You can't use the factor theorem here (at least not over the real numbers). Attempt the factorisation explicitly, and find a contradiction.

We can't use the factor theorem here (at least not over the real numbers.) So, we must attempt the factorisation explicitly. If there were such a factorisation, then the remaining factor must be linear. So, assume, for a contradiction, that

$$4x^3 - 12x^2 + 18 \equiv (x^2 + 1)(Ax + B).$$

Comparing coefficients of x^3 tells us that $A = 1$, but then, comparing coefficients of x, we get $0 = 1$. So, $(x^2 + 1)$ is not a factor.

1376. The diagram shows a cube of unit side length, with a rectangle formed between four vertices.



Three points are chosen at random inside the cube. Find the probability that no two of these points are separated by rectangle $ABCD$.

This is a simpler question than it looks. Place the first point somewhere. Then each subsequent point has a 50% chance of being separated from it.

The rectangle divides the cube in half by volume. Hence, placing the first point anywhere (without loss of generality), the probability that the second two are in its half is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

1377. A sequence has n^{th} term $A_n = 1000 - 85n + 3n^2$. Determine the lowest value A_n in the sequence.

Assuming continuity, use calculus to find the minimum point, then consider the integers either side of it.

Assuming continuity of n, we find the minimum with $\frac{dA}{dn} = 6n - 85 = 0$. This yields $n = 14.16...$. Checking $n = 14$ and $n = 15$, we get $A_{14} = 398$ and $A_{15} = 400$. Hence, the lowest value in the sequence is 398.

1378. State, with a reason, whether the following holds:

$$\frac{d}{dx}\int f(x)\,dx \equiv \int f'(x)dx.$$

Consider the $+c$ in integration.

This does not hold. Since the RHS is an integral, it has an arbitrary $+c$. But the LHS doesn't: its $+c$ is eliminated by subsequent differentiation.

1379. Prove that no four distinct points on a cubic are collinear.

Consider the equation formed were you to solve to find intersections of the cubic and the relevant straight line.

Assume, for a contradiction, that there are four distinct points $x = x_i$ for $i = 1, 2, 3, 4$ on a cubic $y = ax^3 + bx^2 + cx + d$ that are collinear, lying on $y = px + q$. Solving these simultaneously gives $ax^3 + bx^2 + (c-p)x + d - q = 0$. This is a cubic, so can have at most three distinct factors. However, by the factor theorem $(x - x_i)$, for $i = 1, 2, 3, 4$, are distinct factors. This is a contradiction. Hence, no four distinct points on a cubic are collinear. $\square$

1380. Show that the Newton-Raphson iteration for the equation $x^3 + 3x - 1 = 0$ may be written as

$$x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 + 3}.$$

The Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Put the RHS over a common denominator.

Setting $f(x) = x^3 + 3x - 1$, we differentiate to get $f'(x) = 3x^2 + 3$. Hence, the N-R iteration is

$$x_{n+1} = x_n - \frac{x^3 + 3x - 1}{3x^2 + 3}.$$

Putting this over a common denominator, we get

$$x_{n+1} = \frac{x_n(3x^2 + 3) - x^3 - 3x + 1}{3x^2 + 3} = \frac{2x^3 + 1}{3x^2 + 3}.$$

1381. True or false?

(a) Every cubic has a linear factor,

(b) Every quartic has a linear factor,

(c) Every quintic has a linear factor.

Consider the relevant graphs. Any linear factor corresponds to a root. Do cubic, quartic or quintic graphs necessarily have at least one root?

Every polynomial of odd degree must have at least one linear factor, because every polynomial graph of odd degree must cross the x axis at least once. This isn't true of polynomials of even degree, for example $y = x^2 + 1$.

(a) True.

(b) False.

(c) True.

1382. Vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy the following equations:

$$\mathbf{a} + \mathbf{b} = 2\mathbf{i} + 6\mathbf{j},$$
$$\mathbf{a} - 2\mathbf{b} = 8\mathbf{i} - 6\mathbf{j}.$$

Show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Solve simultaneously to find $\mathbf{a}$ and $\mathbf{b}$ explicitly.

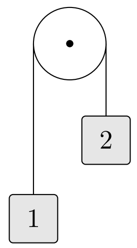
Subtracting the two equations gives $\mathbf{b} = -2\mathbf{i} + 4\mathbf{j}$. Substituting back in, $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$. The gradients of the two vectors are $m_{\mathbf{a}} = \frac{1}{2}$, $m_{\mathbf{b}} = -2$, which are negative reciprocals, so $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

1383. By considering either a unit circle or graphs, prove that $\sin^2(\theta + 180°) + \cos^2(\theta - 180°) = 1$.

Consider the effect on $\sin^2\theta + \cos^2\theta \equiv 1$ when replacing the relevant inputs.

Adding 180° is the same as subtracting 180° on a unit circle. Hence, the identity given is the same as $\sin^2(\theta + 180°) + \cos^2(\theta + 180°) = 1$. Let $\phi = \theta + 180°$, and this is $\sin^2\phi + \cos^2\phi \equiv 1$.

1384. Two blocks are connected by a light, inextensible string, which is passed over a smooth, light, fixed pulley as shown in the diagram. Masses are given in kg. The system is released from rest.

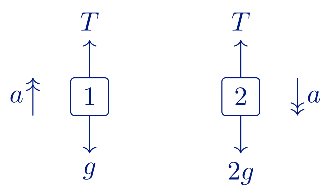


(a) Find the time taken for the masses to move a distance of 1 metre.

(b) At this time, the string snaps. Find the total displacement of the 1 kg mass when it comes to instantaneous rest.

In (a), set up force diagrams and find the acceleration, then use *suvat*. In (b), use *suvat* with $a = -g$.

(a) The force diagrams are



So $T - g = a$ and $2g - T = 2a$. Adding, $a = \frac{1}{3}g$. This gives $1 = \frac{1}{6}t^2$, so $t = \sqrt{6}$ seconds.

(b) The velocity when the string breaks is $\frac{1}{3}\sqrt{6}$. Hence, the subsequent displacement s is given by $0 = (\frac{1}{3}\sqrt{6})^2 - 2gs$. So, $s = 0.03401...$ This is combined with the original displacement of 1 m, giving $s = 1.03$ metres (3sf).

1385. If $\log_q p = x$, write $\left(\sqrt[3]{q}\right)^x$ in terms of p.

Rewrite the original log statement as an index statement, then cube root both sides.

The original statement is $q^x = p$. Cube rooting both sides, we get $\sqrt[3]{q^x} = \sqrt[3]{p}$. Switching the order of the multiplied indices, this gives $\left(\sqrt[3]{q}\right)^x = \sqrt[3]{p}$.

1386. Three white counters and three black counters are placed in a bag. Three of the counters are then drawn out. Find the probability of drawing out

(a) three white counters,

(b) two of one type and one of the other.

Consider the number of successful outcomes in each case.

(a) There is only one successful outcome, so

$$p = \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} = \frac{1}{20}.$$

(b) There are six successful outcomes: *wwb* in any order and *bbw* in any order. Each outcome has the same probability, giving

$$p = 6 \times \frac{3}{6} \times \frac{2}{5} \times \frac{3}{4} = \frac{9}{10}.$$

1387. Solve $\displaystyle\sum_{i=1}^{2} \frac{1}{1 - x^i} = 0$.

Write the sum out explicitly, and multiply up by the denominators.

Written longhand, we have

$$\begin{aligned}
&\frac{1}{1-x} + \frac{1}{1-x^2} = 0 \\
\Longrightarrow\ &\frac{1-x^2}{1-x} + 1 = 0 \\
\Longrightarrow\ &1 + x + 1 = 0 \\
\Longrightarrow\ &x = -2.
\end{aligned}$$

1388. State, with a reason, whether the following holds: "It is possible to have, at a single point, friction and reaction exerted by the same object A, on the same object B, such that the two forces have a component in the same direction."

This is not possible. Find the reason.

This is not possible, by (modern, i.e. not Newton's original) definition of the word "reaction". The terms "reaction" and "friction" are (now) used to refer to the components of a contact force perpendicular and parallel to the surfaces in contact. Hence, they can never have a component in the same direction.

1389. An *Egyptian fraction* is a sum of fractions of the form $\frac{1}{n_i}$, for $n_i \in \mathbb{N}$. Write $\frac{5}{12}$ in this form.

Put $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ over a denominator of 12.

We have $\frac{6}{12}, \frac{4}{12}, \frac{3}{12}, \frac{2}{12}$ and $\frac{1}{12}$ to play with. So, the combination needed is $\frac{5}{12} = \frac{3}{12} + \frac{2}{12} = \frac{1}{4} + \frac{1}{6}$.

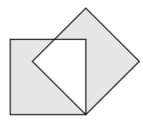
1390. State, with a reason, whether the following gives a well-defined function:

$$f : \begin{cases} [-1, 1] \mapsto [0, 2.7] \\ x \mapsto \sqrt{1 - x^2}. \end{cases}$$

Consider whether every element of the domain maps to an element of the codomain.

This is well defined. With domain $[-1, 1]$, the function $1 - x^2$ has range $[0, 1]$, acceptable inputs for the square root function. Furthermore, they produce outputs in the range $[0, 1]$, making $[0, 2.7]$ a well-defined, albeit slightly bizarre, codomain.

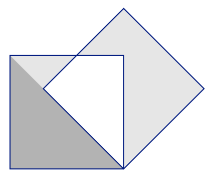
1391. Two unit squares are placed as depicted below. All acute angles in the diagram are 45°.



Determine the exact area of the shaded region.

Split each shaded section up into a pair of right-angled isosceles triangles.

We can split each shaded section as follows:



Considering the diagonal of length $\sqrt{2}$, the small triangle has side length $\sqrt{2} - 1$. Hence, its area is $\frac{1}{2}(\sqrt{2} - 1)^2 = \frac{1}{2}(3 - 2\sqrt{2})$. The darker triangle has area $\frac{1}{2}$. In total, two of each triangle makes $A = 1 + (3 - 2\sqrt{2}) = 4 - 2\sqrt{2}$.

1392. Describe all functions f for which f'' is linear. Express the fact "f'' is linear" algebraically as $f''(x) = ax + b$, and integrate twice. Consider separately the cases where $a = 0$ or $a \neq 0$. Translating, we have $f''(x) = ax + b$. If $a = b = 0$, then integrating twice gives any linear function. If $a = 0, b \neq 0$, then integrating twice gives any quadratic function. If $a \neq 0$, then integrating twice gives any cubic function. Hence, such functions f are polynomial of order at most 3.

1393. A tank in the shape of a cuboid has dimensions $a \times a \times b$. Its volume is 1 cubic metre.

(a) Show that the surface area is given by $A = 2a^2 + 4a^{-1}$.

(b) Hence, show that the minimum value of the surface area is 6 square metres.

In (a), express the surface area in terms of a and b, and then use the volume formula to substitute for b. In (b), differentiate to find points at which A is stationary relative to a.

(a) The surface area is $A = 2a^2 + 4ab$. The volume is $1 = a^2 b$. So, substituting for $b = a^{-2}$, we get $A = 2a^2 + 4a \times a^{-2} = 2a^2 + 4a^{-1}$.

(b) The minimum surface area occurs when A is stationary relative to a. So, we differentiate and set the derivative to zero:

$$\begin{aligned} \frac{dA}{da} &= 4a - 4a^{-2} = 0 \\ \Longrightarrow a^3 &= 1 \\ \Longrightarrow a &= 1. \end{aligned}$$

Substituting back in, this gives $A = 6$ m^2. We can verify that this is a minimum by checking the second derivative at $a = 1$; it has value $12 > 0$, giving a local minimum.

1394. The area of a regular octagon of side length 1 is $A = 2(1 + \sqrt{2})$. Use this result to find the exact distance from the centre to any vertex.

Call the distance r, and use the sine area formula for a triangle on one of the sectors of the octagon.

Splitting the octagon into eight sectors, each has area $\frac{1}{4}(1 + \sqrt{2})$. Each is an isosceles triangle with 45° subtended at the centre, between two radii r. Using the sine area formula, we know that

$$\begin{aligned} \tfrac{1}{2} r^2 \sin 45° &= \tfrac{1}{4}(1 + \sqrt{2}) \\ \Longrightarrow \tfrac{1}{4}\sqrt{2} r^2 &= \tfrac{1}{4}(1 + \sqrt{2}) \\ \Longrightarrow r^2 &= \tfrac{\sqrt{2}}{2} + 1. \end{aligned}$$

Therefore $r = \sqrt{\frac{\sqrt{2}}{2} + 1}$.

1395. State, with justification, whether the curve $y = |x|$ intersects the following curves:

(a) $y = |x + 1|$,

(b) $y = |x| + 1$.

(c) $y = |x + 1| + 1$.

Sketch the curves.

(a) Yes, at $x = -\frac{1}{2}$.

(b) No, because the $+1$ is a vertical translation.

(c) Yes, at all points $x \le -1$.

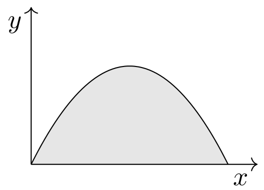
1396. The circles $(x + b)^2 + y^2 = b^2$ and $x^2 + y^2 = 1$ have at least one point of intersection. Find all possible values of b.

The first circle passes through the origin.

The first circle has centre $(-b, 0)$ and passes through the origin (radius b). Hence, if any part of the first circle is on or outside the unit circle, then they will intersect. This occurs whenever $b \ge \frac{1}{2}$.

1397. The parametric curve below has equations

$$x = 2t, \qquad y = 4t - 4t^2.$$



(a) Find the values t_1, t_2 at which $y = 0$.

(b) Write down the value of $\dfrac{dx}{dt}$.

(c) Find the area of the shaded region, using the parametric integration formula

$$A = \int_{t_1}^{t_2} y \frac{dx}{dt} \, dt.$$

In (c), substitute for y and for $\frac{dx}{dt}$, simplifying in terms of t. Then perform the definite integral over t.

(a) Solving $y = 4t - 4t^2 = 0$ gives $t_1 = 0, t_1 = 1$.

(b) $\dfrac{dx}{dt} = 2$.

(c) Substituting for y and $\dfrac{dx}{dt}$,

$$\begin{aligned} A &= \int_0^1 (4t - 4t^2) \cdot 2 \, dt \\ &= \left[4t^2 - \tfrac{8}{3} t^3 \right]_0^1 \\ &= (4 - \tfrac{8}{3}) - (0) \\ &= \tfrac{4}{3}. \end{aligned}$$

1398. By writing over base a, prove that $x^{\log_a y} = y^{\log_a x}$.

Start with the LHS, expressing x as a^*.

Expressing x as a power of a, the LHS is

$$\begin{aligned} & x^{\log_a y} \\ \equiv \; & (a^{\log_a x})^{\log_a y} \\ \equiv\!= \; & a^{\log_a x \log_a y}. \end{aligned}$$

Since this expression is symmetrical in x and y, the RHS must also be equal to it, proving the identity.

1399. In applying for a course, candidates undertake two rounds of testing. Only candidates who pass the first test then sit the second. The probability that any given candidate passes the first test is 0.4, and the probability that any given candidate passes both tests is 0.1.

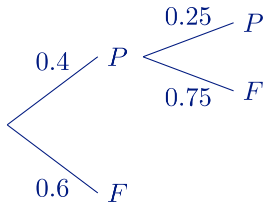
(a) Draw a tree diagram, finding all branch probabilities.

(b) Given that a particular candidate has not been accepted, find the probability that he/she failed on the first test.

(c) Find the probability that, of two randomly chosen candidates, exactly one is accepted.

In (b), restrict the possibility space to those outcomes in which the candidate is not accepted.

(a) Since, $0.4 \times P(\text{passing second}) = 0.1$, we know that $P(\text{passing second}) = 0.25$.



(b) The possibility space is restricted to the lower two outcomes. So, the probability is

$$\frac{P(\text{fail first})}{P(\text{fail})} = \frac{0.6}{0.6 + 0.4 \times 0.75} = \tfrac{2}{3}.$$

(c) $P(\text{acceptance}) = 0.1$, so $X \sim B(2, 0.1)$. Thus, $P(X = 1) = {}^2C_1 \times 0.1 \times 0.9 = 0.18$.

1400. The points (a, a), $(0, a-1)$ and $(8, 5)$ are collinear. Find all possible values of a.

Equate the gradients, as calculated from two pairs of points.

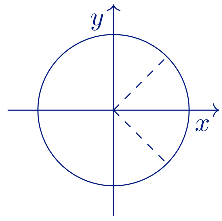
Equating gradients, we have

$$\begin{aligned} \frac{-1}{-a} &= \frac{6-a}{8} \\ \Longrightarrow\ 8 &= a(6-a) \\ \Longrightarrow\ (a-2)(a-4) &= 0 \\ \Longrightarrow\ a &= 2, 4. \end{aligned}$$

1401. Find the six roots of $\cos 3\theta = \dfrac{\sqrt{2}}{2}$ in $[0, 2\pi)$.

Take care to generate all of the extra solutions at the moment you apply the arccos function.

Using the unit circle, extra solutions are generated as reflections in the x axis, as follows. We will need three full rotations around the unit circle, to allow for subsequent division by 3.



Algebraically, this gives

$$\begin{aligned} \cos 3\theta &= \frac{\sqrt{2}}{2} \\ \Longrightarrow\ 3\theta &= ..., \tfrac{\pi}{4}, \tfrac{7\pi}{4}, \tfrac{9\pi}{4}, \tfrac{15\pi}{4}, \tfrac{17\pi}{4}, \tfrac{23\pi}{4}, ... \\ \Longrightarrow\ \theta &= ..., \tfrac{\pi}{12}, \tfrac{7\pi}{12}, \tfrac{9\pi}{12}, \tfrac{15\pi}{12}, \tfrac{17\pi}{12}, \tfrac{23\pi}{12}, ... \end{aligned}$$

1402. Show that the normal to $y = x^2$ at $x = 1$ crosses the curve again at $x = -\frac{3}{2}$.

Find the equation of the normal by differentiating, and then solve simultaneously.

Differentiating, $\frac{dy}{dx} = 2x$. So, $m_{\text{tangent}} = 2$ and $m_{\text{normal}} = -\frac{1}{2}$. The equation of the normal, then, is $y - 1 = -\frac{1}{2}(x-1)$, giving $y = -\frac{1}{2}x + \frac{3}{2}$. Solving simultaneously, we get

$$\begin{aligned} & x^2 = -\tfrac{1}{2}x + \tfrac{3}{2} \\ \Longrightarrow\ & 2x^2 + x - 3 = 0 \\ \Longrightarrow\ & (x-1)(2x+3) = 0 \\ \Longrightarrow\ & x = 1 \text{ or } x = -\tfrac{3}{2} \text{ as required.} \end{aligned}$$

1403. A quadrilateral has sides of length $1, 3, 3, 5$, not necessarily in that order. State, with a reason, whether it is possible for the quadrilateral to

(a) be an isosceles trapezium,

(b) have an interior right-angle,

(c) be cyclic.

All three are possible, but (b) looks like a triangle!

(a) Yes, with sides in order $1, 3, 5, 3$.

(b) Yes, with $1 + 3$ set in a straight line to form a length 4, giving a $(3, 4, 5)$ triangle.

(c) Yes, in the form of part (a).

1404. If $u = \tan 2\theta$, find $\dfrac{du}{d\theta}$.

Use the chain rule, and the standard derivative for tan.

By the chain rule $\dfrac{du}{d\theta} = 2\sec^2 2\theta$.

1405. A sample $\{x_i\}$ is taken, and the sample mean $\bar{x}$ is calculated. Afterwards, a quarter of the x_i values are increased by 10%, and the rest are reduced by 10%. Find the expected percentage change in $\bar{x}$.

Weight the scale factors by $\frac{1}{4}$ and $\frac{3}{4}$.

The expected scale factor ("expected", because we don't know which values are being increased and which reduced) is

$$\frac{1}{4} \times 1.1 + \frac{3}{4} \times 0.9 = 0.95.$$

So, the expected reduction is by 5%.

1406. Two distinct quadratic functions f and g have $f(a) = g(a)$ and $f'(a) = g'(a)$ for some $a \in \mathbb{R}$. Show that the equation $f(x) = g(x)$ has no other root but $x = a$.

Show that the two graphs $y = f(x)$ and $y = g(x)$ are tangent at $x = a$.

Since f and g are quadratic functions, $f(x) = g(x)$ is a quadratic equation, and has a maximum of two roots. Furthermore, we know that the graphs of $y = f(x)$ and $y = g(x)$ are tangent at $x = a$, because they have the same value and the same gradient. Hence, $f(x) = g(x)$ has a *double* root at $x = a$. So, there can be no roots elsewhere.

1407. Solve the equation $\dfrac{4}{\sqrt{x}} = \sqrt{x}(x + 3)$.

This is a quadratic, but consider carefully the validity of the roots you find.

Multiplying up, we get

$$\begin{aligned} 4 &= x(x+3) \\ \implies x^2 + 3x - 4 &= 0 \\ \implies x &= -4, 1. \end{aligned}$$

But $x = -4$ is not in the domain of the square root function, so the solution is $x = 1$.

1408. Prove that the sum of three consecutive squares is never a multiple of 3.

Begin with $(n-1)^2 + n^2 + (n+1)^2$.

Three such squares can be expressed as

$$(n-1)^2 + n^2 + (n+1)^2 \equiv 3n^2 + 2.$$

Hence, since $3n^2$ is a multiple of 3 for any integer n, $(n-1)^2 + n^2 + (n+1)^2$ can never be.

1409. A quadratic function is $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Show that

$$f\left(-\frac{b}{2a} - x\right) \equiv f\left(-\frac{b}{2a} + x\right).$$

Either proceed algebraically or (quicker) consider the axis of symmetry of the parabola $y = f(x)$.

From the formula/differentiating/completing the square, we know that $y = ax^2 + bx + c$ has a line of symmetry at $x = -\frac{b}{2a}$. The inputs $-\frac{b}{2a} - x$ and $-\frac{b}{2a} + x$ are reflections of each other in this line. Hence, they produce identical outputs.

1410. Show that the iteration $a_{n+1} = \sqrt{12a_n - 36}$ has exactly one fixed point, and determine its value.

A fixed point of $x_{n+1} = g(x_n)$ is a root of $x = g(x)$.

For fixed points, we solve $a = \sqrt{12a - 36}$. Squaring both sides, we get $a^2 - 12a + 36 = 0$, which has a double root at $a = 6$. Hence, this is the only fixed point of the iteration.

1411. Disprove the following statement:

If $f(x) \not\equiv g(x)$, then $f^2(x) \not\equiv g^2(x)$.

Find a counterexample, considering $(\pm)^2 = +$. Remember that $f^2(x)$ means "apply f twice".

A counterexample is $f(x) = x$ and $g(x) = 10 - x$. Apply either function twice, and you get $x \mapsto x$.

1412. An error has been found in a statistical procedure: a zero was missed off the end of the quantity $\sum x^2$. State whether the following will increase, decrease or neither when the error is corrected:

(a) the mean,

(b) S_{xx},

(c) the standard deviation.

Consider whether $\sum x^2$ appears in the relevant formulae.

(a) No change: $\sum x^2$ isn't involved in calculation.

(b) Increase: $S_{xx} = \sum x^2 - n\bar{x}^2$.

(c) Increase: $s = \sqrt{\dfrac{S_{xx}}{n}}$.

1413. A student comes across an apparent "proof" that $1 = 0$. It runs as follows:

Let $x = 1$.
This gives $x^2 - x = 0$.
So $x(x - 1) = 0$.
Dividing by $(x - 1)$ yields $x = 0$.
Hence $1 = 0$.

Explain precisely where the logic breaks down.

Consider division by zero.

In the fourth line, there is a division by zero. In algebra, one shouldn't divide by anything without considering whether it could be zero.

1414. Two dice are rolled. Find the probability that the difference between the scores is odd.

Draw a 6×6 possibility space.

The possibility space is

	1	2	3	4	5	6
1		✓		✓		✓
2	✓		✓		✓	
3		✓		✓		✓
4	✓		✓		✓	
5		✓		✓		✓
6	✓		✓		✓	

So, the probability is $\frac{18}{36} = \frac{1}{2}$.

1415. A function f is defined over the reals, and has range $[-a, a]$. Give the ranges of the following:

(a) $x \mapsto \dfrac{1}{a} f(x)$,

(b) $x \mapsto \dfrac{1}{a^2}(f(x))^2$.

Multiplying by a constant scales the range. Squaring, as in (b), renders all outputs positive.

(a) Scaling the outputs gives $[-1, 1]$.

(b) Squaring maps $[-a, a]$ to $[0, a^2]$. Hence, the range of the function given is $[0, 1]$.

1416. Evaluate $\log_p q^2 \times \log_q p^3$.

Use the fact that $\log_a b$ and $\log_b a$ are reciprocal.

Using the fact that $\log_p q$ and $\log_q p$ are reciprocal,

$$\log_p q^2 \times \log_q p^3 = 2 \log_p q \times 3 \log_q p = 6.$$

1417. Show that no (x, y) points satisfy both of

$$x^2 + y^2 = 1,$$
$$2x + 3y > 4.$$

Show that the straight line $2x + 3y = 4$ does not pass through the unit circle.

Solving $x^2 + y^2 = 1$ and $2x + 3y = 4$ gives us $(2 - \frac{3}{2}y)^2 + y^2 = 1$, hence $\frac{13}{4}y^2 - 6y + 3 = 0$. This has discriminant $\Delta = 36 - 4 \cdot \frac{13}{4} \cdot 3 = -3 < 0$. Hence, the straight line passes above the circle, and, since $2x + 3y > 4$ is satisfied by points above the line, no (x, y) points satisfy both conditions.

1418. The surd expression $x^{\frac{1}{3}} + 1$, where $x \in \mathbb{Z}$, may be rationalised by multiplication by a factor of the form $x^{\frac{2}{3}} + px^{\frac{1}{3}} + q$. Find p and q.

Multiply out and ensure that no terms in $x^{\frac{2}{3}}$ or $x^{\frac{1}{3}}$ appear in the product.

We want the following number to be rational:

$$\begin{aligned} &(x^{\frac{1}{3}} + 1)(x^{\frac{2}{3}} + px^{\frac{1}{3}} + q) \\ \equiv\ & x + (1 + p)x^{\frac{2}{3}} + (p + q)x^{\frac{1}{3}} + q. \end{aligned}$$

If x is not a perfect cube, then $x^{\frac{2}{3}}$ is irrational, so we require $(1 + p) = 0$, giving $p = -1$. Likewise with $x^{\frac{1}{3}}$, so $q = 1$.

1419. Solve the equation $\Big[ax^2 - a^2x\Big]_{x=1}^{x=2} = 0$.

This is a quadratic in a.

The notation gives

$$\begin{aligned} & (4a - 2a^2) - (a - a^2) = 0 \\ \implies\ & 3a - 3a^2 = 0 \\ \implies\ & a(1 - a) = 0 \\ \implies\ & a = 0, 1. \end{aligned}$$

1420. Show that the outputs of $h(x) = 4 - (x - 5)^2$ are always exceeded by its inputs.

Sketch $y = 4 - (x - 5)^2$ and $y = x$, showing that the graphs do not intersect.

At a fixed point, the output is equal to the input. Solving to find these, we have $4 - (x - 5)^2 = x$, and so $x^2 - 9x + 29 = 0$. This has discriminant $\Delta = 81 - 4 \times 29 = -35 < 0$, so the graphs $y = h(x)$ and $y = x$ do not intersect. And, since $y = h(x)$ is a negative parabola, it must be below $y = x$ for all x values. Therefore, $x > h(x)$ for all x.

1421. State, with a reason, which of the implications $\implies$, $\impliedby$, $\iff$ links the following statements concerning a polynomial function f:

- $(x - \alpha)$ does not divide exactly into $f(x)$,
- $f(\alpha) \neq 0$.

These are the negations of the two usual statements in the factor theorem.

The factor theorem states that: $(x - \alpha)$ divides exactly into $f(x)$ if and only if $f(\alpha) = 0$. And, if $A \iff B$, it must also be true that $A' \iff B'$. Hence, the implication is $\iff$.

1422. Find the exact angle, in radians, between the hands of a clock at twenty past three.

One hour is $\frac{\pi}{6}$ radians on a clock.

One hour is $\frac{\pi}{6}$ radians on a clock. So, at twenty past three, the minute hand is at $\frac{4\pi}{6}$ radians from 12, and the hour hand is at $(3+\frac{1}{3})\times\frac{\pi}{6} = \frac{5}{9}$ radians from 12. The difference is $\frac{\pi}{9}$ radians.

1423. At the point with x coordinate p, the tangent line to $y = x^3$ has equation $y = mx + c$.

(a) Find m in terms of p.

(b) Hence, determine the equation of the tangent line to the curve $y = x^3$ at $x = p$.

In (a), evaluate the first derivative at $x = p$. In (b), substitute the point (p, p^3) back into $y = mx + c$, with m as found.

(a) We differentiate to get $\frac{dy}{dx} = 3x^2$, which gives $m = 3p^2$ at $x = p$.

(b) So, the equation of a general tangent line is $y = 3p^2x + c$, passing through point (p, p^3) on the curve. Substituting, $p^3 = 3p^2 \cdot p + c$, so $c = -2p^3$. Hence, the equation of the tangent is $y = 3p^2x - 2p^3$.

1424. The interior angles of a quadrilateral form an AP. Give, in radians, the set of possible values for the second largest angle.

The lower bound on the second largest angle is the mean of the four. For the upper bound, consider the fact that the smallest angle cannot be less than zero.

The lower bound is attained if all four angles are equal at $\frac{1}{2}\pi$ radians. For the upper bound, the lowest the smallest angle can be is 0, which gives the angles as $\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi\}$. This value is, however, not attainable. So, the set of possible values is $[\frac{1}{2}\pi, \frac{2}{3}\pi)$ radians.

1425. A function h has domain $\mathbb{R}$ and range $[0, 1]$. State, with a reason, whether the following hold for all constants c:

(a) $h(x) + c \in [c, c + 1]$ for all $x \in \mathbb{R}$,

(b) $x \mapsto h(x) + c$ has range $[c, c + 1]$ over $\mathbb{R}$.

These two statement are not the same, but they have the same truth-value in this case.

The statements are different, but statement (b) implies statement (a). And statement (b) is true, because every value in $[0, 1]$ can be attained by $h(x)$, hence every value in $[c, c+1]$ can be attained by $h(x) + c$. So, statement (a) is also true.

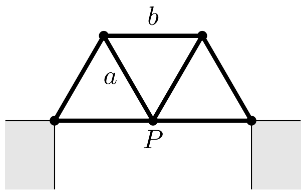
1426. By considering the signs of the factors, solve the following inequality, answering in set notation:

$$(x^2 + 5)(x - 1)(3x + 2) \geq 0.$$

The first factor can be ignored, as it is always positive.

The first factor can be ignored, because $x^2 + 5$ is always positive. The remaining two factors have sign changes at $x = 1$ and $x = -\frac{2}{3}$ respectively. Hence, the solution is $x \in (\infty, -\frac{2}{3}] \cup [1, \infty)$.

1427. A bridge over a stream is built of seven timber beams. The beams may be assumed to be light compared to a heavy load carried by the bridge, which is modelled as being located at point P. You can assume that the joints are freely pinned, i.e. that they are of negligible size, i.e. that no force applied at joint X has a moment around joint X.



Explain how you know, without any calculations, that the beam marked

(a) a must be in tension,

(b) b must be in compression.

Such questions may often be answered by considering what would happen where the relevant beam to disappear.

(a) Beam a and thus its counterpart must be in tension because they are the only beams which can exert a vertical force on the load at P. Since gravity is pulling P down, there must be tension in beam a pulling P up.

(b) If beam b were to disappear, there would be nothing stopping the upper two joints moving towards each other, and hence the load falling. So, beam b must be in compression, holding the upper joints apart.

1428. Show that, if $\log_x y + \log_y x = 2$, then $x = y$.

Use the fact that $\log_x y$ and $\log_y x$ are reciprocals.

The two logarithms are reciprocals of each other. Hence, we have $\log_x y + \frac{1}{\log_x y} = 2$. This is a quadratic in $\log_x y$:

$$
\begin{aligned}
&(\log_x y)^2 - 2\log_x y + 1 = 0 \\
\implies &(\log_x y - 1)^2 = 0 \\
\implies &\log_x y = 1 \\
\implies &x = y.
\end{aligned}
$$

1429. Give a counterexample to the following claim: "If a population is bimodal, then its median and mode cannot be equal."

Note that the modes of a "bimodal" population can be *local* modes, they do not have to both be *global* modes. In other words, they don't have to have precisely the same frequency.

Consider a population consisting of

$$\underbrace{0, ..., 0}_{100}, 1, 2, 3, 4, 5, 6, 7, 8, 9, \underbrace{10, ..., 10}_{50}.$$

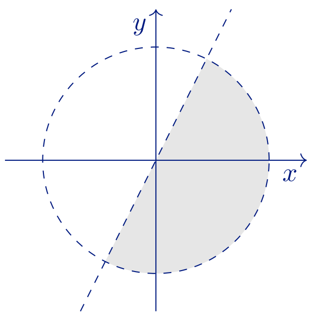
This is bimodal, since 0 and 10 represent distinct peaks. But the mode is 0 and the median is 0.

1430. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$y < 2x, \qquad 4x^2 + 4y^2 < 1.$$

The region is a semicircle.

The boundaries are $y = 2x$ and $x^2 + y^2 = \frac{1}{4}$, which is a circle of radius $\frac{1}{2}$ centred on the origin.



1431. A strip light is suspended from two vertical chains. These have been incorrectly affixed, dividing the length of the light (which is assumed uniform) asymmetrically in the ratio $a : b : c$, where $a \neq c$.

(a) Explain how you know that $\dfrac{a}{a+b+c} \leq \dfrac{1}{2}$.

(b) Show that the ratio of tensions is

$$a + b - c : c + b - a.$$

In (a), consider that the chains must lie either side of the centre of mass. In (b), take moments around the centre of mass.

(a) For equilibrium, the centre of mass must lie between the two chains. Hence, the lengths outside the chains cannot exceed half of the length of the light: $a \leq \frac{1}{2}(a + b + c)$.

(b) The centre of mass is in the middle, since the light is uniform. So, the ratio of distances from the centre is

$$
\begin{aligned}
&b + c - \frac{a+b+c}{2} : a + b - \frac{a+b+c}{2} \\
= &\frac{b+c-a}{2} : \frac{a+b-c}{2} \\
= &b + c - a : a + b - c.
\end{aligned}
$$

Taking moments around the centre of mass, the ratio of tensions must be the reciprocal of the ratio of distances, which gives us the result.

1432. Using a Venn Diagram, or otherwise, simplify

(a) $A \setminus (A \cap B')$.

(b) $(A \cap B') \cup (A' \cap B) \cup (A' \cap B')$.

Consider an element of each of the four regions of the Venn diagram.

(a) Subtracting "not-B" from A gives $A \cap B$.

(b) The three brackets are the three regions of the union $A' \cup B'$.

1433. You are given that the graphs $y = x^2 + 2x + 3$ and $y = x^2 + px + q$ intersect. Write down the sets of possible values for the constants p, q.

Solving simultaneously gives a linear equation.

Solving simultaneously, we get a linear equation. It gives the solution

$$2x + 3 = px + q \implies x = \frac{q-3}{2-p}$$

for all values of p and q except $p = 2$, which would involve division by zero. So, the sets are $p \in \mathbb{R}\setminus\{2\}$ and $q \in \mathbb{R}$.

1434. Solve the equation $\dfrac{2x-1}{3x-2} = \dfrac{4x-3}{5x-4}$.

Multiply up by both denominators.

Multiplying up by both denominators:

$$
\begin{aligned}
&(2x-1)(5x-4) = (3x-2)(4x-3) \\
\implies &10x^2 - 13x + 4 = 12x^2 - 17x + 6 \\
\implies &2x^2 - 4x + 2 = 0 \\
\implies &x = 1.
\end{aligned}
$$

1435. Show that the half-parabola $y = \sqrt{x}$ and the straight line $2y\sqrt{a} = x + a$ are tangent at $x = a$.

Set up an equation to find intersections, and then show that it has a double root at $x = a$.

Solving to find intersections, we substitute for y, which gives a quadratic in $\sqrt{x}$:

$$\begin{aligned} & 2\sqrt{x}\sqrt{a} = x + a \\ \Longrightarrow\ & x - 2\sqrt{x}\sqrt{a} + a = 0 \\ \Longrightarrow\ & (\sqrt{x} - \sqrt{a})^2 = 0 \\ \Longrightarrow\ & x = a. \end{aligned}$$

Furthermore, since this is a double root (squared factor of $\sqrt{x} = \sqrt{a}$), the meeting of the curves must be tangential.

1436. Functions f and g have domains and ranges

$$\begin{aligned} f &: A \longmapsto B, \\ g &: C \longmapsto D. \end{aligned}$$

State, with a reason, what must be true of sets A, B, C, D if fg is to be a well-defined function.

Consider the throughput (output of the inside function, which is the input of the outside function).

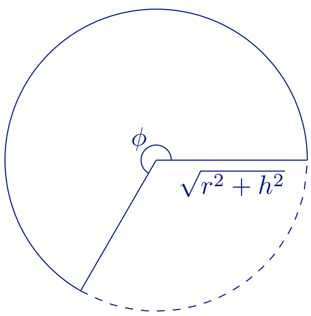
The outputs of function g must be throughput into function f. Hence, every element of set D must also be an element of set A, the permitted inputs of f. So, the necessary relationship is $D \subset A$.

1437. Prove that, if the curved surface of a cone of radius r and height l is unwrapped and laid flat, it forms a sector whose central angle ϕ, given in radians, is

$$\phi = \frac{2\pi r}{\sqrt{r^2 + h^2}}.$$

Calculate the ratio $\phi : 2\pi$, which is the ratio between the circumference of the base of the cone and the circumference of the arc formed when the curved surface is laid flat.

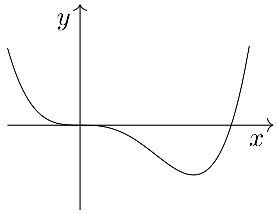
The slant height of the cone, which is given by $l = \sqrt{r^2 + h^2}$, becomes the radius when the curved surface is unwrapped, as follows:



The major arc length is the base circumference of the cone, which is $2\pi r$. The full circumference is $2\pi\sqrt{r^2 + h^2}$. So, this gives us a ratio of angles

$$\begin{aligned} \frac{\phi}{2\pi} &= \frac{2\pi r}{2\pi\sqrt{r^2 + h^2}} \\ \Longrightarrow \phi &= \frac{2\pi r}{\sqrt{r^2 + h^2}}. \end{aligned}$$

1438. State, with a reason, whether each of the equations could possibly be that of the sketched graph.



(a) $y = x^4 - 2x$,
(b) $y = x^4 - 2x^2$,
(c) $y = x^4 - 2x^3$.

Factorise each of the equations, and consider the fact that the graph shown has a triple root at the origin.

The graph shown has a triple root at $x = 0$ and a single root at some positive $x = \alpha$, meaning that its equation must have a triple factor of x and a single factor of $(x - \alpha)$. Only equation (c) has this: $y = x^3(x - 2)$.

1439. For constants $a < b < c < d$, prove that

$$\int_a^c y\,dx + \int_b^d y\,dx = \int_a^d y\,dx + \int_b^c y\,dx.$$

Give the integral of y a name, say $F(x)$.

Let $F(x)$ be the integral of y. Then the LHS is

$$\begin{aligned} & \Big[F(x)\Big]_a^c + \Big[F(x)\Big]_b^d \\ &= F(c) - F(a) + F(d) - F(b). \end{aligned}$$

Similarly, the RHS is

$$\begin{aligned} & \Big[F(x)\Big]_a^d + \Big[F(x)\Big]_b^c \\ &= F(d) - F(a) + F(c) - F(b), \end{aligned}$$

which proves the result.

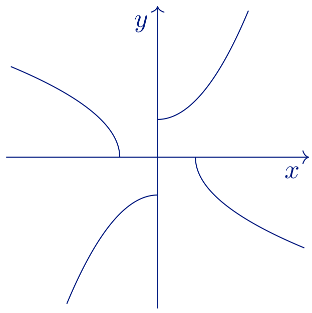
1440. In the positive quadrant, a section of a parabola is defined as $y = x^2 + 1$. This section forms part of a pattern with rotational symmetry order four, centred on the origin. Hence, the section in the quadrant $x \geq 0, y \leq 0$ has equation $y = -\sqrt{x - 1}$.

(a) Sketch the pattern.

(b) Determine the equations of the sections in the remaining two quadrants.

To rotate 180° around the origin, replace (x, y) with $(-x, -y)$.

(a) The pattern is



(b) In each case, we can transform one of the given curves by rotating 180° around the origin, which involves replacing (x, y) with $(-x, -y)$. The equations are $y = \sqrt{-x-1}$ in the second quadrant and $y = -x^2 - 1$ in the third.

1441. Show that $\displaystyle\int_0^1 15\left(x+\sqrt{x}\right)\left(1-\sqrt{x}\right)\,dx = 4.$

Multiply the integrand out, noting the cancellation of x's, and integrate term by term.

Multiplying out, the terms in x cancel. Expressing the square roots as powers of x, then, we have

$$\begin{aligned}
&\int_0^1 15\left(x+\sqrt{x}\right)\left(1-\sqrt{x}\right)\,dx\\
&= 15\int_0^1 x^{\frac{1}{2}} - x^{\frac{3}{2}}\,dx\\
&= 15\left[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}\right]_0^1\\
&= 15\left(\tfrac{2}{3}-\tfrac{2}{5}\right) - 15(0)\\
&= 15\times\tfrac{4}{15}\\
&= 4, \quad \text{as required.}
\end{aligned}$$

1442. Either prove or disprove the following: "If forces of magnitudes $\frac{3}{5}Q$, $\frac{4}{5}Q$, Q act on an object, and it remains in equilibrium, then two of the forces must be perpendicular."

Consider a triangle of forces.

This is true. The object is in equilibrium under the action of three forces, so these forces must form a closed triangle. And, since the magnitudes satisfy

$$\left(\tfrac{3}{5}Q\right)^2 + \left(\tfrac{4}{5}Q\right)^2 = Q^2,$$

this triangle of forces must be right-angled. Hence, the smaller two forces are perpendicular.

1443. Solve $\dfrac{1-\frac{1}{2x}}{\sqrt{1-\frac{1}{4x^2}}} = 2.$

Where there is an inlaid fraction (a fraction within a fraction), it is generally best to multiply top and bottom of the big fraction by the denominator of the little fraction.

We can multiplying the top and bottom of the big fraction by $2x$ and then use a difference of two squares:

$$\begin{aligned}
&\frac{1-\frac{1}{2x}}{\sqrt{1-\frac{1}{4x^2}}} = 2\\
\Longrightarrow\;& \frac{2x-1}{\sqrt{4x^2-1}} = 2\\
\Longrightarrow\;& \frac{2x-1}{\sqrt{2x-1}\sqrt{2x+1}} = 2\\
\Longrightarrow\;& \frac{\sqrt{2x-1}}{\sqrt{2x+1}} = 2\\
\Longrightarrow\;& \frac{2x-1}{2x+1} = 4\\
\Longrightarrow\;& 2x-1 = 8x+4\\
\Longrightarrow\;& x = -\tfrac{5}{6}.
\end{aligned}$$

1444. Express $x^2 + 6x - 8$ as the product of two linear factors containing surds.

Use the factor theorem, i.e. set the expression equal to zero and solve. Then reverse engineer the factors using the roots you find.

Setting to zero, we get

$$x = \frac{-6\pm\sqrt{68}}{2} = 3\pm\sqrt{17}.$$

Hence, by the factor theorem, the original quadratic expression has factors $\left(x-(3\pm\sqrt{17})\right)$. Since the quadratic is monic, these are the only factors, giving

$$x^2+6x-8 \equiv \left(x-3-\sqrt{17}\right)\left(x-3+\sqrt{17}\right).$$

1445. A short-sighted croupier is calling rolls of a die. On each roll, with probability 0.25, he misreads the number by one. This is randomly up or down if both options are available, or else in the only way possible. Show that the probability of the croupier calling a six has been reduced from $\frac{1}{6}$ to $\frac{7}{48}$ by his shortsightedness.

There are now two outcomes which will result in a call of six: a five which is misread upwards or a six which is correctly called. Add the relevant probabilities, conditioning on the die roll first.

There are two successful outcomes. If a five is rolled, then there is a $0.25 \times \frac{1}{2}$ probability that the call will be six; if a six is rolled, then there is a 0.75 probability that the call will be a six. So, the probability, conditioning on the roll, is

$$P(\text{six called}) = \tfrac{1}{6} \times 0.25 \times \tfrac{1}{2} + \tfrac{1}{6} \times 0.75 = \tfrac{7}{48}.$$

1446. Giving your answer in set notation, determine all integers n which satisfy none of the following:

$$n \in (\infty, 0], \quad 2 < n < 5, \quad n^2 > 9.$$

Start by considering the set of integers $\mathbb{Z}$. Then eliminate any values in the set that satisfy any one of the given statements.

Considering the (negation of the) first statement, we need $n \in [1, 2, ...)$. From the second, 3 and 4 are ruled out, leaving $[1, 2, 5, 6, 7, ...)$. The third statement then rules out $5, 6, ...$, leaving the values that satisfy none of the three as $n \in \{1, 2\}$.

1447. The function $S(n)$ is defined, for a sequence u_i, by

$$S(n) = \sum_{i=1}^{n} u_i.$$

Find the following sums, giving your answers in terms of the function S and the variable n:

(a) $\displaystyle\sum_{i=1}^{n}(u_i + 1)$, (b) $\displaystyle\sum_{i=1}^{n}(2u_i + i)$.

In each case, you can distribute the sum over the two terms. In other words, you can split the sum up. You get $\sum 1$, in (a), and $\sum i$, in (b). These are both standard results (they can also be derived by consideration of arithmetic progressions).

(a) Splitting the sum up

$$\begin{aligned} &\sum_{i=1}^{n}(u_i + 1) \\ &\equiv \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} 1 \\ &\equiv S(n) + n. \end{aligned}$$

(b) Here, we take a factor of two out, and use the standard sum of the first n integers:

$$\begin{aligned} &\sum_{i=1}^{n}(2u_i + i) \\ &\equiv 2\sum_{i=1}^{n} u_i + \sum_{i=1}^{n} i \\ &\equiv 2S(n) + \tfrac{1}{2}n(n+1). \end{aligned}$$

1448. Two particles have positions given by

$$x_1 = \frac{1}{1-t}, \qquad x_2 = \frac{2}{1-2t}.$$

Determine whether they collide.

The particles collide if $x_1 = x_2$. So, set up an equation and solve it.

Collision occurs if $x_1 = x_2$. This gives

$$\begin{aligned} &\frac{1}{1-t} = \frac{2}{1-2t} \\ \implies &1 - 2t = 2 - 2t \\ \implies &1 = 2. \end{aligned}$$

This is impossible. Hence, the two particles do not collide.

1449. Assume, for the sake of a contradiction, that $\log_2 3$ is rational, and can therefore be written as $\frac{a}{b}$, where $a, b \in \mathbb{N}$.

(a) Show that $2^a = 3^b$.

(b) Hence, by considering prime factors, prove that $\log_2 3$ is irrational.

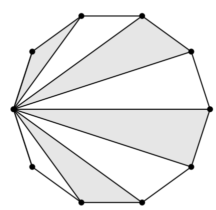
This is a standard proof of the irrationality of a logarithm. In (b), show that the two sides can only be equal if $a = b = 0$, which is a contradiction.

(a) Assume, for a contradiction, that $\log_2 3$ is a rational number, and can therefore be written as $\frac{a}{b}$, where $a, b \in \mathbb{N}$. This gives

$$\begin{aligned} &\log_2 3 = \tfrac{a}{b} \\ \implies &2^{\frac{a}{b}} = 3 \\ \implies &2^a = 3^b \text{ (raising to the power } b\text{)}. \end{aligned}$$

(b) The only prime factor of the LHS 2^a is 2, so it can have no factors of 3. Hence neither does 3^b. This means $b = 0$, which is a contradiction. Therefore, $\log_2 3$ is irrational. $\square$

1450. In the diagram below, four shaded triangles have been constructed inside a regular decagon.



Prove that half of the decagon is shaded.

Use a symmetry argument.

The horizontal diameter is a line of symmetry of the shape. Furthermore, in each symmetrical pair of triangles, exactly one is shaded. So, half of the decagon is shaded, as required.

1451. Find all values of x for which, with respect to x, the expression $x^5 - 2x^4 + x^3$ is stationary.

If an expression E is stationary with respect to x, then $\frac{d}{dx}(E) = 0$. Factorise the resulting equation.

For stationary values:

$$\begin{aligned} &\tfrac{d}{dx}(x^5 - 2x^4 + x^3) = 0 \\ \Longrightarrow\ & 5x^4 - 8x^3 + 3x^2 = 0 \\ \Longrightarrow\ & x^2(5x^2 - 8x + 3) = 0 \\ \Longrightarrow\ & x^2(5x-3)(x-1) = 0 \\ \Longrightarrow\ & x \in \{0, \tfrac{3}{5}, 1\}. \end{aligned}$$

1452. A function g has $g'(x) = a$ for all $x \in \mathbb{R}$, where a is a constant. Prove that, for any sample of real numbers $\{x_i\}$, the mean of $\{g(x_i)\}$ is $g(\bar{x})$.

This is a simpler result than it looks. Integrate $g'(x) = a$ to get a general linear function g, then you can do the algebra explicitly.

Integrating $g'(x) = a$, we have a generic linear function $g(x) = ax+b$. The mean of the set $\{g(x_i)\}$ is given, then, by

$$\begin{aligned} &\tfrac{1}{n}\sum_{i=1}^{n} g(x_i) \\ =\ & \tfrac{1}{n}\sum_{i=1}^{n} a(x_i) + b \\ =\ & a\tfrac{1}{n}\sum_{i=1}^{n}(x_i) + \tfrac{1}{n}\sum_{i=1}^{n}(x_i)b \\ =\ & a\bar{x} + b \\ =\ & g(\bar{x}), \text{ as required.} \end{aligned}$$

1453. By setting up and solving simultaneous equations, express the fraction $\frac{36}{323}$ in the form $\frac{1}{a} + \frac{1}{b}$, where a and b are prime numbers.

Since both $\frac{36}{323}$ and $\frac{a+b}{ab}$ are in their lowest terms, you can equate the numerators and equate the denominators of $\frac{36}{323} = \frac{a+b}{ab}$.

The fraction $\frac{36}{323}$ is in its lowest terms; $\frac{a+b}{ab}$ is too, as neither a nor b, being primes, can be a factor of $a + b$. Hence, we can equate the numerators and denominators, giving $36 = a + b$ and $323 = ab$. This can be solved algebraically as a quadratic, or else by factorising $323 = 17 \times 19$. These sum to 36. So

$$\frac{36}{323} = \frac{1}{17} + \frac{1}{19}.$$

1454. The masses of a population of rodents are modelled with a normal distribution, in units of grams, of $M \sim N(245, 120^2)$. Two such rodents are caught in the same trap.

(a) With this model, find the probability that
   i. both weigh more than 245 grams.
   ii. neither weighs more than 250 grams.

(b) Give two reasons why these probabilities are unlikely to be useful to a biologist studying the rodents.

In (b), consider the applicability of the normal model and the (in)dependence of the trials.

(a) i. Since 245 is the mean, $P(M > 245) = \frac{1}{2}$. So the probability is $\frac{1}{2}^2 = \frac{1}{4}$.
   ii. Using a calculator cumulative distribution function, we find $P(M < 250) = 0.516618$. Squaring this gives the required probability as 0.267 (3sf).

(b) Firstly, there is no reason to think that masses can be modelled with a normal distribution: the normal distribution only applies in specific circumstances, and very rarely for individuals in biology. Secondly, since the rodents were caught in the same trap, it is very unlikely that the two trials are independent, which we have used in multiplying probabilities.

1455. Solve $\sin\left(2x + \dfrac{\pi}{3}\right) = \cos\left(2x + \dfrac{\pi}{3}\right)$ for $x \in [0, 2\pi]$.

You don't need to use a compound angle formula here, although you could. Instead, divide both sides by cos, and simplify the LHS to tan.

Using the identity $\tan x \equiv \frac{\sin x}{\cos x}$, we have

$$\begin{aligned} &\sin\left(2x + \tfrac{\pi}{3}\right) = \cos\left(2x + \tfrac{\pi}{3}\right) \\ \Longrightarrow\ & \tan\left(2x + \tfrac{\pi}{3}\right) = 1 \\ \Longrightarrow\ & 2x + \tfrac{\pi}{3} = ..., \tfrac{5\pi}{4}, \tfrac{9\pi}{4}, \tfrac{13\pi}{4}, \tfrac{17\pi}{4}, ... \\ \Longrightarrow\ & x = ..., \tfrac{11\pi}{24}, \tfrac{23\pi}{24}, \tfrac{35\pi}{24}, \tfrac{47\pi}{24}, ... \end{aligned}$$

1456. A student writes: "When you step on the pedals of a bicycle, a frictional force is generated, which acts backwards on the back wheel of the bicycle." Explain whether this is correct.

It is a true that a frictional force is created, but the student is nevertheless wrong.

Friction is indeed generated, which manifests as a Newton III pair of forces. But the student has the directions wrong. The two forces generated are a frictional force backwards on the ground, and a frictional force of equal magnitude forwards on the bicycle.

1457. Assuming that a, b, c, d are distinct real numbers, write down the roots of the following equation:

$$\frac{(x^3 - a^3)(x^3 + b^3)}{(x^3 - c^3)(x^3 + d^3)} = 0.$$

If a fraction is zero, then its numerator is zero.

Setting the numerator to zero, we get $x = a$ or $x = -b$. And, because a, b, c, d are distinct, the denominator is non-zero, so both are roots.

1458. The sequence a_n is given as arithmetic. Prove that $b_n = 10^{a_n}$ is geometric.

Substitute a generic AP term $a_n = a + (n-1)d$ into the formula for b_n.

Substituting a generic AP term $a_n = a + (n-1)d$ into the formula for b_n, we get

$$10^{a+(n-1)d} = 10^a \times (10^d)^{n-1}.$$

This is the standard ordinal formula $u_n = ar^{n-1}$ of a geometric progression, with first term 10^a and common ratio 10^d.

1459. Two vectors $\mathbf{p}$ and $\mathbf{q}$ are given in terms of the usual perpendicular unit vectors as

$$\mathbf{p} = 3\mathbf{i} + 4\mathbf{j},$$
$$\mathbf{q} = 4\mathbf{i} - 3\mathbf{j}.$$

(a) Give the magnitudes of $\mathbf{p}$ and $\mathbf{q}$,

(b) Express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

In (a), use Pythagoras. In (b), solve for $\mathbf{i}$ and $\mathbf{j}$ simultaneously.

(a) By Pythagoras, $|\mathbf{p}| = |\mathbf{q}| = 5$.

(b) We solve simultaneously by elimination:

$$3\mathbf{p} = 9\mathbf{i} + 12\mathbf{j}$$
$$4\mathbf{q} = 16\mathbf{i} - 12\mathbf{j}$$

Adding gives $3\mathbf{p} + 4\mathbf{q} = 25\mathbf{i}$. Hence,

$$\mathbf{i} = \tfrac{1}{25}(3\mathbf{p} + 4\mathbf{q})$$
$$\mathbf{j} = \tfrac{1}{25}(4\mathbf{p} - 3\mathbf{q}).$$

1460. Show that the following equation defines a family of circles:

$$\int x\,dx + \int y\,dy = 0.$$

Perform the integrals, gathering the $+c$'s together into a single constant.

Integrating, we get

$$\tfrac{1}{2}x^2 + c_1 + \tfrac{1}{2}y^2 + c_2 = 0.$$

Combining the constants and multiplying by two, this can be expressed as $x^2 + y^2 = c$, which is the equation of a family of circles of varying radius, centred on the origin.

1461. The scores a and b on two exams, out of A and B marks respectively, are to be combined into one mark M, given out of a hundred. The exams (as opposed to the individual marks) are to have equal weighting. Find a formula for M.

Scale each exam to be out of 50, then add.

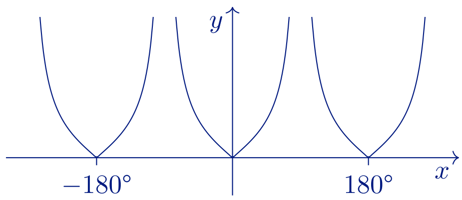
Each exam is scaled to total 50 marks by the scale factors $\frac{50}{A}$, $\frac{50}{B}$. Then, adding gives the formula:

$$M = \frac{50a}{A} + \frac{50b}{B}.$$

1462. Sketch $y = |\tan x|$.

Sketch $y = \tan x$ first, then consider the effect of the modulus function on those regions below the x axis.

Applying the modulus function to the outputs of the tan function takes all (x, y) points on the graph $y = \tan x$ for which $y < 0$ and reflects them in the x axis, giving



1463. Show that, if the following simultaneous equations are to be satisfied, then $z = 2$.

$$x - y + z = 3,$$
$$2x - z = 2y.$$

These can be considered as a pair of simultaneous equations in z and $x - y$.

In general, this problem would not be possible to solve. However, both equations can be written in terms of z and $x-y$, which we can call a.

$$\begin{aligned} a+z&=3, \\ 2a-z&=0. \end{aligned}$$

Solving, we find $a=1, z=2$.

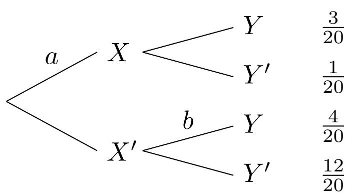
1464. Show that the discriminant of $y=(px+1)(qx+1)$ is $(p-q)^2$.

Multiply out, find Δ and simplify.

Multiplying out gives $pqx^2+(p+q)x+1=0$. So, the discriminant Δ is

$$\begin{aligned} &(p+q)^2-4pq \\ &=p^2+2pq+q^2-4pq \\ &=p^2-2pq+q^2 \\ &=(p-q)^2, \text{ as required.} \end{aligned}$$

1465. Events X and Y have probabilities as represented on the following tree diagram:



(a) Find probabilities a and b.

(b) Find $P(X \cup Y')$.

In (a), find a first, using the upper two branches, then find b using the third. In (b), consider the complement of the union.

(a) The top two branches have total probability $\frac{4}{20}=\frac{1}{5}$, which is the value of a. Using this, $P(X' \cap Y)=\frac{4}{20}$ gives $\frac{4}{5}b=\frac{4}{20}$, so $b=\frac{1}{4}$.

(b) $P(X \cup Y')=1-P(X' \cap Y)=\frac{16}{20}=\frac{4}{5}$.

1466. By first locating a root with the Newton-Raphson method, determine the roots of the following equation:

$$0.04x^3-0.07x^2=0.62x+0.15.$$

Rearrange to $f(x)=0$ and find one (rational) root with NR. Then, by the factor theorem, take out the relevant factor, of the form $(ax+b)$, and solve the remaining quadratic exactly.

Multiplying by 100 and writing as $f(x)=0$, the Newton-Raphson iteration is

$$x_{n+1}=x_n-\frac{4x^3-7x^2-62x-15}{12x^2-14x-62}.$$

Running the iteration with $x_0=0$ gives $x_n \to -\frac{1}{4}$. So, by the factor theorem, we know $(4x+1)$ is a factor. We can factorise, therefore, as

$$\begin{aligned} &4x^3-7x^2-62x-15=0 \\ \Longrightarrow\ &(4x+1)(x^2-2x-15)=0 \\ \Longrightarrow\ &(4x+1)(x+3)(x-5)=0 \\ \Longrightarrow\ &x=-3,-\tfrac{1}{4},5. \end{aligned}$$

1467. Write down $\displaystyle\int e^{-x}\,dx$.

This is a standard result that follows from the chain rule. Since the integral of e^x is e^x+c, the integral of e^{-x} is...?

Since the integral of e^x is e^x, by the (reverse) chain rule, we require an extra minus sign:

$$\int e^{-x}\,dx=-e^{-x}+c.$$

1468. Four values x_1, x_2, x_3, x_4 are chosen at random and independently from the interval $[0,1]$. Write down the probability that $x_1+x_2<x_3+x_4$.

This is easier than it might look. Since $[0,1]$ is a continuous set, the probability of equality is zero. So, there are only two directions the inequality can go, which are symmetrical.

Since $[0,1]$ is a continuous set, the probability of equality is zero. So, there are only two directions the inequality can go. These are equally likely, since x_1+x_2, x_3+x_4 is symmetrical. Hence, $p=\frac{1}{2}$.

1469. State, with a reason, whether the argument below, concerning a trio of statements $\{A,B,C\}$, holds: "If $A \implies B$ and $C \implies B$, then $A \implies C$."

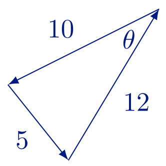
The statement isn't true; find a counterexample.

This is not true. A counterexample is $A: x=1$, $B: x^2=1$ and $C: x=-1$.

1470. A mass is in equilibrium with three forces acting on it. These forces have magnitudes $5, 10, 12$ N. Using a triangle of forces, or otherwise, determine the obtuse angle between the 10 and 12 N forces.

Use the cosine rule.

Setting up the triangle of forces, we have



The cosine rule gives

$$\theta = \arccos \frac{10^2 + 12^2 - 5^2}{2 \cdot 10 \cdot 12} = 24.146...^\circ.$$

We want the obtuse angle, so $\theta = 155.9$ (1dp).

1471. Solve the following simultaneous equations:

$$\begin{aligned} x + y + z &= 0, \\ x + y - z &= 1, \\ x - y + z &= 2. \end{aligned}$$

Use equations 1 and 3 to eliminate both x and z.

We can eliminate both x and z by subtracting equation 3 from equation 1. This gives $2y = -2$, so $y = -1$. The first two equations are then

$$\begin{aligned} x + z &= 1 \\ x - z &= 2. \end{aligned}$$

Solving these gives (x, y, z) as $(\frac{3}{2}, -1, -\frac{1}{2})$.

1472. Show that, for any set of data, $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x}) = 0$.

Split the sum up, and replace $\bar{x}$ with its definition in terms of $\sum x_i$ and n.

This is true by definition of $\bar{x}$. We can prove the result algebraically by splitting the sum up:

$$\begin{aligned} &\sum_{i=1}^{n}(x_i - \bar{x}) \\ = &\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} \\ = &\; n\bar{x} - n\bar{x} \\ = &\; 0. \end{aligned}$$

1473. The quadratic functions f, g have discriminants which are zero and positive respectively. For each of the following equations, write down the possible numbers of distinct roots:

(a) $f(x)g(x) = 0$,

(b) $f(x)g(x)^2 = 0$,

(c) $f(x)^4 g(x)^7 = 0$.

The discriminants tell us that $f(x) = 0$ has exactly one root, and $g(x) = 0$ has exactly two. These may possibly be the same, however.

We know that $f(x) = 0$ has exactly one root, and $g(x) = 0$ has exactly two. The root of $f(x) = 0$ may also be a root of $g(x) = 0$, or all three may be distinct. This gives two possibilities in each case.

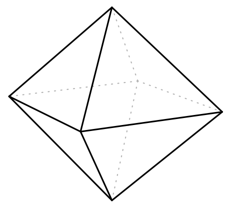
(a) 2 or 3 roots,

(b) 2 or 3 roots,

(c) 2 or 3 roots.

The powers on the factors $f(x)$ and $g(x)$ do not affect the possible numbers of roots.

1474. A regular octahedron is shown below.



Two of the faces of the octahedron are selected at random. Find the probability that these faces share an edge.

It doesn't make any difference which face is picked first, as the octahedron is symmetrical. So, pick an arbitrary face, and then consider the second choice.

The octahedron is symmetrical, so we can pick the first face without loss of generality. There are then seven faces remaining, of which three share an edge with the first. So, the probability is $\frac{3}{7}$.

1475. Simplify $(e^x + 1)^4 - (e^x - 1)^4$.

Use the binomial expansion.

Using the binomial expansion gives

$$(e^x \pm 1)^4 = e^{4x} \pm 4e^{3x} + 6e^{2x} \pm 4e^x + 1.$$

When we subtract, the even terms cancel, leaving

$$(e^x + 1)^4 - (e^x - 1)^4 = 8e^{3x} + 8e^x.$$

1476. General linear simultaneous equations are given below, for constants a, b, c, d, p, q with $ad - bc \neq 0$.

$$\begin{aligned} ax + by &= p, \\ cx + dy &= q. \end{aligned}$$

(a) Solve to find formulae for x and y.

(b) Explain the condition $ad - bc \neq 0$.

Use either elimination or substitution, as usual.

(a) Multiplying for elimination, we have

$$\begin{aligned} adx + bdy &= dp, \\ bcx + bdy &= bq. \end{aligned}$$

This gives $x = \frac{dp-bq}{ad-bc}$. Substituting back in and simplifying, we get $y = \frac{cp-aq}{ad-bc}$.

(b) The condition avoids division by zero. If $ad - bc = 0$, then the equations represent parallel straight lines: there are either no points of intersection or else infinitely many.

1477. Show that $\dfrac{1}{1+\sqrt[4]{2}} = 2^{\frac{1}{4}} - 2^{\frac{1}{2}} + 2^{\frac{3}{4}} - 1$.

Start with an expression: the denominator of the LHS multiplied by the RHS. Simplify to get 1.

We need to show that the denominator of the LHS and the RHS multiply to give 1. Writing as powers of x, we have

$$\begin{aligned} &\left(1 + 2^{\frac{1}{4}}\right)\left(2^{\frac{1}{4}} - 2^{\frac{1}{2}} + 2^{\frac{3}{4}} - 1\right) \\ =\ &\left(2^{\frac{1}{4}} - 2^{\frac{1}{2}} + 2^{\frac{3}{4}} - 1\right) + \left(2^{\frac{1}{2}} - 2^{\frac{3}{4}} + 2 - 2^{\frac{1}{4}}\right) \\ =\ &-1 + 2 \\ =\ &1, \text{ as required.} \end{aligned}$$

1478. Three cards are dealt, with replacement, from a standard deck. Determine which, if either, of the following events has the greater probability: the three cards being, in any order,

- an ace, a king and a queen,
- two fours and a five.

Calculate the probabilities explicitly.

The first event is twice as likely:

$$\begin{aligned} P(\text{ace, king, queen}) &= 3! \times \left(\tfrac{1}{13}\right)^3 \\ P(4,4,5) &= {}^3C_1 \times \left(\tfrac{1}{13}\right)^3. \end{aligned}$$

1479. Show algebraically that $18x^3 + 33x^2 = 7x + 2$ has three roots, all of which are rational.

Use the polynomial solver on your calculator to find the roots of the equation, and then, using the factor theorem, reverse engineer a factorisation.

Using a polynomial solver on the calculator, the cubic has roots at $x = -2, -\frac{1}{6}, \frac{1}{3}$. Using these, we can reconstruct an algebraic solution:

$$\begin{aligned} & 18x^3 + 33x^2 - 7x - 2 = 0 \\ \Longrightarrow\ & (x+2)(6x+1)(3x-1) = 0 \\ \Longrightarrow\ & x = -2, -\tfrac{1}{6}, \tfrac{1}{3}. \end{aligned}$$

1480. A parabola is given by $y = ax^2 + bx + c$. Write down the equations of the graphs obtained when the original parabola is

(a) reflected in the x axis,

(b) reflected in the y axis.

(c) reflected in the line $y = x$.

In each case, consider reflection as a replacement of the variables. In (a), replace y by $-y$.

(a) Replace y by $-y$, giving $-y = ax^2 + bx + c$, or equivalently $y = -ax^2 - bx - c$.

(b) Replace x by $-x$, giving $y = ax^2 - bx + c$,

(c) Switch x and y, giving $x = ay^2 + by + c$.

1481. By factorising, solve the equation

$$(x+2)^3(x-1) + (x+2)(x-1)^3 = 0.$$

Factorise immediately, without multiplying out.

There are factors to be taken out immediately, without multiplying out:

$$\begin{aligned} & (x+2)^3(x-1) + (x+2)(x-1)^3 = 0 \\ \Longrightarrow\ & (x+2)(x-1)\left[(x+2)^2 + (x-1)^2\right] = 0 \\ \Longrightarrow\ & (x+2)(x-1)\left[2x^2 + 2x + 5\right] = 0. \end{aligned}$$

The discriminant of the quadratic factor is $\Delta = 2^2 - 4 \cdot 2 \cdot 5 = -36 < 0$, so there are only two real roots, $x = -2$ and $x = 1$.

1482. Two dice are rolled; neither shows a six. Find the probability that the combined score is at least six.

Set up a six by six possibility space, then restrict it according to the information given.

The information given restricts the 36 outcomes of the possibility space as follows:

	1	2	3	4	5	6
1					✓	✓
2				✓	✓	✓
3			✓	✓	✓	✓
4		✓	✓	✓	✓	✓
5	✓	✓	✓	✓	✓	✓
6	✓	✓	✓	✓	✓	✓

There are 15 successful outcomes in the restricted possibility space, so the probability is $\frac{15}{25} = \frac{3}{5}$.

1483. If $a = 2^{x+1}$ and $b = 4^{x-1}$, write a in terms of b.

Use index laws; it's easier if you don't convert to logarithms.

Using index laws, we have

$$\begin{aligned} a &= 2^{x+1} \\ &= 2 \times 2^x \\ &= 2 \times \sqrt{4^x} \\ &= 2 \times 2 \times \sqrt{4^x \div 4} \\ &= 4\sqrt{b}. \end{aligned}$$

1484. By calculating the discriminant of a suitable quadratic equation, show that $y = 9 - 2(x-3)^2$ and $y = 2x^2$ are tangent to each other.

If two quadratics are tangent to each other, the equation formed to find their intersections must have a double root at the point of tangency.

Solving for intersections, $9-2(x-3)^2 = 2x^2$, which simplifies to $4x^2 - 12x + 9 = 0$. The discriminant is $\Delta = 144 - 4 \times 4 \times 9 = 0$; therefore, this equation has a double root. Hence, the point of intersection must be a point of tangency.

1485. Show that $\cos(\arcsin x) = \sqrt{1-x^2}$.

Begin by setting $\arcsin x = y$ and rearranging.

Setting $\arcsin x = y$, we get $x = \sin y$. Therefore,

$$\begin{aligned} &\sqrt{1-x^2} \\ =\ &\sqrt{1-\sin^2 y} \\ =\ &\sqrt{\cos^2 y} \\ =\ &\cos y \\ =\ &\cos(\arcsin x). \end{aligned}$$

1486. If $\dfrac{d}{dx}(x^2 + y^2) = 0$, find $\dfrac{dy}{dx}$ in terms of x and y.

Differentiate (implicitly) term by term.

Differentiating implicitly, we treat the term y^2 as a composition of two functions: the inside function is $x \mapsto y$, the outside function is "squaring". The chain rule, then, gives

$$\begin{aligned} &\frac{d}{dx}(x^2 + y^2) = 0 \\ \Longrightarrow\ & 2x + 2y\frac{dy}{dx} = 0 \\ \Longrightarrow\ & \frac{dy}{dx} = -\frac{x}{y}. \end{aligned}$$

1487. A set of forty data is summarised with mean $\bar{x}$ and standard deviation $\sigma \neq 0$. A check shows, however, that the lowest five data points were, in fact, duplicated, and only thirty-five data should have been recorded. State, with a reason, what effect rectifying this mistake will have on

(a) the sample mean,

(b) the sample standard deviation.

In each case, there will be either a certain increase or decrease.

(a) With a set of the low-valued data removed, the sample mean will increase.

(b) With five extreme-valued data removed, the sample standard deviation will decrease.

1488. Find the value of the constant b, if the following may be written as a quadratic function of x:

$$\frac{2x^3 - x^2 + bx - 3}{2x+1}.$$

This is the factor theorem in disguise.

Since the fraction may be written as a quadratic, the denominator must divide into the numerator. Hence, by the factor theorem, $x = -\frac{1}{2}$ is a root of the numerator, giving $-\frac{1}{2}b - \frac{7}{2} = 0$. So $b = -7$.

1489. Show that the equation

$$\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} = 0$$

has precisely two distinct real roots.

Multiply up by all three denominators.

Multiplying up, we get

$$\begin{aligned} &(x+1)(x+2) + x(x+2) + x(x+1) = 0 \\ \Longrightarrow\ & x^2 + 3x + 2 + x^2 + 2x + x^2 + x = 0 \\ \Longrightarrow\ & 3x^2 + 6x + 2 = 0. \end{aligned}$$

This is a quadratic with discriminant $\Delta = 12 > 0$, so it has two distinct real roots.

1490. Evaluate $\displaystyle\sum_{i=1}^{50} \log_8(2^i)$.

Simplify the summand, in order to ascertain what kind of series it is.

The summand is $\log_8(2^i) = i\log_8 2 = \frac{1}{3}i$. Hence, we have an arithmetic series with first term $a = \frac{1}{3}$, common difference $d = \frac{1}{3}$ and $n = 50$ terms. We can use the standard formula:

$$S_{50} = \tfrac{50}{2}\left(\tfrac{2}{3} + 49 \cdot \tfrac{1}{3}\right) = 425.$$

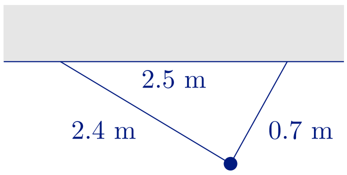
1491. A 10 kg audio rig, modelled as a particle, is hung from two cables, lengths 0.7 m and 2.4 m, which are attached to two points, 2.5 m apart, on the horizontal ceiling of a concert space.

(a) Find the angles of inclination of the ropes.

(b) Draw a triangle of forces, and hence find the tensions in the two ropes.

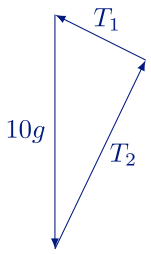
You may find it helpful to draw all three of: a picture, a force diagram, and a triangle of forces.

(a) The physical situation is



This is a Pythagorean triangle. So, the angles of inclination (from the horizontal) are given by $\arccos \frac{24}{25} = 16.26°$ and $\arcsin \frac{24}{25} = 73.74°$.

(b) The triangle of forces, modelling the tensions and the weight, is as follows. [Note the larger tension T_2 corresponding to the shorter string.]



This is also right-angled, with sides in ratio $7 : 24 : 25$. Therefore, $T_1 = \frac{70}{25}g = 2.8g$ and $T_2 = \frac{240}{25}g = 9.6g$.

1492. By evaluating the second derivative, or otherwise, determine whether $y = x^4 - x^3$ is concave, convex or neither at the origin.

A curve is concave wherever $\frac{d^2y}{dx^2} > 0$ and convex wherever $\frac{d^2y}{dx^2} < 0$.

Differentiating, we get

$$\begin{aligned} y &= x^4 - x^3 \\ \Longrightarrow \frac{dy}{dx} &= 4x^3 - 3x^2 \\ \Longrightarrow \frac{d^2y}{dx^2} &= 12x^2 - 6x. \end{aligned}$$

Evaluating the second derivative at $x = 0$ gives a value 0. Hence, the curve is neither concave nor convex at the origin.

1493. Find constants $a, b, c, d > 0$ such that

$$(a + bx)^3 \equiv c + 225x + dx^2 + 27x^3.$$

Find b first, then a, then c and d.

This is an identity, so we can equate coefficients. This gives $b = 3$ immediately. Considering the x term, we get $3a^2b = 225$, so (positive) $a = 5$. We can then perform the binomial expansion, giving $c = 125$ and $d = 3ab^2 = 135$.

1494. Describe the transformation that takes the graph $y = (x-p)^2 + q$ onto the graph $y = (x-p+1)^2 + q$.

This is an input transformation, so ask "what has x been replaced by?"

Since $(x - p + 1) = (x + 1 - p)$, the transformation has replaced x with $x + 1$. This is a translation by vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

1495. Find formulae for the sum of

(a) the first n even integers,

(b) the first n odd integers.

Each can be considered as an arithmetic series or, alternatively, expressed in terms of the standard result $\sum i = \frac{1}{2}n(n + 1)$.

We can use the standard formula for the sum of the first n integers $\sum i = \frac{1}{2}n(n + 1)$.

(a) $\displaystyle\sum_1^n 2i = 2\sum_1^n i = n(n + 1)$.

(b) $\displaystyle\sum_1^n (2i - 1) = 2\sum_1^n i - n = n(n + 1) - n = n^2$.

Alternatively, each can be considered as an AP.

1496. A locus is defined, in terms of two points $A : (1, 3)$ and $B : (5, 1)$, as the set of points P for which $|AP| = |BP|$. Determine the Cartesian equation of the locus.

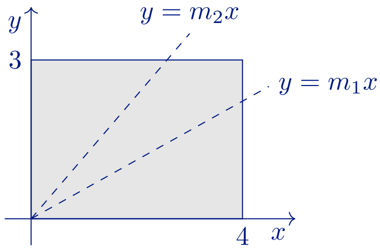
The set of points equidistant from two points is the perpendicular bisector.

We need the perpendicular bisector of AB. Since $m_{AB} = -\frac{1}{2}$ and the midpoint of AB is at $(3, 2)$, this has equation $y = 2x - 4$.

1497. A rectangle has vertices at $(0, 0)$, $(4, 0)$, $(0, 3)$ and $(4, 3)$. A line $y = mx$ is drawn. Determine the two values of m such that the line divides the area of the rectangle in the ratio $1 : 2$.

Sketch the two possibilities for the scenario (they are different), and, in each case, use coordinate geometry to set up an equation in m.

The two possible lines are drawn below:



Line $y = m_1x$ intersects the rectangle at $(4, 4m_1)$, creating a triangle of area $\frac{1}{2} \cdot 4 \cdot 4m_1 = 8m_1$. The area of the rectangle is 12, so we require $8m_1 = 4$. Hence, $m_1 = \frac{1}{2}$.

Line $y = m_2x$ intersects the rectangle at $(\frac{3}{m_2}, 3)$, creating a triangle of area $\frac{1}{2} \cdot \frac{3}{m_2} \cdot 3 = \frac{9}{2m_2}$. As before, we require $\frac{9}{2m_2} = 4$. Hence, $m_2 = \frac{9}{8}$.

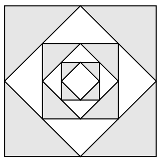
1498. Find the probability that six consecutive dice rolls give six different scores.

Multiply together six probabilities (or five, since the first is 1).

Beginning with the first roll, whose outcome has no effect:

$$P(\text{all different}) = 1 \times \tfrac{5}{6} \times \tfrac{4}{6} \times \tfrac{3}{6} \times \tfrac{2}{6} \times \tfrac{1}{6} = \tfrac{5}{324}.$$

1499. A symmetrical fractal pattern uses rotated squares inscribed in squares, as shown.



By evaluating an infinite geometric sum, find the limit of the ratio of shaded to unshaded area.

Set the outer square to have side length 1, and calculate the total shaded area as $\frac{1}{2} + ...$ Then use $S_\infty = \frac{a}{1-r}$.

Giving the outer square side length 1, the outer set of shaded triangles has area $\frac{1}{2}$. The length scale factor to the first inscribed square is $1/\sqrt{2}$, so the LSF to the second inscribed square is $\frac{1}{2}$. Hence, the area scale factor to the second set of shaded triangles is $\text{LSF}^2 = \frac{1}{4}$. So, the total shaded area is an infinite geometric series with first term $a = \frac{1}{2}$ and common ratio $r = \frac{1}{4}$. Using the standard sum:

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}.$$

The ratio of areas is, therefore, $2 : 1$.

1500. The tangent to the curve $y = (1 + x^2)^{-1}$ at $x = 1$ crosses the x axis at Q. Find the coordinates of Q.

Use the chain rule to find the tangent line.

Using the chain rule to differentiate,

$$y = (1 + x^2)^{-1} \implies \frac{dy}{dx} = -2x(1 + x^2)^{-2}.$$

Evaluating at $x = 1$ gives $m_{\text{tangent}} = -\frac{1}{2}$. The coordinate is $(1, \frac{1}{2})$, so the tangent has equation $y - \frac{1}{2} = -\frac{1}{2}(x - 1)$. At Q, $y = 0$, which gives $Q : (2, 0)$.

1501. A bird of weight W is standing in equilibrium on a roof sloped at 30° to the horizontal. Find the following, in terms of W:

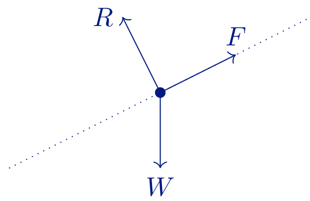
(a) the normal reaction on the bird's feet,

(b) the friction on the bird's feet,

(c) the total contact force on the bird's feet.

Draw a force diagram, but note that the answer to (c) can be written down without calculation.

The normal reaction acts perpendicular to the roof, the friction parallel to it.



(a) Perpendicular to the roof, $R = \frac{\sqrt{3}}{2}W$.

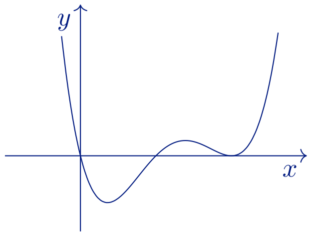
(b) Parallel to the roof, $R = \frac{1}{2}W$.

(c) This can be written down without calculation: the total contact force must be W.

1502. A positive polynomial of even degree has single roots at 0 and 1, a double root at 2, and no other roots. Sketch its main features.

The quartic $y = x(x - 1)(x - 2)^2$ is the minimal example of such a polynomial.

The quartic $y = x(x - 1)(x - 2)^2$ is the minimal example of such a polynomial. It crosses the x axis at $x = 0$ and $x = 1$, and is tangent to it at $x = 2$.



1503. The *golden ratio* ϕ is defined as $\phi = \frac{1}{2}(1+\sqrt{5})$. Find, in the form $x^2 + px + q = 0$ for $p, q \in \mathbb{Z}$, the quadratic equation of which ϕ is a root.

Considering the quadratic formula, find the other root, and then multiply out a factorised version of the equation.

According to the quadratic formula, if p and q are to be integers, the second root must be $\frac{1}{2}(1-\sqrt{5})$. The factor theorem, then, gives the quadratic as

$$\begin{aligned} &\left(x - \tfrac{1}{2}(1+\sqrt{5})\right)\left(x - \tfrac{1}{2}(1-\sqrt{5})\right) = 0 \\ \Longrightarrow\ & x^2 - x - 1 = 0. \end{aligned}$$

1504. Variables X_1 and X_2 have identical, independent binomial distributions $X_1, X_2 \sim B(4, 0.5)$. State whether the following are distributed binomially; if so, give their distributions in the form $B(n, p)$:

(a) $2X_1$,

(b) $X_1 + X_2$,

(c) $X_1 - X_2$.

Consider the list of possible outcomes.

(a) Not binomial: the outcomes are all even.

(b) Since X_1, X_2 are independent, this is binomial: eight identical trials as opposed to just four. The distribution is $X_1 + X_2 \sim B(8, 0.5)$.

(c) Not binomial: there are negative outcomes.

1505. The graph $y = \dfrac{x^2+4}{x+1}$ has an oblique asymptote.

(a) Find A, B, C such that

$$\frac{x^2+4}{x+1} \equiv Ax + B + \frac{C}{x+1}.$$

(b) Hence, show that the oblique asymptote has equation $y = x - 1$.

In (a), write $x^2 + 4$ as $x(x+1) - (x+1) + 5$ and then split the fraction up. In (b), consider the behaviour as $x \to \infty$ (or $x \to -\infty$).

(a) We write the numerator as a multiple of the denominator plus a constant remainder:

$$\begin{aligned} &\frac{x^2+4}{x+1} \\ \equiv\ & \frac{(x-1)(x+1)+5}{x+1} \\ \equiv\ & \frac{(x-1)(x+1)}{x+1} + \frac{5}{x+1} \\ \equiv\ & x - 1 + \frac{5}{x+1}. \end{aligned}$$

(b) As $x \to \pm\infty$, the fraction becomes negligible. Hence, the curve approaches $y = x - 1$, which is therefore an oblique asymptote.

1506. A projectile is launched horizontally from the point $(0, c)$ with speed u ms^{-1}. Show that the equation of the trajectory is given by

$$y = c - \frac{gx^2}{2u^2}.$$

Set up *suvat* vertically and horizontally, ignoring the c and using x and y for vertical displacement. Eliminate t to find a Cartesian equation, and then translate the curve to take account of the starting point.

Vertical and horizontal *suvat*:

s_y	y	s_x	x
u_y	0	u_x	u
v_y	n/a	v_x	n/a
a_y	$-g$	a_x	0
t	t	t	t

These give $y = -\frac{1}{2}gt^2$ and $x = ut$. Rearranging to $t = \frac{x}{u}$, we substitute for t. Adding $+c$ to translate the curve by vector $\binom{0}{c}$ gives the required result:

$$y = c - \frac{gx^2}{2u^2}.$$

1507. A sequence has n^{th} term $u_n = 4n^2 - 18n - 40$, for $n \geq 1$. Find the value of the first non-negative term.

To find the term, solve $u_n = 0$.

Solving u_n, we get

$$n = \frac{18 \pm \sqrt{964}}{8} = -1.63..., 6.13...$$

Hence, since the ordinal formula is a positive quadratic, the first positive term will be at $n = 7$. Substituting, its value is $u_7 = 30$.

1508. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $xyz = 0 \implies xy = 0$,

(b) $xy = 0 \implies xyz = 0$.

Statement (a) is false. Find a counterexample.

Statement (a) is false; $(1, 1, 0)$ is a counterexample.

1509. The result of an experiment is modelled with Y, where Y has a normal distribution with mean 3 and variance 6. Find $P(Y^2 > 4Y)$.

Solve the inequality $Y^2 > 4Y$ algebraically before considering the normal distribution.

Algebraically, $Y^2 > 4Y$ is satisfied whenever $Y \in (-\infty, 0) \cup (4, \infty)$. Evaluating the probability of Y being in these regions, using symmetry and then a cumulative distribution function, we get

$$p = 0.5 + 0.3415... = 0.842 \text{ (3sf)}.$$

1510. You are given that the following curve intersects the x axis at $(k, 0)$:

$$y = \frac{x^2}{1+x} + k.$$

Determine all possible values of k.

Substitute $(k, 0)$ and solve.

Substituting $(k, 0)$, we require

$$\begin{aligned} & 0 = \frac{k^2}{1+k} + k. \\ \implies & 0 = k^2 + k(1+k) \\ \implies & 0 = k(2k+1) \\ \implies & k = 0, -\tfrac{1}{2}. \end{aligned}$$

1511. Separate the variables in the following differential equation, writing it in the form $f(y)\frac{dy}{dx} = g(x)$ for some functions f and g:

$$e^{2x+3y}\frac{dy}{dx} = 2.$$

Split the exponential up using an index law.

We can split the exponential up using the relevant index law, and then divide:

$$\begin{aligned} & e^{2x+3y}\frac{dy}{dx} = 2 \\ \implies & e^{2x}e^{3y}\frac{dy}{dx} = 2 \\ \implies & e^{3y}\frac{dy}{dx} = 2e^{-2x}. \end{aligned}$$

1512. Solve for n in $\displaystyle\sum_{k=1}^{n}(2k-1) = 100$.

This is an arithmetic series.

This is an arithmetic series with first term $a = 1$, common difference $d = 2$ and n terms. So, we have

$$\begin{aligned} & \tfrac{1}{2}n\big[2 + 2(n-1)\big] = 100 \\ \implies & n^2 = 100. \end{aligned}$$

Therefore, $n = 10$.

1513. State, with justification, which of the implications $\implies$, $\impliedby$, $\iff$ could replace the question marks below, linking statements concerning $x \in \mathbb{R}$:

$$0 = x^2 + x + 1 \quad ??? \quad \sqrt{x} = x + 1.$$

Square both sides of the second statement.

The implication does not go forwards: the root $x = \frac{1}{2}(-1 - \sqrt{5}) < 0$ is a counterexample. In the other direction, if we square both sides of the second statement, we get

$$\begin{aligned} & \sqrt{x} = x + 1 \\ \implies & x = x^2 + 2x + 1 \\ \implies & 0 = x^2 + x + 1. \end{aligned}$$

Hence, the direction of implication is $\impliedby$.

1514. Show that the ordinal formula for the sequence $u_n = 4u_{n-1}$, $u_3 = 24$ can be expressed as

$$u_n = 3 \times 2^{2n-3}.$$

This is a geometric progression. Use the standard formula for the n^{th} term and then manipulate the algebra.

This is a GP with common ratio $r = 4$. The first term, therefore, is $a = u_1 = \frac{1}{16}u_3 = \frac{3}{2}$. Quoting the ordinal formula for a GP:

$$\begin{aligned} u_n &= \tfrac{3}{2} \times 4^{n-1} \\ &= \tfrac{3}{2} \times (2^2)^{n-1} \\ &= \tfrac{3}{2} \times 2^{2n-2} \\ &= 3 \times 2^{2n-3}. \end{aligned}$$

1515. Write down the periods of the following functions, defined in degrees:

(a) $\sin x$,

(b) $\sin 3x$,

(c) $\sin 2x + \sin 3x$,

(d) $\sin 2x + \sin 4x$.

In (b), consider the transformation associated with replacement of x by $3x$. In (c) and (d), you then need to consider the lowest common multiple of the individual periods.

(a) $360°$,

(b) $\frac{1}{3} \times 360° = 120°$,

(c) $\text{lcm}(180°, 120°) = 360°$,

(d) $\text{lcm}(180°, 90°) = 180°$.

1516. Prove that, if a quadratic graph $y = x^2 + bx + c$ has its vertex at $x > 0$, then $b < 0$.

Using the quadratic formula, or by differentiation, find the equation of the line of symmetry of the graph.

Setting $\frac{dy}{dx} = 2x + b = 0$, the vertex of the parabola is at $x = -\frac{b}{2}$. We are told that $x > 0$, hence, $-\frac{b}{2} > 0$. Therefore $b < 0$.

1517. A function $g(x)$ has second derivative 0 for all x. The graph $y = g(x)$ passes through $(-1, 2)$ with gradient 4.

(a) Find $g(x)$.

(b) Solve the equation $g(x) = g(1 - x)$.

Since the second derivative is zero, the graph is a straight line (it has zero curvature).

(a) The second derivative is zero everywhere, so g is linear. Its gradient is 4. Substituting $(-1, 2)$ gives $g(x) = 4x + 6$.

(b) Our equation is

$$\begin{aligned} & g(x) = g(1-x) \\ \Longrightarrow\ & 4x + 6 = 4(1-x) + 6 \\ \Longrightarrow\ & x = \tfrac{1}{2}. \end{aligned}$$

This result could also be written down without calculation as the line of symmetry in which $y = g(x)$ and $y = g(1 - x)$ are reflections.

1518. Describe the transformation which takes the graph $y = f(x)$ onto the graph $f(y) = x$.

The roles of x and y have been reversed.

Since the roles of x and y have been reversed, the transformation is a reflection in the line $y = x$.

1519. Show that $\displaystyle\int_0^4 x(x+1)(x+2)\,dx = 144$.

Multiply the brackets out first, then integrate.

Multiplying out and then integrating,

$$\begin{aligned} & \int_0^4 x(x+1)(x+2)\,dx \\ &= \int_0^4 x^3 + 3x^2 + 2x\,dx \\ &= \left[\tfrac{1}{4}x^4 + x^3 + x^2\right]_0^4 \\ &= (64 + 64 + 16) - (0) \\ &= 144. \end{aligned}$$

1520. A set of data is analysed, and one or more outliers are removed from it. State, with a reason, whether the following necessarily hold:

(a) the new IQR is smaller,

(b) the new standard deviation is smaller.

Both are measures of spread, but only one takes account of all of the data.

(a) This may or may not be true. The IQR doesn't take account of the magnitudes of the highest and lowest values, so it is possible that it will be unaffected by such a removal.

(b) This is definitely true. The standard deviation takes account of all data items. Hence, the removal of one or more extreme values must reduce its value as a measure of spread.

1521. Show that $y = 4x^4 - 8x^2$ has minima at $y = -4$.

Set the first derivative to zero to find stationary points, then evaluate the second derivative to classify them as minima (or use the shape of the curve).

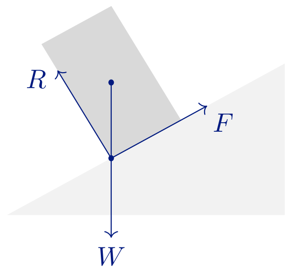
For stationary points, we set the first derivative to zero: $16x^3 - 16x = 0$. This gives $x = -1, 0, 1$. We can test these values with the second derivative $48x^2 - 16$; the respective values are $32, -16, 32$. Hence, the stationary points at ± 1 are local minima. Their y coordinate is $y = -4$.

1522. A uniform cylinder of radius r and length l is placed with one of its circular faces on a rough slope. Assuming that friction is high enough so that no slippage occurs, show that the greatest angle of inclination θ on which the cylinder can remain in equilibrium is

$$\theta = \arctan \frac{2r}{l}.$$

Consider the line of action of the weight when the cylinder is on the point of toppling: it must pass through the lowest point in contact with the slope.

The cross-section is a $2r$ (diameter in contact with the slope) by l rectangle. Since it is uniform, the centre of mass is in the middle. At the greatest angle of inclination, the cylinder is on the point of toppling, which means the reaction force and the weight must both have lines of action passing through the lowest point on the cylinder:



The acute angle between the reaction and the weight is given by $\tan\theta = \frac{2r}{l}$. This is also the angle of inclination, so $\theta = \arctan\frac{2r}{l}$ as required.

1523. Show that it is impossible for a quadratic graph $y = ax^2 + bx + c$, where $a \neq 0$, to pass through the three points $(4, 0)$, $(5, 2)$, and $(7, 6)$.

Show that the points are collinear.

The three points are collinear, lying on $y = 2x - 8$. Assume, for a contradiction, that they all lie on a parabola $y = ax^2 + bx + c$. Substituting for y to find intersections with the line $y = 2x - 8$, we get a quadratic equation in x with three distinct roots $x = 4, 5, 7$. This is impossible. Hence, no quadratic graph passes through these three points.

1524. A set of data, whose mean is 2 and whose standard deviation is 1, is given as follows:

x	0	1	2	3
f	12	14	a	b

Find a and b.

Solve for b using the mean, then for a using the standard deviation.

The value of the mean gives

$$\frac{0\times 12 + 1\times 14 + 2a + 3b}{12 + 14 + a + b} = 2,$$

which simplifies to $b = 38$. Then, we can use the formula for the variance, with $n = 64 + a$:

$$s^2 = \frac{\sum x^2 - n\bar{x}^2}{n}.$$

$$\begin{aligned}\text{So,}\quad 1 &= \frac{14 + 4a + 9\times 38 - (64+a)2^2}{64+a}\\ \implies 64 + a &= 356 + 4a - 4(64+a)\\ \implies a &= 36.\end{aligned}$$

1525. Solve $x^{\frac{5}{3}} + 7776 = 275x^{\frac{5}{6}}$.

This is a disguised quadratic in $x^{\frac{5}{6}}$.

Since $x^{\frac{5}{3}} = (x^{\frac{5}{6}})^2$, this is a disguised quadratic in $x^{\frac{5}{6}}$. The formula gives

$$\begin{aligned}&x^{\frac{5}{3}} - 275x^{\frac{5}{6}} + 7776 = 0\\ \implies\ &x^{\frac{5}{6}} = \frac{275 \pm \sqrt{275^2 - 4\cdot 7776}}{2}\\ \implies\ &x^{\frac{5}{6}} = 32 \quad\text{or}\quad x^{\frac{5}{6}} = 243\\ \implies\ &x = 32^{\frac{6}{5}} \quad\text{or}\quad x = 32^{\frac{6}{5}}\\ \implies\ &x = 64 \quad\text{or}\quad x = 729.\end{aligned}$$

1526. Two balls of similar size, with masses m and $2m$, are dropped simultaneously from the same height. The air resistance on each ball is modelled as a constant $\frac{1}{4}mg$.

(a) Find a formula for the distance d between the two in the ensuing motion.

(b) Find the predicted time taken for this distance to exceed 100 metres.

(c) Give a reason why this value is unlikely to be accurate in any physical scenario.

In (a), find the accelerations using $F = ma$, and then set up *suvat* equations for the displacement of each ball. In (c), consider the modelled speed of the balls once they are 100 metres apart.

(a) The accelerations are given by $mg - \frac{1}{4}mg = ma$ and $2mg - \frac{1}{4}mg = 2ma$. These yield $a_1 = \frac{3}{4}g$ and $a_2 = \frac{7}{8}g$. Since the initial conditions are the same for the two balls, we can use relative acceleration, which is $a = \frac{7}{8}g - \frac{3}{4}g = \frac{1}{8}g$. With this value, the distance between them is

$$d = \tfrac{1}{2}\cdot\tfrac{1}{8}gt^2 = \frac{gt^2}{16}.$$

(b) Setting $d = 100$, we get $t = 12.8$ s (3sf).

(c) At this time, the heavier ball is modelled to be travelling at $v = 12.8 \times \frac{7}{8}g > 100$ m/s. Assuming air resistance to remain constant at such speeds is certainly inaccurate.

1527. You are given that the equations $x + y = a$ and $2x + by = 10$, for constants $a, b, c \in \mathbb{R}$, have no simultaneous solutions (x, y).

(a) Show that $b = 2$.

(b) Determine all possible values for a, giving your answer in set notation.

See the equations as a pair of parallel lines.

(a) If these equations are to have no simultaneous solutions, then they must be a pair of distinct parallel lines. Equating the gradients, we get $\frac{b}{2} = 1$ as required.

(b) All values of a are permitted, except $a = 5$ in which case the lines are identical and there are infinitely many solutions. So, $a \in \mathbb{R} \setminus \{5\}$.

1528. "An x intercept of $y = f(x)$ must be an x intercept of $y = f(x)^2$." True or false?

At an x intercept, $f(x) = 0$.

At an x intercept, $f(x) = 0$, which implies that $f(x)^2 = 0$. So the statement is true.

1529. The four points A, B, C, D, have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. M is the midpoint of AB, and N is the midpoint of CD. Show that the position vector of the midpoint of MN is the mean of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

Find the position vectors of M and N using means, then find the mean of them.

The position vector of a midpoint is the mean of the two position vectors. So, the position vector of M is $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and the position vector of N is $\mathbf{n} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$. Hence, the midpoint P of MN has position vector

$$\begin{aligned}\mathbf{p} &= \tfrac{1}{2}(\mathbf{m} + \mathbf{n}) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}(\mathbf{a} + \mathbf{b}) + \tfrac{1}{2}(\mathbf{c} + \mathbf{d})\right) \\ &= \tfrac{1}{4}\left(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}\right),\end{aligned}$$

which is the mean, as required.

1530. Prove that a quadratic curve and a cubic curve, each of the form $y = f(x)$, must intersect.

Express the curves in generic algebraic form, and set up an equation to solve for intersections.

Setting up generic curves $y = ax^2 + bx + c$ and $y = px^3 + qx^2 + rx + s$, where $a, p \neq 0$, we can solve for intersections with

$$px^3 + (q - a)x^2 + (r - b)x + s - r = 0.$$

Since $p \neq 0$, this is a cubic equation. And, being a polynomial of odd degree, it must have at least one root. Hence, the curves must intersect. $\square$

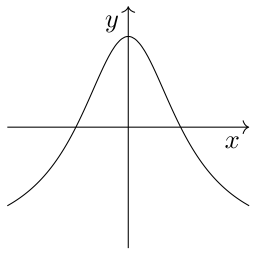
1531. The edges of a regular pentagon are coloured red or blue. Show that, if rotations and reflections are not counted as distinct, there are eight colourings.

You can list these by cases, according to whether there are 0,1,2,3,4,5 blue edges.

Listing by number of blue edges:

0 :	RRRRR	
1 :	BRRRR	
2 :	BBRRR,	BRBRR
3 :	BBBRR,	BBRBR
4 :	BBBBR	
5 :	BBBBB	

1532. Prove that $y = \dfrac{1}{x^2 + 1}$ cannot generate the graph:



Consider the x axis intercepts.

The graph shown crosses the x axis. However, the algebraic fraction $\frac{1}{1+x^2}$ could never be zero, since its numerator is non-zero. Hence, it can't produce such a graph.

1533. Solve $(3x - 1)^2 - (3x - 1)^3 = 0$.

This should be solved without multiplying out: take out a factor of $(3x - 1)^2$.

We can factorise directly:

$$\begin{aligned} & (3x - 1)^2 - (3x - 1)^3 = 0 \\ \implies & (3x - 1)^2(1 - (3x - 1)) = 0 \\ \implies & x = \tfrac{1}{3}, \tfrac{2}{3}.\end{aligned}$$

1534. Determine the number of elements in the set

$$\{x \in \mathbb{N} : x^2 < 20\} \cap \{x \in \mathbb{N} : x^3 < 50\}.$$

First, solve the boundary equations $x^2 = 20$ and $x^3 = 50$. Then list the constituent sets.

The boundaries are at $\sqrt{20}$, which lies between 4 and 5, and $\sqrt[3]{50}$, which lies between 3 and 4. Hence, the set is

$$\begin{aligned} &\{-4,-3,...,3,4\} \cap \{...,1,2,3\} \\ =\ &\{-4,-3,-2,-1,0,1,2,3\} \end{aligned}$$

It contains 8 elements.

1535. In this question, $X_1 \sim B(n_1, p)$ and $X_2 \sim B(n_2, p)$ are independent binomial variables with the same trial probability. State, with a reason, whether the following variables are binomially distributed. If so, give the distribution in the form $B(n, p)$.

(a) $X_1 + X_2$,
(b) $X_1 X_2$,
(c) $n_1 + n_2 - X_1 - X_2$.

The variables in (a) and (c) are binomial.

(a) This is binomial: a set of $n_1 + n_2$ independent trials with constant probability of success p. Hence $X_1 + X_2 \sim B(n_1 + n_2, p)$.
(b) This is not binomial. For starters, its outcomes do not form a list of integers $\{0, 1, ..., n\}$.
(c) Using part (a), this is binomial. Since $X_1 + X_2$ is the number of successes in $n_1 + n_2$ trials, this variable $n_1 + n_2 - X_1 - X_2$ is the number of failures. Its distribution is $B(n_1 + n_2, 1 - p)$.

1536. "The y axis is tangent to $(x+1)^2 + (y+1)^2 = 1$." True or false?

Find the radius and centre of the circle.

This is true. The equation is that of a circle centre $(-1, -1)$ with radius 1. Hence, it is tangent to the y axis at $(-1, 0)$.

1537. State, with a reason, whether the following holds: "A hypothesis test for the probability of success p in a binomial distribution $X \sim B(20, p)$ is to be carried out, testing $H_0 : p = \frac{1}{4}$ and $H_1 : p > \frac{1}{4}$ at the 1% level. Assuming H_0, with c denoting the critical value and x the number of successes,

$$P(c \le x) < 0.01 < P(c - 1 \le x).$$

This is true; explain why.

True. In such a one-tail test, the critical region is of the form $c \le x$. The value c is chosen such that $P(c \le x) < 0.01$ (getting an outcome in the region $c \le x$ is less probable than 1%), and $P(c - 1 \le x) > 0.01$ (getting an outcome in the larger region $x \ge c - 1$ is more probable than 1%).

1538. Determine which of the points $(2, -2)$ and $(-6, 2)$ is closer to the locus $x^2 + 6x + y^2 + 2y - 20 = 0$.

Being closer to a circle is the same as having a distance from the centre closer to the radius.

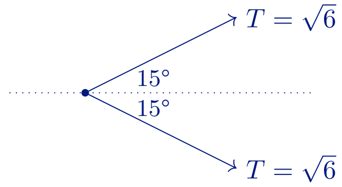
Since the locus is a circle, we can test distances to the centre. Completing the square for x and y gives $(x+3)^2 + (y+1)^2 = 30$, so the centre is $(-3, -1)$ and the the radius is $\sqrt{30}$. The distances from the centre, then, are $\sqrt{26}$ and $\sqrt{20}$. Since both points lie inside the circle, the point further away from the centre is closer: this is $(2, -2)$.

1539. A light string is passed over a smooth pulley. The sections of string either side of the pulley make an angle 30° with each other, and the tension in the rope is $\sqrt{6}$ N. Show that the force F exerted on the pulley by the rope is given by $F = 3 + \sqrt{3}$ N. You may assume the following exact value:

$$\cos 15° = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

Resolve the forces along their line of symmetry.

The forces exerted by the string are:



Perpendicular to the line of symmetry, there is no resultant component of force. Parallel to it, the forces combine to give

$$\begin{aligned} &2 \times \sqrt{6} \cos 15° \\ =\ &\frac{2\sqrt{6}(\sqrt{6} + \sqrt{2})}{4} \\ =\ &\frac{6 + 2\sqrt{3}}{2} \\ =\ &3 + \sqrt{3} \text{ N}, \quad \text{as required.} \end{aligned}$$

1540. By setting up simultaneous equations, find integers p and q such that

$$\left(p\sqrt{3} + q\sqrt{5}\right)^2 = 155 - 40\sqrt{15}.$$

Expand and equate rational and irrational parts.

Expanding, $3p^2 + 2pq\sqrt{15} + 5q^2 = 155 - 40\sqrt{15}$. Since $p, q \in \mathbb{Z}$, we can equate the rationals and coefficients of $\sqrt{15}$, giving simultaneous equations:

$$\begin{aligned} 3p^2 + 5q^2 &= 155 \\ 2pq &= -40. \end{aligned}$$

Substituting $q = -\frac{20}{p}$ gives

$$\begin{aligned} 3p^2 + 5 \times \frac{400}{p^2} &= 155 \\ \implies 3p^4 - 155p^2 + 2000 &= 0 \\ \implies (3p^2 - 80)(p^2 - 25) &= 0. \end{aligned}$$

The first bracket gives non-integers, so $p = \pm 5$. Substituting yields $q = \mp 4$.

1541. "For any quadrilateral, there exists a circle which passes through all four vertices." Disprove this.

Give a specific counterexample.

For a counterexample, consider the vertices of a square. Since three of these points define a circle, moving one of the vertices away from the centre will produce a non-cyclic quadrilateral.

1542. Find y in simplified terms of x, if

(a) $3^y = 3^x \cdot 9^x$,
(b) $e^y = e^x \div e^{2x-5}$.

Rearrange to $a^p = a^q \implies p = q$.

(a) The RHS is $3^x \cdot 3^{2x} = 3^{3x}$, so $y = 3x$.
(b) The RHS is $e^{x-(2x-5)} = e^{-x+5}$, so $y = -x + 5$.

1543. A normal is drawn to the curve $y = x^3 - x$ at the origin. Show that this normal, together with the curve, encloses two regions of area 1.

Differentiate to find the equation of the normal at the origin. Then solve this simultaneously with the curve to find the intersections, and set up a definite integral for one of the regions.

The derivative is $\frac{dy}{dx} = 3x^2 - 1$, so $m_{\text{tangent}} = -1$ at the origin, giving $m_{\text{normal}} = 1$. The equation of the normal as $y = x$. To find intersections, then, we solve $x^3 - x = x$, giving $x = 0, \pm\sqrt{2}$.

The area of the region $x \in \left[0, \sqrt{2}\right]$, for which the normal is above the curve, can be found by integrating the following difference:

$$\begin{aligned} A &= \int_0^{\sqrt{2}} x - (x^3 - x)\,dx \\ &= \int_0^{\sqrt{2}} 2x - x^3\,dx \\ &= \left[x^2 - \tfrac{1}{4}x^4\right]_0^{\sqrt{2}} \\ &= (2-1) - (0) \\ &= 1. \end{aligned}$$

By symmetry, the other region has the same area.

1544. Two invertible functions have $f(a) = g(-a) = 0$, and $f(-a) = g(a) = -a$. Write down

(a) a root of g,
(b) a fixed point of f,
(c) a root of g^2.

A root satisfies $f(x) = 0$, a fixed point satisfies $f(x) = x$.

(a) $x = -a$ is a root of g, since $g(-a) = 0$,
(b) $x = -a$ is a fixed point of f, since $f(-a) = -a$,
(c) $x = a$ is a root of g^2, as $g(g(a)) = g(-a) = 0$.

1545. A rectangle has dimensions $x \times y$. Starting from $x = 2, y = 10$, its lengths change at constant rates of 1 and -1 units per second respectively.

(a) Using the product rule, or otherwise, show that the rate of change of area is given by

$$\frac{dA}{dt} = y - x.$$

(b) Determine whether, at the beginning, the area increases or decreases.
(c) Find the stationary value of the area.

Use the product rule $\dfrac{d}{dt}(xy) = \dfrac{dx}{dt}y + x\dfrac{dy}{dt}$.

(a) Differentiating the area with respect to time, we need the product rule, which gives the sum of the components of the rate of change of area due to changes in x and changes in y:

$$\begin{aligned} \frac{dA}{dt} &= \frac{dx}{dt}y + x\frac{dy}{dt} \\ &= 1 \times y - x \times 1 \\ &= y - x, \text{ as required.} \end{aligned}$$

(b) Substituting $x = 2, y = 10$, the rate of change of area is $\frac{dA}{dt}\big|_{t=0} = 10 - 2 = 8$ square units/second, so area increases initially.

(c) The area is stationary when $\frac{dA}{dt} = y - x = 0$, i.e. when $x = y$. Since x is increasing from 2 and y is decreasing at the same rate from 10, they are equal at $x = y = 6$, when the area is 36 square units.

1546. Solve $\dfrac{x}{x+1} - \dfrac{x-1}{x} = \dfrac{1}{12}$.

Multiply up by the denominators, taking care with the minus signs.

Multiplying up by the denominators,

$$\begin{aligned} & \frac{x}{x+1} - \frac{x-1}{x} = \frac{1}{12} \\ \implies & 12x^2 - 12(x-1)(x+1) = x(x+1) \\ \implies & 0 = x^2 + x - 12 \\ \implies & x = -4, 3. \end{aligned}$$

1547. A particle, following a period of time of duration t moving with constant acceleration a, ends up with final velocity v.

(a) Show that the velocity V at any time T is given by the formula $V = aT + v - at$.

(b) Use definite integration with respect to T to prove that, over this period, the displacement s is given by

$$s = vt - \tfrac{1}{2}at^2.$$

In (a), set up a linear expression for the velocity $V = aT + c$ and substitute to find the $+c$. In (b), integrate between $T = 0$ and $T = t$. In both, note carefully the difference in meaning between v (final velocity) and V (variable velocity), and between t (final time) and T (variable time).

(a) The formula for the velocity V is linear in T, with constant gradient a. It is $V = aT + c$. Substituting the known value $T = t, V = v$, we get $c = v - at$, which gives $V = aT + c - at$.

(b) The total displacement, then, is given by the definite integral between $T = 0$ and $T = t$:

$$\begin{aligned} s &= \int_{T=0}^{T=t} aT + v - at \, dT \\ &= \left[\tfrac{1}{2}aT^2 + vT - atT\right]_{T=0}^{T=t} \\ &= \left(\tfrac{1}{2}at^2 + vt - at^2\right) - (0) \\ &= vt - \tfrac{1}{2}at^2. \end{aligned}$$

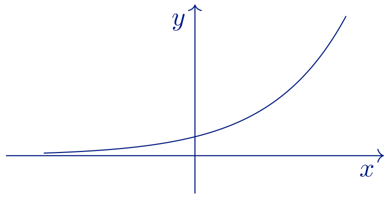
1548. Sketch examples of curves with these properties:

(a) increasing and convex everywhere,

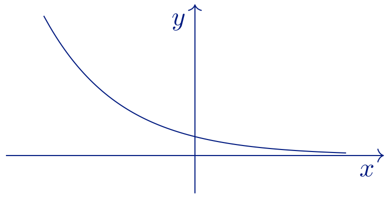
(b) decreasing and convex everywhere.

Increasing/decreasing refers to the sign of the first derivative, i.e. the gradient. Convex/concave refers to the sign of the second derivative, i.e. the rate of change of the gradient or curvature.

(a) Exponential growth, such as the curve, $y = e^x$ has positive first and second derivatives:



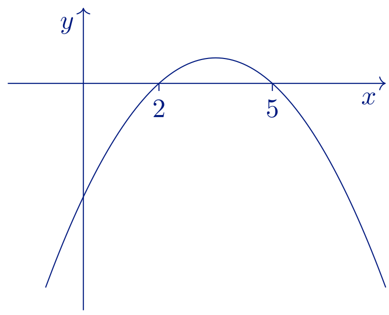
(b) Exponential decay, such as the curve $y = e^{-x}$, has negative first, positive second derivative:



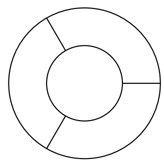
1549. For a quadratic function f, the definite integral of $f(x)$, between $x = a$ and $x = b$, is maximised when $a = 2$ and $b = 5$. Sketch $y = f(x)$.

The area under the quadratic has a maximum, so it must be a negative parabola. Consider the contribution of the sections below the x axis.

The area under the quadratic has a maximum, so $y = f(x)$ is a negative parabola. Furthermore, it must have roots at $x = 2$ and $x = 5$, so that any domain larger than $[2, 5]$ causes a reduction in the value of the definite integral. Hence, the sketch is



1550. The regions of the following diagram are randomly coloured, each red, green or blue.



Write down the probability that no two regions sharing a border are coloured the same.

This is a key map in the *Four Colour Theorem*: it requires four different colours to colour it.

Since each region borders every other region, the probability is zero. You need four colours to have no two like regions bordering each other.

1551. Simplify $\ln(2\sqrt{e^n}) - \ln 2$.

Use log laws.

Using laws of logarithms and then the definition of the natural logarithm as $\ln x := \log_e x$,

$$\begin{aligned} &\ln(2\sqrt{e^n}) - \ln 2 \\ =\ &\ln \frac{2e^{\frac{1}{2}n}}{2} \\ =\ &\ln e^{\frac{1}{2}n} \\ =\ &\tfrac{1}{2}n. \end{aligned}$$

1552. The cubic $y = x^3 + x^2$ is reflected in the line $y = 4$.

(a) By finding the stationary points of the cubic, show that the cubic crosses the line only once.

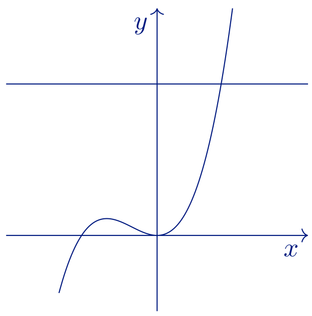
(b) Sketch cubic and line on the same set of axes.

(c) By considering reflection in $y = 4$ as a combination of reflection in $y = 0$ and translation by some vector $\binom{0}{k}$, determine the equation of the reflected cubic.

In (c), reflection in $y = 4$ is the same as reflection in $y = 0$ followed by translation by vector $\binom{0}{8}$.

(a) To find stationary points, we set $3x^2 + 2x = 0$, giving $(0, 0)$ and $(-\frac{2}{3}, \frac{4}{27})$. Since both of its turning points are below $y = 4$, the cubic can cross $y = 4$ only once.

(b) Positive cubic and line are:



(c) The relevant vector is $\binom{0}{8}$. Hence, the reflected cubic has equation $y = -x^3 - x^2 + 8$.

1553. To generate the formula for the volume of a sphere of radius r, the following integral is set up

$$V = \int_{-r}^{r} \pi(r^2 - x^2)\, dx.$$

(a) By considering a circle of radius r in an (x, y) plane, describe the physical significance of the integrand.

(b) Prove that $V = \frac{4}{3}\pi r^3$.

In (a), consider the equation of a circle radius r in the form $y^2 = r^2 - x^2$.

(a) A circle centred on the origin with radius r has equation $y^2 = r^2 - x^2$. So, the integrand is πy^2, which is the area of a circular cross-section through a sphere of radius r. The integral adds up all such discs, from $-r$ to r, to produce the volume of the sphere.

(b) Integrating definitely,

$$\begin{aligned} V &= \int_{-r}^{r} \pi(r^2 - x^2)\, dx \\ &= \pi\Big[r^2x - \tfrac{1}{3}x^3\Big]_{-r}^{r} \\ &= \pi\Big(\big(r^3 - \tfrac{1}{3}r^3\big) - \big(-r^3 + \tfrac{1}{3}r^3\big)\Big) \\ &= \tfrac{4}{3}\pi r^3 \quad \text{Q.E.D.} \end{aligned}$$

1554. Solve the equation $\left(\tan x - \frac{\sqrt{3}}{3}\right)\left(2\cos x - \sqrt{2}\right) = 0$, giving all values $x \in [0, 2\pi)$.

This is already factorised, so can be treated as two separate equations. There are four roots.

The equation is factorised, so we can split it. This gives firstly $\tan x = \frac{\sqrt{3}}{3} \implies x = \frac{\pi}{6}, \frac{7\pi}{6}$ and also $\cos x = \frac{\sqrt{2}}{2} \implies x = \frac{\pi}{4}, \frac{7\pi}{4}$. So, the solution is $x \in \{\frac{\pi}{6}, \frac{\pi}{4}, \frac{7\pi}{6}, \frac{7\pi}{4}\}$.

1555. Explain the (very large!) error in this appraisal of Newton's Laws:

> For an object in equilibrium, weight and reaction force are equal and opposite. This is due to Newton's third law.

Think of a counterexample to the first sentence: that initial claim is not always true. And when it is true, it has nothing to do with Newton III!

The claim isn't true: if an object is also being held up by, say, tension in a rope, then the reaction force will be lower than the weight.

The justification with NIII is, therefore, incorrect. A weight (gravitational) force is never the NIII pair of a reaction (contact) force: the NIII pair of gravity is always gravity (one on object A, one on object B, the Earth); the NIII pair of a contact force is always a contact force (one on object A, one on object B, the ground).

In situations in which the initial claim is correct, i.e. for objects in equilibrium with only weight and reaction acting on them, the justification for these forces being equal in magnitude is, in fact, Newton I/II: the resultant force must be zero.

1556. For $g : x \mapsto \dfrac{2}{\sqrt{3-2x}} + 1$, find g^{-1}.

Set up an equation with the output equal to y, then rearrange to make x the subject.

To find the inverse function g^{-1}, we set the output to y and rearrange:

$$\begin{aligned} y &= \frac{2}{\sqrt{3-2x}} + 1 \\ \Longrightarrow \sqrt{3-2x} &= \frac{2}{y-1} \\ \Longrightarrow 3-2x &= \frac{4}{(y-1)^2} \\ \Longrightarrow x &= \frac{3}{2} - \frac{2}{(y-1)^2}. \end{aligned}$$

Switching inputs and outputs, we have

$$g^{-1} : x \mapsto \frac{3}{2} - \frac{2}{(x-1)^2}.$$

1557. Either prove or disprove the following statement:

$$P(A \mid B) > P(A) \iff P(A \mid B') < P(A).$$

This is true. Consider the boundary scenario in which the LHS, say, is an equality.

This is true. The boundary condition of the LHS is $P(A \mid B) = P(A)$, which is exactly the condition for independence of events A and B. The same is true of the RHS. Hence, we know that

$$P(A \mid B) = P(A) \iff P(A \mid B') = P(A).$$

Consider departure from this equality. If knowing that B occurs *increases* the probability of A (LHS of the original inequality), then knowing that B doesn't occur must *decrease* the probability of A (RHS of the original inequality) and vice versa. Hence, the result holds.

1558. Show that, if x and y are related exponentially by $y = ae^{kx}$, then $\ln y$ and x are related linearly.

Take natural logs of both sides.

Taking natural logarithms of both sides,

$$\begin{aligned} & y = ae^{kx} \\ \Longrightarrow \quad & \ln y = \ln\left(ae^{kx}\right) \\ \Longrightarrow \quad & \ln y = \ln a + \ln\left(e^{kx}\right) \\ \Longrightarrow \quad & \ln y = \ln a + kx. \end{aligned}$$

This is a linear relationship between $\ln y$ and x: k is the gradient, $\ln a$ the additive constant.

1559. You are given that $f(x) = 2$ has exactly one root, which is at $x = p$, and that $f(x) = 3$ has exactly one root, which is at $x = q$. Giving your answers in terms of p and q, solve the following equations:

(a) $f(4x-1) = 2$,

(b) $\big(f(x+1)-2\big)\big(2f(x)-6\big) = 0$.

In (a), the logic you need is $f(*) = 2 \implies * = p$. In (b), solve two equations.

(a) Since p is the only value for which $f(p) = 2$, we have $4x - 1 = p$, therefore $x = \frac{1}{4}(p+1)$.

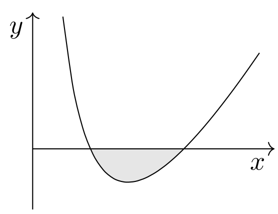
(b) Since the equation is already factorised, either $f(x+1) = 2$ or $f(x) = 3$. Hence, the solution is $x = p - 1$ or $x = q$.

1560. Show that, for any real-valued random variable X, $P(X^2 + 1 > X) = 1$.

This isn't a probability question, in fact. It's a question of pure algebra.

To show $P(X^2+1 > X) = 1$, we need to show that $X^2 + 1 > X$ always holds. The boundary equality is then $X^2 - X + 1 = 0$, which has discriminant $\Delta = -3 < 0$. Hence, $X^2 + 1 \neq X$. Furthermore, since X^2+1 is a positive quadratic, it must always be greater than X, regardless of the distribution of X. This proves the result.

1561. The curve below is $y = \dfrac{1}{\sqrt{x}} + x - 2$.



(a) By writing the relevant equation as a cubic in $\sqrt{x}$, show that the curve crosses the x axis at $x = \frac{1}{2}(3 - \sqrt{5})$ and $x = 1$.

(b) Hence, find the area of the shaded region.

In (a), set to 0 and multiply by $\sqrt{x}$. Then spot an obvious root and factorise, or else use a solver function. In (b), integrate definitely between the intercepts.

(a) Solving for the x intercepts,

$$\begin{aligned}\frac{1}{\sqrt{x}} + x - 2 &= 0\\ \implies x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 1 &= 0.\end{aligned}$$

This is a cubic in $x^{\frac{1}{2}}$, with a root at $x^{\frac{1}{2}} = 1$. So, we can factorise, giving

$$\begin{aligned}&(x^{\frac{1}{2}} - 1)(x + x^{\frac{1}{2}} - 1) = 0\\ \implies & x^{\frac{1}{2}} = 1 \text{ or } x^{\frac{1}{2}} = \tfrac{1}{2}(-1 + \sqrt{5}),\end{aligned}$$

ignoring the negative root of the quadratic. Squaring gives $x = 1$ or $x = \frac{1}{2}(3 - \sqrt{5})$.

(b) We set up a definite integral between the x intercepts:

$$\begin{aligned}&\int_{\frac{1}{2}(3-\sqrt{5})}^{1} \frac{1}{\sqrt{x}} + x - 2\,dx\\ =& \left[2x^{\frac{1}{2}} + \tfrac{1}{2}x^2 - 2x\right]_{\frac{1}{2}(3-\sqrt{5})}^{1}\\ =& \tfrac{1}{2} - 0.54508...\\ =& -0.04508...\end{aligned}$$

The signed area is negative, because below the x axis). The area, therefore, is 0.0451 (3sf).

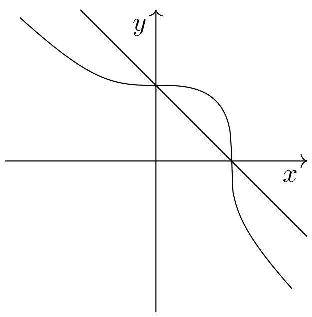
1562. Write $x^4 + x^2 + 1$ in terms of $(x^2 + 1)$.

Start with $(x^2 + 1)^2$ to match the x^4 term, then consider the term in x^2.

We need $(x^2 + 1)^2$ to match the x^4 term, which gives $x^4 + 2x^2 + 1$. So, we subtract $(x^2 + 1)$ to match the x^2 term. The constant term is then 1.

$$x^4 + x^2 + 1 \equiv (x^2 + 1)^2 - (x^2 + 1) + 1.$$

1563. The graph below shows $x + y = 1$ and $x^3 + y^3 = 1$.



Use the graph to provide counterexamples to the following implications:

(a) $x + y > 1 \implies x^3 + y^3 > 1$,

(b) $x^3 + y^3 > 1 \implies x + y > 1$.

The inequalities are satisfied by the points above and to the right of the relevant boundary line.

Choosing points close to the y intercept at $(0, 1)$, counterexamples are:

(a) $(0.2, 0.9)$: above $x + y = 1$, below $x^3 + y^3 = 1$.

(b) $(-0.2, 1.1)$: above $x^3 + y^3 = 1$, below $x + y = 1$.

1564. The binomial expansion of $(x + a)^5$ is given by $x^5 - 15x^4 + bx^3 + ...$, where $a, b \in \mathbb{Z}$. Determine the values of a and b.

Use -15 to find a, then expand to find b.

The coefficient -15 is ${}^5C_4 a$, which gives $a = -3$. Then, the coefficient of x^3 is ${}^5C_3 a^2$, so $b = 90$.

1565. A random number generator produces values Y_i distributed normally, with mean and variance 100. A sample of fifty values is taken.

(a) Write down the distribution of $\bar{Y}$.

(b) Find $P(99 \leq \bar{Y} \leq 101)$.

Use the standard result that, if $X \sim N(\mu, \sigma^2)$, then the mean of a random sample of size n has distribution $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

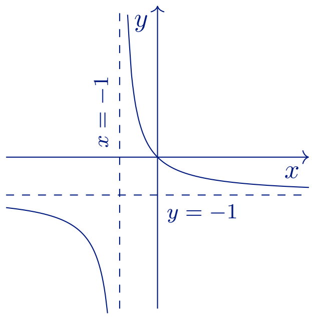
(a) Using the standard result for the distribution of the sample mean, $\bar{Y} \sim N(100, 2)$.

(b) Calculating a normal cumulative distribution, $P(99 \leq \bar{Y} \leq 101) = 0.52049... = 0.520$ (3sf).

1566. Sketch $(x + 1)(y + 1) = 1$.

Consider this graph as a transformation of $xy = 1$.

The graph $xy = 1$ is the standard reciprocal. We have replaced x by $(x + 1)$ and y by $(y + 1)$, enacting a translation by vector $\binom{-1}{-1}$. The resulting graph passes through the origin.



1567. An object is moving in one dimension of space with constant acceleration. It has (t, x) values $(0, 2)$, $(2, 4)$, and $(4, 8)$ during its motion.

(a) Determine the object's acceleration.

(b) Find x in terms of t.

(c) Find the average speed for $t \in [0, 10]$.

In (a), set up simultaneous equations in u and a using the period $t \in [0, 2]$ and the period $t \in [0, 4]$. In (c), find the distance travelled over the 10 seconds.

(a) Taking u to be the initial velocity and a to be the constant acceleration, we get simultaneous equations, each using $s = ut + \frac{1}{2}at^2$:

$$\begin{aligned} t \in [0, 2]: &\quad 2 = 2u + 2a \\ t \in [0, 4]: &\quad 6 = 4u + 8a \end{aligned}$$

Solving these, $u = \frac{1}{2}$ and $a = \frac{1}{2}$.

(b) Using the values above and the initial position $x = 2$, the formula is $x = 2 + \frac{1}{2}t + \frac{1}{4}t^2$.

(c) At $t = 10$, $x = 32$. Hence, the displacement is $s = 30$, giving an average speed of 3.

1568. Four lines are given as $x - 3y = 0$, $x - 3y = 10$, $3x + y = 10$, and $3x + y = 0$. Show that the lines enclose a square, whose area is 10 square units.

Find the coordinates of the vertices, or else use a symmetry argument.

The first two lines have gradient $\frac{1}{3}$, the last two have gradient -3. Hence, the shape is a rectangle. One vertex is at the origin, so we need only find the two adjacent vertices, which are at $(3, 1)$ and $(1, -3)$. Each is situated a distance $\sqrt{10}$ from the origin, proving that the shape enclosed is a square of area 10 square units.

1569. Describe all functions f for which f' is quadratic.

Integrate a cubic, and you get...?

If f' is quadratic, then f is a cubic function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

1570. In the solving of a separable differential equation, the following line has been reached:

$$\int 3x(x + 2)\, dx = \int \frac{1}{y}\, dy.$$

Show that $|y| = Ae^{x^3+3x^2}$, for some constant A.

Integrate each side, gathering the two constants of integration into one $+c$ on the LHS.

Combining the constants of integration into one $+c$ on the left-hand side, and using the standard integral of the natural logarithm,

$$\begin{aligned} \int 3x(x + 2)\, dx &= \int \frac{1}{y}\, dy \\ \Longrightarrow\ x^3 + 3x^2 + c &= \ln |y| \\ \Longrightarrow\ |y| &= e^{x^3+3x^2+c} \\ \Longrightarrow\ |y| &= e^c e^{x^3+3x^2} \\ \Longrightarrow\ |y| &= Ae^{x^3+3x^2}. \end{aligned}$$

We rename e^c as a (positive) constant A.

1571. A sequence is defined iteratively as $x_{n+1} = ax_n + b$, for some $a, b \in \mathbb{R}$. Three consecutive terms of the sequence are $8, 25, 93$. Determine a and b.

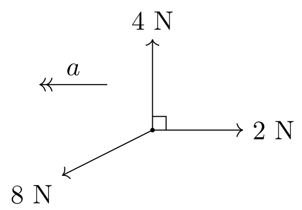
Set up simultaneous equations, using the steps $x_1 \mapsto x_2$ and $x_2 \mapsto x_3$.

Converting the information into equations,

$$\begin{aligned} 25 &= 8a + b \\ 93 &= 25a + b. \end{aligned}$$

Solving simultaneously, $a = 4$ and $b = -7$.

1572. An object of mass 2 kg has forces acting on it as shown in the diagram. Acceleration is antiparallel to the 2 Newton force.



Find the obtuse angle between the 8 and 2 Newton forces, and the exact magnitude of the acceleration.

Resolve perpendicular to the acceleration to find the angle, then resolve parallel to the acceleration.

Perpendicular to the acceleration, $8 \sin \theta - 4 = 0$, where θ is the acute angle between the 8 N force and the acceleration. This gives $\theta = 30°$. Hence, the obtuse angle between the 8 N and 2 N forces is 150°. Resolving parallel to the acceleration, $8 \cos 30° - 2 = 2a$, which gives $a = 2\sqrt{3} - 1$ ms^{-2}.

1573. A tangent is drawn to the curve $y = x^2$ at the point (p, p^2). Show that the equation of the tangent is $y = 2px - p^2$.

Differentiate to find the gradient in terms of p.

The gradient of the tangent is $2x\,|_{x=p} = 2p$. Then, using $y - y_0 = m(x - x_0)$, the equation of a general tangent is $y - p^2 = 2p(x - p)$, which simplifies to $y = 2px - p^2$, as required.

1574. Two of the following statements are true for all $a, b, c, d \in \mathbb{R}$; the other two are not. Identify and disprove the false statements.

(a) $a = b \implies (a - b)(c - d) = 0$,

(b) $a = b \implies \dfrac{a - b}{c - d} = 0$,

(c) $(a - b)(c - d) = 0 \implies a = b$,

(d) $\dfrac{a - b}{c - d} = 0 \implies a = b$.

Statements (b) and (c) are the false ones.

(b) is false. If $a = b$, but $c = d = 0$, then the fraction on the RHS is undefined, not zero.

(c) is false. The counterexample is $a \neq b$, $c = d$.

1575. By considering the distance from the origin, show that no part of the parametric curve $y = \sin t$, $x = \sin 2t$ lies outside the circle $x^2 + y^2 = 2$.

Consider the range of the sine function.

The range of the sine function is $[-1, 1]$. This means that the inputs of the two sine functions are not, in fact, relevant. The furthest the curve could possibly be from the origin is $\sqrt{1^2 + 1^2} = \sqrt{2}$, which is the radius of the given circle. Hence, the result is proved.

1576. At a surgery, the ages of 10 walk-in patients are recorded. The mean of this data is $\bar{x} = 66.1$ years and the standard deviation is 14.6 years. An overly zealous employee decides to model the ages with a normal distribution.

(a) Find, according to the employee's model, the probability that the next walk-in patient is

i. under 30 years of age,

ii. between 50 and 70 years of age.

(b) Give two reasons why these probabilities are unlikely to be useful in making predictions.

In (a), use the cumulative distribution function on a calculator. In (b), consider the skewness of the distribution and the size of the sample.

(a) Using the cumulative distribution function on a calculator, assuming $X \sim N(66.1, 14.6^2)$,

i. $P(X < 30) = 0.00671$ (3sf),

ii. $P(50 < X < 70) = 0.470$ (3sf).

(b) There are many reasons! Firstly, the sample is too small to give reliable estimates for μ and σ. Secondly, and of broader application, the default position is that a normal distribution should **not** be used, and you need very good reasons for supposing that one does apply. In this case, we have the opposite. For example, since there are upper and lower bounds (whose origins are different) on age, the distribution of the population of ages of walk-in patients is guaranteed not to be symmetrical.

1577. Show that, if $x + y > 2$, then $x^2 + y^2 > 2$.

This can be easily proven by sketching the relevant boundary graphs.

The relevant boundary graphs are a straight line $x + y = 2$ and a circle $x^2 + y^2 = 2$. These are tangent to one another at the point $(1, 1)$. The line, therefore, is always above and to the right of the circle. Hence, if we are above and to the right of the line, i.e. if the first inequality holds, then we must be outside the circle, i.e. the second inequality must also hold.

1578. By solving a suitable quadratic equation, factorise the expression $39867x^2 - 93574xy - 57893y^2$.

Use the factor theorem.

Equating this expression to zero, we solve to find $x = \frac{277}{97}$ and $x = -\frac{209}{411}$. Converting these using the factor theorem, we can factorise the expression as

$$(97x - 277)(411x + 209).$$

A quick check that $97 \times 411 = 39867$ suffices to show that no constant factors are needed.

1579. Identical projectiles are dropped from the same point, with a 1 second delay. Prove that, in the subsequent motion, the distance between them grows, theoretically, without bound.

Find the position at time t of each, taking $t = 0$ to be the time at which the second projectile is dropped. The first projectile, then, will have fallen for $t + 1$ seconds at time t.

Taking $t = 0$ to be the time at which the second projectile is dropped, the first projectile has fallen for $t + 1$ seconds at time t. Assuming projectile motion, the difference in vertical position, then, is

$$\tfrac{1}{2}g(t + 1)^2 - \tfrac{1}{2}gt^2 = \tfrac{1}{2}g(t + 1).$$

This is a linear function of t; it grows, therefore, according to the model, without bound.

1580. Determine whether the function $x \mapsto \ln(1+x+x^2)$ is well defined over the domain $\mathbb{R}$.

Logarithms are undefined for negative inputs.

Logarithms are undefined for negative inputs. But $1+x+x^2$ is a positive quadratic with discriminant $-3 < 0$. Hence, its value is always positive. The overall function, therefore, is well defined over $\mathbb{R}$.

1581. Three vectors are given as

$$\mathbf{p} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

(a) Write down the angle between vectors $\mathbf{p}$ and $\mathbf{q}$.

(b) The magnitudes of $\mathbf{p}$ and $\mathbf{r}$ are related as $|\mathbf{p}| = k|\mathbf{r}|$. Determine the value of k.

In (a), the vectors share no components. In (b), use Pythagoras.

(a) Since $\mathbf{p}$ and $\mathbf{q}$ share no components, they are perpendicular. Hence, the angle between them is a right angle.

(b) Using Pythagoras's theorem, this equation is $12 = k\sqrt{1^2+2^2+2^2}$, giving $k = 4$.

1582. A six-sided die and a twelve-sided die are rolled together. Find the probability that the score on the six-sided die is larger.

The possibility space is getting a bit big to draw it explicitly. Visualise it, and count outcomes.

The possibility space has $6 \times 12 = 72$ outcomes in it. Of these, there are $1+2+3+4+5 = 15$ successful. So, the probability is $p = \frac{15}{72} = \frac{5}{24}$.

1583. Show that, if $y = \tan px$ for some constant p, then

$$\frac{dy}{dx} = p + py^2.$$

Differentiate by the chain rule, then use a Pythagorean trig identity.

The chain rule gives $\frac{dy}{dx} = p\sec^2 px$. Then, using the Pythagorean trig identity $1+\tan^2 x = \sec^2 x$, we get $\frac{dy}{dx} = p + p\tan^2 px$. Substituting $y = \tan^2 px$ yields the required result.

1584. A pupil is trying to prove that $\sqrt{2}$ is irrational. He begins with the line "Assume, for a contradiction, that $\sqrt{2}$ cannot be written in the form $\frac{p}{q}$, where p and q are integers with $\text{hcf}(p,q) = 1$." Explain the error that has been made, and give a corrected opening line.

A proof by contradiction begins with the *opposite* of what one is trying to prove.

The pupil has begun with a statement of what he is trying to prove. That's not right. We start with the negation of the result to be proved, and try to reach something impossible. For example: if I was a parrot, then I would be able to fly; I can't fly, so, I must not be a parrot. Corrected, the statement is "Assume, for a contradiction, that $\sqrt{2}$ **can** be written in the form $\frac{p}{q}$, where p and q are integers with $\text{hcf}(p,q) = 1$."

1585. A function f with the following property is sought:

$$\int_0^x f(t)\,dt \equiv \frac{2}{f(x)}.$$

(a) A function $f(x) = x^k$ is proposed. Show that

$$\tfrac{1}{k+1}x^{k+1} \equiv 2x^{-k}.$$

(b) Hence, determine k.

In (a), substitute the proposed solution. In (b), spot the value of k which makes the two sides identical. Or substitute $x = 0$.

(a) Substituting the proposed function,

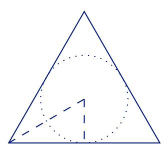
$$\begin{aligned} &\int_0^x t^k\,dt \equiv \frac{2}{x^k} \\ \Longrightarrow\ &\left[\tfrac{1}{k+1}t^{k+1}\right]_{t=0}^{t=x} \equiv 2x^{-k} \\ \Longrightarrow\ &\tfrac{1}{k+1}x^{k+1} \equiv 2x^{-k}. \end{aligned}$$

(b) Since this is an identity, the two sides must be equivalent for all values of x. Substituting $x = 0$ gives $\frac{1}{k+1} = 2$, hence $k = -\frac{1}{2}$. This can easily be verified to satisfy the identity for all values of x.

1586. Find the area of the largest equilateral triangle which can rotate freely while remaining inside an equilateral triangle of unit area.

Find the length scale factor between the triangles.

The length scale factor between the two triangles is that between the centre-and-vertex and centre-and-midpoint of an equilateral triangle:



The length scale factor is $\sin 30° = \frac{1}{2}$, so the area scale factor is $\frac{1}{4}$. This, therefore, is the area.

1587. A function f is defined over the reals, and has range $[1, \infty)$. Give the ranges of the following:

(a) $x \mapsto f(x) + 1$,

(b) $x \mapsto 2f(x) + 1$,

(c) $x \mapsto af(x) + b$, where $a > 0$,

(d) $x \mapsto af(x) + b$, where $a < 0$.

In (a),(b),(c), simply transform the lower bound 1. Only in (d) does the form of the range change.

In the first three, the range transforms linearly and positively, so we need only transform the lower bound. In (d), the range is also reversed.

(a) $[2, \infty)$

(b) $[5, \infty)$

(c) $[a + b, \infty)$

(d) $(-\infty, a + b]$

1588. Two mechanics students speak as follows:

"If two particles, moving in 1D, are connected by an inextensible string, then, for application of $F = ma$, they can be treated as a single object."

"That is only true if the connector is rigid."

State, with a reason, who is correct.

Consider the fact that a string can go slack.

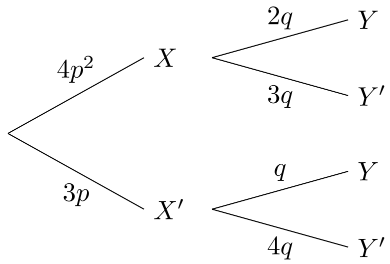
In general, two particles can only be considered as a single system in $F = ma$ if they have the same acceleration. In this question, this is not true if the string goes slack. So the first student is right only some of the time. But this cannot happen if the connector is rigid, so the second student is fully correct.

1589. Find the equation of the normal to the parabola $x = y^2 + 2y - 6$ at $y = -2$.

Differentiate to find $\frac{dx}{dy}$.

Firstly, we calculate the relevant point to be $(-6, -2)$. Now, differentiating with respect to y, we get $\frac{dx}{dy} = 2y + 2$. Evaluating at $y = -2$, we have $M_{\text{tangent}} = -2$, noting that M is a rate of change of x with respect to y. This gives $M_{\text{normal}} = \frac{1}{2}$. So, $x = My + C$ is $x = \frac{1}{2}y + C$. Substituting $(-6, -2)$ gives $x = \frac{1}{2}y - 5$ or $y = 2x + 10$.

1590. Events X and Y have probabilities as represented on the following tree diagram, for constants p, q.



(a) Find p and q.

(b) Determine $P(X \mid Y)$.

In (a), use the fact that probabilities are positive and sum to 1. In (b), restrict the possibility space.

(a) Since probabilities are positive and sum to 1, we have $4p^2 + 3p = 1$, giving $p = \frac{1}{4}$, and $q = \frac{1}{5}$.

(b) Restricting the possibility space to the first and third branches, we have
$$P(X \mid Y) = \frac{4p^2 \times 2q}{4p^2 \times 2q + 3p \times q} = \frac{2}{5}.$$

1591. Find constants A, B to make this an identity:
$$\frac{1}{x^4 - x^2} \equiv \frac{A}{x^2} + \frac{B}{x^2 - 1}.$$

Multiply both sides by $x^2 - x^2$, and then equate coefficients.

Multiplying up, we need $1 \equiv A(x^2 - 1) + Bx^2$. Equating the coefficients of x^2 gives $A + B = 1$. Equating the constant terms gives $1 = -A$. Hence, $A = -1$ and $B = 2$.

1592. Show that $y = x^2$ and $y = 8 - (x - 4)^2$ are tangent to each other.

Use the discriminant, or factorise, to show that there is a double root at the intersection.

For intersections,
$$\begin{aligned} & x^2 = 8 - (x - 4)^2 \\ \Longrightarrow\ & 2x^2 - 8x + 8 = 0 \\ \Longrightarrow\ & (x - 2)^2 = 0. \end{aligned}$$

Since $x = 2$ is a double root, the curves cannot cross at $x = 2$; they must just touch. Hence, the two parabolae are tangent at that point.

1593. By squaring the equations and subtracting, find all possible values of R satisfying both $R \sec\theta = 13$ and $R \tan\theta = 5$.

Use the second Pythagorean trig identity.

Squaring and subtracting, we get
$$R^2(\sec^2\theta - \tan^2\theta) = 13^2 - 5^2 = 144.$$

Using the second Pythagorean trig identity, the trigonometric factor is 1, so $R = \pm 12$.

1594. The equations $f(x) = 0$ and $g(x) = 0$, where f and g are quadratic functions, have the same solution set S. The equation $f(x) = g(x)$ is denoted E. State, with a reason, whether these claims hold:

(a) "E has solution set S",

(b) "the solution set of E contains S",

(c) "the solution set of E is a subset of S".

Consider the fact that f and g may be the same quadratic function.

If f and g are distinct, then E has solution set S. However, if f and g are identical, then E has solution set $\mathbb{R}$.

(a) False.

(b) True.

(c) False.

1595. An AP starts $\sqrt{q} - 2$, $2\sqrt{q} + 1$, q. Find q.

Equate the differences and solve a quadratic.

Equating the differences,

$$\begin{aligned} & 2\sqrt{q} + 1 - (\sqrt{q} - 2) = q - (2\sqrt{q} + 1) \\ \Longrightarrow\ & q - 3\sqrt{q} - 4 = 0 \\ \Longrightarrow\ & (\sqrt{q} - 4)(\sqrt{q} + 1) = 0 \\ \Longrightarrow\ & \sqrt{q} = 4, -1. \end{aligned}$$

There is no q with $\sqrt{q} = -1$, so $q = 16$.

1596. A student writes: "The driving force on a car is really a frictional force. We call it a driving force only by convention." State, with a reason, whether this is true.

Visualise the car floating in space.

This is correct. The argument can be pictured (surprisingly enough) with the car floating in space. The engine still whirrs round, but there is nothing to push the wheels forward. It is the frictional force of the road on the wheels (driving force) that does that.

1597. Using calculus, prove that, for a fixed perimeter P, the rectangle of greatest area is a square.

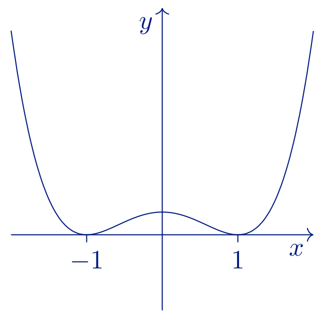
Express A in terms of the length x (and the constant P). Then differentiate and set the derivative to zero to optimise the area.

The area is given by $A = x(\frac{1}{2}P - x)$, where x is one of the lengths. Differentiating with respect to x, we get $\frac{dA}{dx} = \frac{1}{2}P - 2x$. Setting this to zero, the area is optimised at $x = \frac{1}{4}P$, which corresponds to a square as required.

1598. Sketch $\sqrt{y} = x^2 - 1$.

Square both sides to get a quartic.

Squaring and factorising, $y = (x - 1)^2(x + 1)^2$. This is a positive quartic with double roots at ± 1. Hence, the curve is



1599. Explain why the following statement is not true: "The standard deviation of a combined sample cannot be bigger than the standard deviations of the individual samples."

Consider a sample from a bimodal distribution.

A counterexample is the two samples $\{0, 0, 0, 0\}$ and $\{1, 1, 1, 1\}$. Standard deviation is zero for each sample, but non-zero for their combination.

1600. Show that $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$.

Put the fraction in its lowest terms before you take the limit.

We cannot take the limit as it stands, because the numerator tends to zero. But, factorising the top, we can cancel a factor of $(x - 1)$, giving

$$\lim_{x \to 1} (x + 1).$$

At this point, we can take the limit, which is 2.

1601. A functional instruction is defined as

$$f : x \mapsto \frac{\sqrt{x}}{1 + \sqrt{x}}.$$

(a) Write down the largest real domain over which f may be defined.

(b) Show that, where f^{-1} is defined, its rule is

$$f^{-1} : x \longmapsto 1 + \frac{2x - 1}{(x - 1)^2}.$$

(c) Show that $y = 1$ is not in the range of f.

In (b), group the $\sqrt{x}$ terms and take it out as a factor. In (c), showing that $y = 1$ is not in the range of f is the same as showing that $x = 1$ is not in the domain of f^{-1}.

(a) The domain is $[0, \infty)$.

(b) Switching inputs and outputs, we rearrange to make y the subject:

$$\begin{aligned} & x = \frac{\sqrt{y}}{1+\sqrt{y}} \\ \Longrightarrow\ & x + x\sqrt{y} = \sqrt{y} \\ \Longrightarrow\ & x = \sqrt{y}(1-x) \\ \Longrightarrow\ & \frac{x}{1-x} = \sqrt{y} \\ \Longrightarrow\ & \frac{x^2}{(1-x)^2} = y. \end{aligned}$$

We can then express the numerator x^2 as $(x-1)^2 + 2x - 1$. Splitting the fraction up, we have the required result

$$x \longmapsto 1 + \frac{2x-1}{(1-x)^2}.$$

(c) If $y = 1$ were in the range of f, it would be in the domain of f^{-1}. But f^{-1} involves division by $(x-1)$, so is undefined at $x = 1$. Hence, $y = 1$ is not in the range of f.

1602. Prove that no quadratic function can have three distinct fixed points.

Consider the type of equation $f(x) = x$ is.

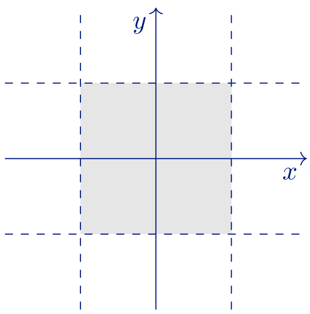
To find the fixed points of a quadratic function f, we solve $f(x) = x$. This is a quadratic equation. Hence, it has a maximum of two roots. This proves that no quadratic function can have three distinct fixed points.

1603. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$x^2 < 4, \qquad y^2 < 4.$$

The region is a square.

Each region is a corridor between two parallel lines $x = \pm 2$ and $y = \pm 2$. So, we have



1604. On average, four in five people at a retirement home, which has population 200, are vaccinated against flu. In a given year, 2% of those who have been vaccinated, and 15% who haven't been, come down with flu.

(a) Represent this information on a tree diagram.

(b) In any given year, find the expected number who come down with flu among

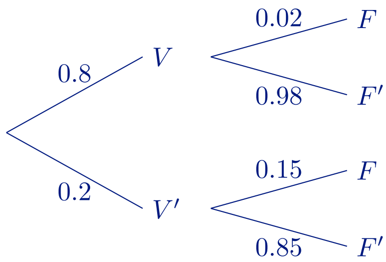
i. those not vaccinated,

ii. everyone.

(c) Show that, among those who come down with flu, the ratio of vaccinated to not vaccinated people is 8 : 15.

Part (c) is a condition probability. On the tree diagram, restrict the possibility space to those branches representing "coming down with flu".

(a) The tree diagram is



(b) The expectations are np in each case:

i. $40 \times P(F \mid V') = 40 \times 0.15 = 6$ people,

ii. $6 + 160 \times P(F \mid V) = 9.2$ people.

(c) Restricting the possibility space to F,

$$P(V \mid F) = \frac{0.8 \times 0.02}{0.2 \times 0.15 + 0.8 \times 0.02} = \frac{8}{23}.$$

So, the ratio is 8 : 15.

Alternatively, this could have been calculated directly with expectations. The ratio 3.2 : 6 is the same as the ratio 8 : 15.

1605. Find the following limits, defined in radians.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin^3 x}{x^2}$,

(b) $\displaystyle\lim_{x\to 0} \frac{\sin^3 x}{x^3}$,

(c) $\displaystyle\lim_{x\to 0} \frac{\sin^3 x}{x^4}$.

Use the small-angle approximation $\sin x \approx x$, for small x in radians.

Using the small-angle approximation $\sin x \approx x$, which becomes exact in the limit as $x \to 0$,

(a) $\displaystyle\lim_{x\to 0} \frac{x^3}{x^2} = \lim_{x\to 0} x = 0.$

(b) $\lim_{x \to 0} \frac{x^3}{x^3} = \lim_{x \to 0} 1 = 1.$

(c) $\lim_{x \to 0} \frac{x^3}{x^4} = \lim_{x \to 0} \frac{1}{x} = \infty.$

1606. Find, in simplified terms of x, the mean of

$$\left(\sqrt{x} + \sqrt{x-1}\right)^2 \text{ and } \left(\sqrt{x} - \sqrt{x-1}\right)^2.$$

Just multiply out and simplify the sum.

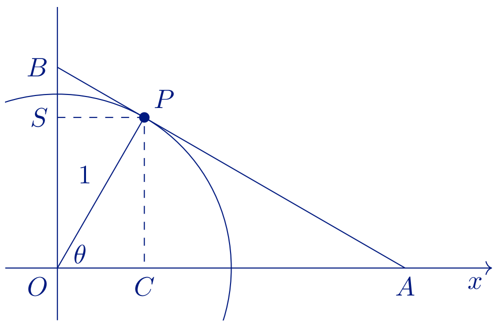
The mean is

$$\begin{aligned} &\tfrac{1}{2}\left[\left(\sqrt{x} + \sqrt{x-1}\right)^2 + \left(\sqrt{x} - \sqrt{x-1}\right)^2\right] \\ \equiv\ &\tfrac{1}{2}\left[x + 2\sqrt{\text{etc.}} + x - 1 + x - 2\sqrt{\text{etc.}} + x - 1\right] \\ \equiv\ &\tfrac{1}{2}(4x-2) \\ \equiv\ &2x - 1. \end{aligned}$$

1607. A unit circle is drawn around the origin O, and a point P is drawn on its circumference at $(\cos\theta, \sin\theta)$, where $\theta \in (0, 90°)$. A tangent is drawn to the circle at point P, which crosses the x and y axes at A and B. Sketch the diagram, and show that the six trigonometric functions sin, cos, tan, cosec, sec, and cot are all represented as lengths on it.

The point $(\cos\theta, \sin\theta)$ gives the first two. Then the length AB is split into $\tan\theta$ and $\cot\theta$, and the lengths OA and OB are the other two.

The sine and cosine functions are represented by definition with the dashed lines and equivalently by lengths OS and OC.



Then the remaining lengths can be found using the fact that every one of the (many!) triangles on the diagram is similar.

$$\begin{aligned} OS &= \sin\theta \\ OC &= \cos\theta \\ BP &= \tfrac{\sin\theta}{\cos\theta} = \tan\theta \\ AP &= \tfrac{\cos\theta}{\sin\theta} = \cot\theta \\ OA &= \tfrac{1}{\cos\theta} = \sec\theta \\ OB &= \tfrac{1}{\sin\theta} = \operatorname{cosec}\theta \end{aligned}$$

1608. Solve $\sum_{j=1}^{3} x^j = 0$.

Write the sum out longhand.

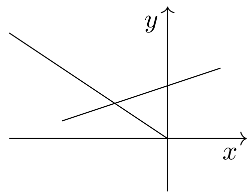
Longhand, the sum is $x^3 + x^2 + x = 0$, which has a common factor of x. The remaining quadratic is $x^2 + x + 1 = 0$, which has discriminant $-3 < 0$. Hence, the only root is $x = 0$.

1609. A particular binomial distribution $B(n, p)$ is best approximated by the normal distribution $N(30, 22.5)$. Find n and p.

The mean and variance of a binomial distribution are $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = npq$.

The mean and variance of a binomial distribution are $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = npq$. So, we have simultaneous equations $np = 30$ and $npq = 22.5$. Dividing gives $q = \frac{3}{4}$, so $p = \frac{1}{4}$ and $n = 120$.

1610. On the graph below, two line segments defined parametrically by $\mathbf{r}_1 = (3s-3)\mathbf{i} + (2-2s)\mathbf{j}$, for $s \in [0,1]$ and $\mathbf{r}_2 = t\mathbf{i} + (1 + \frac{1}{3}t)\mathbf{j}$, for $t \in [-2, 1]$ are depicted, to scale.



Verify that the line segments trisect each other.

Since the equations are linear, the values of the parameters at the intersection must divide their respective domains in the ratio $1 : 2$.

The equations are linear, so the parameters must trisect their respective domains. Considering the endpoints on the diagram, we need $s = \frac{2}{3}$ and $t = -1$. Substituting, we get $\mathbf{r}_1 = \mathbf{r}_2 = -\mathbf{i} + \frac{2}{3}\mathbf{j}$, verifying the result.

1611. A student writes: "If a rope is extensible, then the tensions exerted on its two ends should not be modelled as equal." State, with a reason, whether this is correct or not.

It is not correct.

This is not correct. Inextensibility is required to assume that the *accelerations* of both ends are the same. For the *tensions* to be the same, it is only required that 1 the rope be light and 2 that no friction acts on it, i.e. that any pulleys etc. are smooth. This can be true even if the rope is extensible.

1612. An isosceles triangle has two sides of length 5, and area 12. Show that there are two possible values for the third length, both of which are integers.

Use the sine area formula, then the Pythagorean trig identity, then the cosine rule.

The area formula gives $12 = \frac{1}{2}5^2 \sin\theta$, where θ lies between the isosceles lengths. Hence, $\sin\theta = \frac{24}{25}$. By the Pythagorean trig identity, this gives two possible values for $\cos\theta$, namely $\pm\frac{7}{25}$. The cosine rule then produces

$$c^2 = 5^2 + 5^2 \mp 2 \cdot 5^2 \tfrac{7}{25} = 36 \text{ or } 64.$$

So, $c = 6$ or $c = 8$ as required.

1613. Express 3^{2x+4} in terms of 9^x.

Write 3 as $9^{\frac{1}{2}}$, and use index laws.

Writing 3 as $9^{\frac{1}{2}}$ and using an index law, we have $3^{2x+4} = 9^{x+2}$. Then, splitting with another index law, we have $81 \cdot 9^x$.

1614. Write down the value of $\displaystyle\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$.

It's a semicircle!

This is the area below a $y \geq 0$ semicircle of radius r, centred on O. Hence, the area is $\frac{1}{2}\pi r^2$.

1615. The letters of the words HEALTH are jumbled up and rearranged at random.

(a) Explain why there are 5! arrangements where the pairing EA appears in that order.
(b) Hence, find the number of arrangements where the vowels are next to each other.

Think of EA as a single "letter".

(a) Grouping EA as a single "letter", we want the number of rearrangements of five entities: H, EA, L, T, H. There are 5! of these.
(b) This doubles the previous answer, for EA and AE, giving 240.

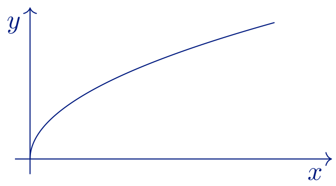
1616. Sketch the following graphs, for $k \in \mathbb{N}$:

(a) $y = x^{\frac{1}{2k}}$,
(b) $y = x^{\frac{1}{2k+1}}$.

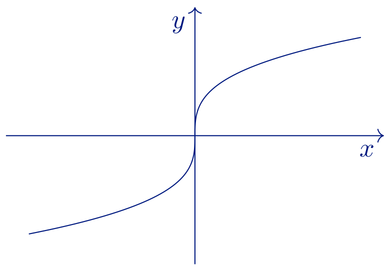
These can be modelled on the prototype even and odd cases $y = \sqrt{x}$ and $y = \sqrt[3]{x}$, which are reflections of (parts of) $y = x^2$ and $y = x^3$ in the line $y = x$.

These nth-root curves look different, depending on the parity (evenness or oddness) of the power. Each is a reflection of the relevant $y = x^n$ curve in $y = x$, but the even powers are restricted to positive inputs.

(a) Curves of the type $y = \sqrt{x}$:



(b) Curves of the type $y = \sqrt[3]{x}$:



1617. Two unit circles are drawn, with equations $x^2 + y^2 = 1$ and $x^2 + (y-4)^2 = 1$. A third unit circle is then drawn so that all three circles are equidistant. Find the two possible equations of the third circle.

Since the circles all have the same unit radius, their being equidistant is equivalent to their centres being equidistant.

Since the circles all have the same unit radius, their being equidistant is equivalent to their centres being equidistant. The first two centres are at $(0, 0)$ and $(0, 4)$. The third centre, therefore, must be at $(x, 2)$, a distance of 4 away from the origin. Hence, $x^2 + 2^2 = 16$, giving $x = \pm\sqrt{12}$. So, the possible equations are $(x \mp \sqrt{12})^2 + (y - 2)^2 = 1$.

1618. It is given that the expression x^2y is constant. Find $\frac{dy}{dx}$ in simplified terms of x and y.

Translate into algebra, then differentiate using the product rule.

In algebra, we have $x^2y = k$. Differentiating by the product rule gives $2xy + x^2\frac{dy}{dx} = 0$, which we can rearrange to $\frac{dy}{dx} = -\frac{2y}{x}$, for $x \neq 0$.

1619. Take g=10 ms^{-2} in this question.

A particle is projected vertically upwards from ground level at speed 20 ms^{-1}. Two seconds later, another particle is projected in the same manner. Find the height at which the two particles collide.

Use t and $t+2$ as the time variables.

Taking $t=0$ at projection of the second particle, the times since projection are given by $t+2$ and t. Equating the heights, then, we get

$$\begin{aligned} & 20(t+2)-5(t+2)^2 = 20t-5t^2 \\ \Longrightarrow\ & 40-20t-20=0 \\ \Longrightarrow\ & t=1 \text{ seconds.} \end{aligned}$$

So, the height is $h = 20\cdot 1 - 5\cdot 1^2 = 15$ m.

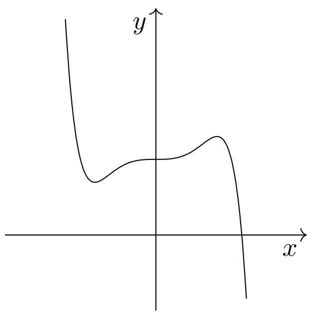
1620. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning real numbers x and y:

- $y = \log_2 x$,
- $x = 2^y$.

Logarithms are the inverses of exponentials.

The implication is $\Longleftrightarrow$: these are two ways of writing exactly the same statement, in logarithm or index language.

1621. The graph below is of the function $f(x) = x^3 - x^7 + 1$, which has a root close to $x = 1.1$.



(a) Calculate $f'(0.8)$.

(b) With reference to this value, explain why the Newton-Raphson iteration, with $x_0 = 0.8$, takes a long time to converge to the root of $x^3 - x^7 + 1 = 0$.

In (b), calculate and consider the value of x_1, with reference to the tangent at $x_0 = 0.8$

(a) Differentiating, we have $f'(x) = 3x^2 - 7x^6$. Hence, $f'(0.8) = 0.084992$.

(b) This is a very shallow positive gradient. Hence, when the tangent is drawn, it will give a large, negative value for x_1. In fact, $x_1 =\approx -15$. The iteration will then take a long time to work its way back from this point.

1622. By rewriting 2 as e^k, find the factor by which areas are scaled when the graph $y = e^x$ is transformed to $y = 2^x$.

Since the transformation is a one-dimensional stretch, the area scale factor is the same as the length scale factor.

We can write $y = 2^x = (e^{\ln 2})^x = e^{x\ln 2}$. Hence, in transforming $y = e^x \longmapsto y = 2^x$, the x input has been replaced by $x \ln 2$. The transformation is a stretch in the x direction, scale factor $\frac{1}{\ln 2}$. Since this is only a one-dimensional stretch, areas are scaled by the same factor.

1623. An equilateral triangle is being enlarged. Its area is $\sqrt{48}$ cm^2, and is increasing at a rate of $\sqrt{3}$ cm^2/s.

(a) Write the perimeter in terms of the area.

(b) Hence, show that, at this instant, the perimeter is increasing at $\frac{3}{2}$ cm/s.

In (b), differentiate part (a) with respect to t.

(a) In terms of length l, the area is $A = \frac{\sqrt{3}}{4}l^2$. Hence, making l the subject and multiplying by 3, the perimeter is given by

$$P = 2\cdot 3^{\frac{3}{4}} \cdot \sqrt{A}.$$

(b) To find the rate, we differentiate with respect to t. We need implicit differentiation, i.e. the chain rule on the RHS:

$$\begin{aligned} \frac{dP}{dt} &= 2\cdot 3^{\frac{3}{4}} \cdot \tfrac{1}{2}A^{-\frac{1}{2}} \cdot \frac{dA}{dt} \\ &= 3^{\frac{3}{4}} A^{-\frac{1}{2}} \frac{dA}{dt}. \end{aligned}$$

Substituting values for A and $\frac{dA}{dt}$ gives

$$\frac{dP}{dt} = 3^{\frac{3}{4}} \sqrt{48}^{-\frac{1}{2}} \sqrt{3} = \frac{3}{2}, \text{ as required.}$$

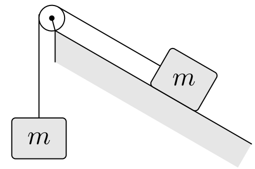
1624. State, with a reason, whether the following gives a well-defined function:

$$g : \begin{cases} [-1,1] \mapsto \mathbb{R} \\ x \mapsto \dfrac{1}{\sqrt{1-x^2}}. \end{cases}$$

Substitute the boundary values.

It is not well defined. For the boundary values $x = \pm 1$, the square root is well defined as $\sqrt{1-1}$, but this then gives division by zero. The domain should be reduced to $(-1, 1)$.

1625. Two masses are connected by a light, inextensible string, which is passed over a smooth, light, fixed pulley as shown in the diagram. One mass sits on a rough slope of inclination 30°. The system is in equilibrium.



(a) Find the set of possible values of μ.

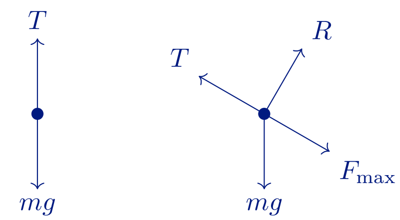
(b) State, with a reason, whether your answer would have been different had the string been

i. extensible,

ii. heavy.

Draw force diagrams for the two objects, setting $F_{\max} = \mu R$ for the limiting case. For the direction of friction, consider what would happen if the slope were smooth.

(a) Force diagrams, with no acceleration, are as follows. This is limiting equilibrium, with $F_{\max} = \mu R$ acting down the slope.



Resolving for equilibrium perpendicular to the slope, $R = \frac{\sqrt{3}}{2}mg$, so $F_{\max} = \mu\frac{\sqrt{3}}{2}mg$. We can now resolve along the string for the system:

$$\begin{aligned} mg - \mu\tfrac{\sqrt{3}}{2}mg &= 0 \\ \implies \mu &= \tfrac{2}{\sqrt{3}}. \end{aligned}$$

So, we require $\mu \in \left[\frac{2}{\sqrt{3}}, \infty\right)$.

(b) i. The answer would be unchanged. With the system in equilibrium, an extensible string acts exactly like an inextensible one.

ii. The answer would almost certainly change. Depending on the density of the string and its length above and below the pulley, the minimal μ could increase or decrease.

1626. The interior angles of an irregular hexagon are in geometric progression, and the largest is $\frac{128\pi}{63}$ rad.

(a) Considering the largest angle as the first term a, show that the common ratio r satisfies

$$128r^6 - 252r + 124 = 0.$$

(b) Using Newton-Raphson, solve this equation.

(c) Hence, show that the smallest angle is $\frac{4}{63}\pi$ rad.

In (a), find the sum of the interior angles of a hexagon, in radians, and set it equal to the sum of a geometric series.

(a) The sum of the interior angles is $(n-2)\pi$, so, for a hexagon, 4π. Equating this with the sum of our GP, in which $a = \frac{128}{63}\pi$,

$$\begin{aligned} \frac{\frac{128}{63}\pi(1-r^6)}{1-r} &= 4\pi \\ \implies 128 - 128r^6 &= 252 - 252r \\ \implies 128r^6 - 252r + 124 &= 0. \end{aligned}$$

(b) The N-R iteration is

$$x_{n+1} = x_n - \frac{128x_n^6 - 252x_n + 124}{768x^5 - 252}.$$

Since we know that the common ratio must be less than 1, we start with, let's say, $x_0 = 0.75$. This tends to $r = \frac{1}{2}$.

(c) Using this common ratio, the smallest angle is

$$r_6 = \tfrac{128}{63}\pi \times \tfrac{1}{2}^5 = \tfrac{4}{63}\pi.$$

1627. If $y = \tan^2 3x$, find $\left.\dfrac{dy}{dx}\right|_{x=\frac{1}{18}\pi}$ exactly.

Differentiate with the chain rule. Then evaluate using $\tan\frac{1}{6}\pi = \frac{\sqrt{3}}{3}$ and $\cos\frac{1}{6}\pi = \frac{\sqrt{3}}{2}$.

The chain rule gives

$$\begin{aligned} \frac{dy}{dx} &= 2\tan 3x \cdot \sec^2 3x \cdot 3 \\ &= 6\tan 3x \sec^2 3x. \end{aligned}$$

With an input of $x = \frac{1}{18}\pi$, we have $3x = \frac{1}{6}\pi$. The exact values we need, therefore, are $\tan\frac{1}{6}\pi = \frac{\sqrt{3}}{3}$ and $\cos\frac{1}{6}\pi = \frac{\sqrt{3}}{2}$. Reciprocating the latter gives $\sec\frac{1}{6}\pi = \frac{2}{\sqrt{3}}$. This produces

$$\frac{dy}{dx} = 6 \cdot \frac{\sqrt{3}}{3} \cdot \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{8\sqrt{3}}{3}.$$

1628. The IQ scores of a large population are modelled with a normal distribution $X \sim N(100, 15^2)$. A random sample of size 40 is taken.

(a) Write down the distribution of $\bar{X}$, the mean score of the sample.

(b) Find $P(|\bar{X} - 100| > 5)$.

In (b), calculate the negation, i.e. the the probability that the departure of the mean from 100 is less than 5.

(a) The standard result is $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

(b) Calculating with the above distribution,

$$\begin{aligned} & P(|\bar{X} - 100| > 5) \\ &= 1 - P(95 \le \bar{X} \le 105) \\ &= 1 - 0.9649... \\ &= 0.0350 \text{ (3sf)} \end{aligned}$$

1629. Determine the number of roots of the equation

$$(x^2 + 4x + 4)(x^4 + 4x^2 + 4) = 0.$$

Factorise fully.

We factorise fully, recognising the right-hand factor as a quadratic in x^2,

$$\begin{aligned} & (x^2 + 4x + 4)(x^4 + 4x^2 + 4) = 0 \\ \Longrightarrow\ & (x + 2)^2(x^2 + 2)^2 = 0. \end{aligned}$$

The double factor $(x+2)^2$ gives one repeated root at $x = -2$. But $(x^2 + 2)^2$ gives no roots, as $x^2 + 2 = 0$ has none. So there is one in total.

1630. At takeoff, a fighter jet accelerates at $(\mathbf{i}+2\mathbf{j}+\mathbf{k})g$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors with $\mathbf{k}$ vertical. Find the magnitude of the contact force on a pilot of mass m.

In this system of coordinates, the weight of the pilot is $-mg\mathbf{k}$.

There are two forces on the pilot: the reaction force $\mathbf{R}$ and the weight $-mg\mathbf{k}$. Since the pilot is accelerating with the plane, $\mathbf{a} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k})g$. So, $\mathbf{F} = m\mathbf{a}$ is

$$\begin{aligned} & \mathbf{R} - mg\mathbf{k} = m(\mathbf{i} + 2\mathbf{j} + \mathbf{k})g \\ \Longrightarrow\ & \mathbf{R} = \big(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\big)mg \\ \Longrightarrow\ & |\mathbf{R}| = \sqrt{1^2 + 2^2 + 2^2}\, mg \\ &= 3mg. \end{aligned}$$

1631. Find the range of $x \mapsto (e^x - 1)^3$.

Find the range of $e^x - 1$ first.

Assuming the domain $\mathbb{R}$, the range of e^x is $(0, \infty)$, so the range of $e^x - 1$ is $(-1, \infty)$. Since the cubing function is one-to-one and since $(-1)^3 = -1$, the range of $(e^x - 1)^3$ is also $(-1, \infty)$.

1632. Show that, if $4x^5 - 7x^3 + 2x + 1$ is expressed as $(2x - 1)f(x)$, then $f(x)$ is not polynomial.

This is another way of saying "Show that $(2x - 1)$ isn't a factor of $4x^5 - 7x^3 + 2x + 1$."

For $f(x)$ to be polynomial, $(2x - 1)$ would need to be a factor of $4x^5 - 7x^3 + 2x + 1$. To show that this is not the case, we can use the factor theorem:

$$4x^5 - 7x^3 + 2x + 1\,\big|_{x=\frac{1}{2}} = \tfrac{5}{4} \neq 0.$$

Hence, $f(x)$ cannot be polynomial.

1633. The graph $|x| + |y| = 1$ is translated by the vector $a\mathbf{i}+b\mathbf{j}$. Write down the equation of the new graph.

Variables x and y appear symmetrically as inputs in this equation, so each component of translation should be thought of as a replacement.

Translation by vector $a\mathbf{i}$ corresponds to replacing x by $x - a$, and translation by $b\mathbf{j}$ to replacing y by $y-b$. Hence, the new equation is $|x-a|+|y-b| = 1$.

1634. Velocity v takes the following values at times t:

t	0	2	4	6
v	0	1.2	4.8	v_6

(a) Show that the data above is inconsistent with an assumption of constant acceleration.

(b) Show that the data is consistent with $a \propto t$, and hence find v_6.

In (b), use calculus.

(a) Change in velocity during the first two seconds is 1.2, and during the next two seconds is $4.8 - 1.2 = 3.6$. This is not constant acceleration.

(b) Setting up $a = kt$, we integrate to $v = \frac{1}{2}kt^2 + c$. Substituting $t = 0, v = 0$ gives $c = 0$. Then $t = 2, v = 1.2$ gives $v = 0.3t^2$. This produces the correct value of $t = 4, v = 4.8$, so the model is consistent. Hence, we can calculate $v_6 = 0.3 \times 6^2 = 10.8$.

1635. Show that $\displaystyle\int_0^1 \frac{35(1 + x)^3}{\sqrt{x}}\, dx = 192.$

Expand binomially and split the fraction up.

Expanding binomially and integrating,

$$\begin{aligned}
&\int_0^1 \frac{35(1+x)^3}{\sqrt{x}}\,dx \\
&= 35\int_0^1 x^{-\frac{1}{2}} + 3x^{\frac{1}{2}} + 3x^{\frac{3}{2}} + x^{\frac{5}{2}}\,dx \\
&= 35\Big[2x^{\frac{1}{2}} + 2x^{\frac{3}{2}} + \tfrac{6}{5}x^{\frac{5}{2}} + \tfrac{2}{7}x^{\frac{7}{2}}\Big]_0^1 \\
&= 35\Big(2 + 2 + \tfrac{6}{5} + \tfrac{2}{7}\Big) \\
&= 192, \text{ as required.}
\end{aligned}$$

1636. Simplify $(-2,2] \cap [-3-k^2, 3+k^2)$, for $k \in \mathbb{R}$.

Consider the fact that k^2 must be positive.

Since $k^2 \geq 0$, the interval $[-3-k^2, 3+k^2)$ must include the whole set $(-2,2]$. The intersection, therefore, is simply $(-2,2]$.

1637. A binomial hypothesis test has acceptance region $\{3, 4, ..., 12, 13\}$. A student writes: "In the sample, $x = 2$, which lies in the critical region, so there is sufficient evidence to reject the null hypothesis." State, with a reason, whether this sentence is valid.

It is valid.

This is valid. Irrespective of the hypotheses, the critical region is the complement of the acceptance region. So, $x = 2$ is indeed in the critical region. And, if a sample statistic lies in the critical region, then there is sufficient evidence to reject the null hypothesis (at the given significance level).

1638. Find all pairs of values (x, y) which satisfy

$$\begin{aligned}
2\sqrt{x} + \sqrt{y} &= 15, \\
4\sqrt{y} - \sqrt{x} &= 24.
\end{aligned}$$

Solve simultaneously for $\sqrt{x}$ and $\sqrt{y}$ as usual, then deal with the square roots afterwards.

These are linear simultaneous equations in $\sqrt{x}$ and $\sqrt{y}$. Solving for these by elimination, ① + 2 × ② gives $9\sqrt{y} = 63$. So $y = 49$. Substituting back in, we get $x = 16$.

1639. For some real constant k, an equation E is given as $kx^2 + (k+1)x + (k+2) = 0$. Determine **three** values of k for which E has exactly one root.

One value comes from the equation not being a quadratic at all; the others need the discriminant.

The first value is $k = 0$, for which the equation is linear: $x = -2$ is the one root. If $k \neq 0$, then we have a quadratic, which has exactly one root if $\Delta = (k+1)^2 - 4k(k+2) = 0$. Solving this quadratic for k, we get $k = -1 \pm \frac{2\sqrt{3}}{3}$.

1640. "The line $y = x$ is normal to the curve $y = 4x - x^2$." True or false?

Solve for intersections, and test the gradient.

Solving for intersections, we need $x = 4x - x^2$, which gives $x = 0, 3$. The gradient of the curve at those points is $4 - 2x\,|_{0,3} = 4, -2$. Neither of these is normal to $y = x$, so the statement is false.

1641. For a game, a circle is to be divided up into three sectors, such that the probabilities of a needle, spun at the centre of the circle, landing on the three sectors are in the ratio $1:3:5$.

(a) Find the angles subtended by the sectors.

(b) Show that the probability of two successive spins giving different results is $\frac{46}{81}$.

The two parts are unrelated. In (b), use a tree diagram approach. (You don't need to draw the tree diagram, however.)

(a) The angles subtended are $\frac{1}{9} \times 360 = 40°$, and then, multiplying by the ratios, $120°$ and $200°$.

(b) Conditioning on the first spin,

$$\tfrac{1}{9} \times \tfrac{8}{9} + \tfrac{3}{9} \times \tfrac{6}{9} + \tfrac{5}{9} \times \tfrac{4}{9} = \tfrac{46}{81}.$$

1642. Factorise $3e^{4x} - 1 - 2e^{2x}$.

This is a quadratic in e^{2x}.

This is a quadratic is e^{2x}:

$$\begin{aligned}
&3e^{4x} - 2e^{2x} - 1 \\
&\equiv (3e^{2x} + 1)(e^{2x} - 1).
\end{aligned}$$

1643. A number theorist sets up a function to answer the following question: "If a real number x can be expressed as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ have no common factors, what are p and q?" Write down

(a) the largest possible real domain,

(b) a suitable codomain,

(c) the range, with the domain given

A function is an input-to-output process: the domain is the set of inputs, the codomain is the coverall set of outputs (their type), and the range is the specific set of outputs attainable.

(a) The inputs of the process are numbers x. Only rational numbers can be analysed, however. So the largest real domain is $\mathbb{Q}$.

(b) Outputs of the process are pairs of integers. So, a suitable codomain is $\{(p, q) : p, q \in \mathbb{Z}\}$. (Any set containing $\mathbb{Z}$ would do in place of $\mathbb{Z}$ here, e.g. $\mathbb{R}$).

(c) The range, which could be expressed variously, is $\{(p, q) : p, q \in \mathbb{Z},\ \text{where}\ \text{hcf}(p, q) = 1\}$.

1644. Prove that, if a projectile is launched from and returns to ground level, then the angle below the horizontal at which it lands is equal to the angle above the horizontal with which it was launched.

This can be done simply by symmetry, or explicitly using *suvat*.

The equation of the trajectory of a projectile is a parabola $y = f(x)$. Such a curve is symmetrical about a line $x = k$ through the vertex. Since the points at which the projectile is launched and lands are both at ground level, they are symmetrical in this line. Hence, the tangents (velocities) at those points are at the same angle to the horizontal.

1645. True or false?

(a) $x^2 + 1$ has no linear factors,

(b) $x^2 + x + 1$ has no linear factors,

(c) $x^2 + 2x + 1$ has no linear factors.

Test the discriminants.

These have linear factors if they have roots. Hence, testing the discriminant:

(a) $\Delta = -4 < 0$, true.

(b) $\Delta = -3 < 0$, true.

(c) $\Delta = 0$, false.

1646. A mathematics student cannot find an answer to the problem "Solve the simultaneous equations $4x + 6y = 1$ and $y = -\frac{2}{3}x + \frac{1}{6}$." Explain what the correct interpretation of this fact is, giving the solution set.

The equations are the same.

The two equations are rearrangements of each other, meaning that they are the same equation. Hence, the solution is infinite: any point lying on the line $4x + 6y = 1$ satisfies both equations. So, the solution set is $\{(x, y) \in \mathbb{R}^2 : 4x + 6y = 1\}$.

1647. Show that $(x-1)^2 + (y+2)^2 = 1$, $x^2 + y^2 = 2$ and $(x+2)^2 + (y+2)^2 = 10$ are concurrent.

"Concurrent" means that all three circles meet at a single point. Find the intersections of the first two circles and test them in the third.

We find the intersections of the first two circles. Multiplying out, we have $x^2 - 2x + y^2 + 4y = -4$. Subtracting $x^2 + y^2 = 2$ gives $-2x + 4y = -6$, so $x = 2y + 3$. Substituting into the second circle, $(2y + 3)^2 + y^2 = 2$, so $y = -1, -\frac{7}{5}$. This gives the intersections at $(1, -1)$ and $(\frac{1}{5}, -\frac{7}{5})$. Checking these, $(1, -1)$ lies on all three circles, which proves that they are concurrent.

1648. A function is given, for $x \in \mathbb{R}$, by

$$f : x \mapsto \ln(1 + x^2).$$

(a) Find $f(1)$ and $f'(1)$.

(b) Use these values to show that, at $x = 1$, the linear function that best approximates $f(x)$ is $g(x) = x - 1 + \ln 2$.

In (a), use the chain rule. In (b), "$g(x)$ is the linear function that best approximates $f(x)$ at $x = 1$" is equivalent to saying "$y = g(x)$ is the tangent line to $y = f(x)$ at $x = 1$".

(a) $f(1) = \ln 2$. We then differentiate by the chain rule to get $f'(x) = \frac{2x}{1+x^2}$. So, $f'(1) = 1$.

(b) The tangent line, using $y - y_0 = m(x - x_0)$, is $y - \ln 2 = 1(x - 1)$, which simplifies to $y = x - 1 + \ln 2$. Hence, $g(x) = x - 1 + \ln 2$ is the linear function which best approximates $f(x)$ at $x = 1$.

1649. Find the two values of $\displaystyle\lim_{x\to\pm\infty} \frac{e^x}{e^x + e^{-x}}$.

For the positive limit, divide top and bottom by e^x.

For the limit $x \to +\infty$, we divide top and bottom by e^x, giving

$$\lim_{x\to\infty} \frac{1}{1 + e^{-2x}}.$$

Since e^{-2x} tends to zero, the limit is 1.

For the other limit, we multiply top and bottom by e^x, giving

$$\lim_{x\to-\infty} \frac{e^{2x}}{e^{2x} + 1}.$$

The numerator tends to 0 and the denominator to 1, so the limit is 0.

1650. Simplify the expressions $a^{\log_a xy}$ and $a^{\log_a x+\log_a y}$, and hence prove the law of logarithms

$$\log_a xy = \log_a x + \log_a y.$$

Equate the simplified expressions.

By definition, $a^{\log_a xy} = xy$. Then, using an index law, $a^{\log_a x+\log_a y} = a^{\log_a x}a^{\log_a y} = xy$. Hence, the expressions are equivalent, giving

$$\begin{aligned} a^{\log_a xy} &= a^{\log_a x+\log_a y} \\ \implies \log_a xy &= \log_a x + \log_a y. \end{aligned}$$

This proves the result.

1651. The shortest path between two smooth curves is always normal to both. This question concerns the curves $y = 4 - x - x^2$ and $y = x^2 - 17x + 74$.

(a) Show that $3y = x + 5$ is normal to both.

(b) Hence, show that the length of the shortest path between the curves is $\sqrt{40}$.

In (a), solve simultaneously for intersections, and test the gradients. In (b), use Pythagoras.

(a) Solving for intersections, we get $(\frac{7}{3}, \frac{8}{9})$, $(1, 2)$ for the first curve, and $(7, 4)$, $(\frac{31}{3}, \frac{46}{9})$ for the second. Looking for the negative reciprocal of $\frac{1}{3}$, we differentiate and evaluate:

$$-1 - 2x\,|_{x=1} = -3 \quad\text{and}\quad 2x - 17\,|_{x=7} = -3.$$

Hence, the line is normal to the curves at $(1, 2)$ and $(7, 4)$.

(b) Therefore, the shortest distance between the curves is $d = \sqrt{6^2 + 2^2} = \sqrt{40}$.

1652. A sample $\{x_i\}$ of size n has mean $\bar{x}$ and variance s^2. In terms of these quantities, find the mean of

(a) $\{2x_i + 3\}$,

(b) $\{x_i^2\}$.

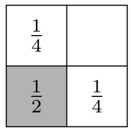
In (a), the mean transforms linearly. In (b), express the mean of x_i^2 algebraically using sigma notation. Rearrange the usual variance formula to make this expression the subject.

(a) The means transforms linearly: $2\bar{x} + 3$.

(b) The mean of x_i^2 is given by $\frac{1}{n}\sum x_i^2$. We can make this the subject of the variance formula:

$$\begin{aligned} s^2 &= \frac{\sum x_i^2 - n\bar{x}^2}{n} \\ \implies \frac{1}{n}\sum x_i^2 &= s^2 + \bar{x}^2. \end{aligned}$$

1653. A computer programmer is modelling a random walk on a 2×2 grid. At any iteration, one square of the four is shaded. At the next iteration, the probabilities, relative to the current position, are as shown in the diagram.



Find the probability that, after two iterations,

(a) the shading is at the opposite corner,

(b) the shading is where it began.

Count successful outcomes.

(a) There are two successful outcomes, LL and RR. So, $p = 2 \times \frac{1}{4}^2 = \frac{1}{8}$.

(b) There are three successful outcomes, LR, RL and NN. So, $p = 2 \times \frac{1}{4}^2 + \frac{1}{2}^2 = \frac{3}{8}$.

1654. You are given that the two lines $3x + ky = 0$ and $2x = (k^2 - 1)y$ are perpendicular to one another. Determine all possible values of k.

Find the gradient of the first line, and then use its negative reciprocal.

The gradient of the first line is $-\frac{3}{k}$. So the gradient of the second must be $\frac{k}{3}$. This gives

$$\begin{aligned} \frac{2}{k^2 - 1} &= \frac{k}{3} \\ \implies 6 &= k^3 - k \\ \implies (k - 2)(k^2 + 2k + 3) &= 0. \end{aligned}$$

The quadratic factor has $\Delta = 2^2 - 4 \cdot 3 = -8 < 0$, so $k = 2$.

1655. For aid distribution, bales of supplies, each of mass m kg, are being dropped from an aeroplane. The aeroplane is in level flight travelling at a constant 40 ms^{-1}. Assume that, once the bales are in the air, air resistance due to their horizontal speed will generate a constant horizontal force of $2m$ N on each one. The bales spend 8 seconds in the air.

(a) Find the horizontal distance between dropping point and landing point.

(b) The bales will split apart if they land with a speed greater than 30 ms^{-1}. What is the greatest vertical speed with which they can safely land?

In (a), find the horizontal acceleration using $F = ma$, then use *suvat*. In (b), find the horizontal speed at landing, and use Pythagoras.

(a) Horizontal $F = ma$ is $2m = ma$, giving $a = 2$ ms^{-2} backwards. The distance is then $s = 40 \cdot 8 + \frac{1}{2}(-2)8^2 = 256$ metres.

(b) The horizontal speed at landing is $40 - 8 \times 2 = 24$ ms^{-1}. Considering then a velocity triangle with 30 ms^{-1} as the hypotenuse, the greatest vertical speed is $v = \sqrt{30^2 - 24^2} = 18$ ms^{-1}.

1656. Solve $49^x - 56 \cdot 7^x + 343 = 0$.

This is a quadratic in 7^x.

This is a quadratic in 7^x:

$$\begin{aligned} & 49^x - 56 \cdot 7^x + 343 = 0 \\ \Longrightarrow\ & (7^x - 7)(7^x - 49) = 0 \\ \Longrightarrow\ & 7^x = 7, 49 \\ \Longrightarrow\ & x = 1, 2. \end{aligned}$$

1657. For positive real numbers a, b, the quantities $\frac{a+b}{2}$ and $\sqrt{ab}$ are known as the *arithmetic mean* and *geometric mean*. Explain the relationship of these means to arithmetic and geometric progressions.

Consider the mean of terms u_n and u_{n+2}.

Consider three terms u_n, u_{n+1}, u_{n+2} in a progression. The middle term u_{n+1} may be thought of as an average of the outer two u_n and u_{n+2}.

In an AP, $u_{n+2} - u_{n+1} = u_{n+1} - u_n$. This can be rearranged to $u_{n+1} = \frac{1}{2}(u_n + u_{n+2})$, which is therefore known as the *arithmetic mean*.

In a GP, $u_{n+2}/u_{n+1} = u_{n+1}/u_n$. With $u_i > 0$, this can be rearranged to $u_{n+1} = \sqrt{u_n u_{n+2}}$, which is therefore known as the *geometric mean*.

1658. Three variables are related as follows:

$$\begin{aligned} b^2x - y &= 0, \\ x + by &= 1. \end{aligned}$$

Show that $y = x^{\frac{1}{3}}(x-1)^{\frac{2}{3}}$.

Substitute for b and rearrange.

The second equation, assuming $y \neq 0$, is $b = \frac{1-x}{y}$. Substituting this into the first, we get

$$\begin{aligned} & \left(\frac{1-x}{y}\right)^2 x - y = 0 \\ \Longrightarrow\ & y^3 = x(1-x)^2 \\ \Longrightarrow\ & y = x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}. \end{aligned}$$

1659. Two dice have been rolled, with scores X_1 and X_2. Given that $|X_1 - X_2| = 1$, find $P(X_1 + X_2) = 7$.

Draw a six by six probability space, and restrict it. This is a regular conditional probability, albeit couched in fancy language.

The possibility space, with restriction, is

	1	2	3	4	5	6
1						✓
2					✓	
3				✓		
4			✓			
5		✓				
6	✓					

Hence, the probability is $\frac{2}{10} = \frac{1}{5}$.

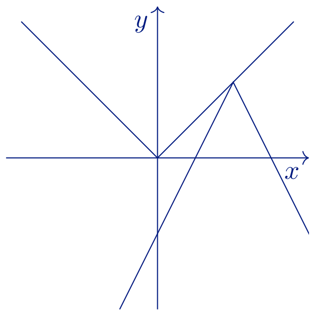
1660. The modulus function is defined over $\mathbb{R}$ as

$$|x| = \begin{cases} -x, & x \in (-\infty, 0) \\ x, & x \in [0, \infty) \end{cases}$$

Show that the graphs $y = |x|$ and $y = 2 - |4 - 2x|$ intersect at precisely one point.

Find the coordinates of the vertex of the second graph, by solving $4 - 2x = 0$.

The second graph has its vertex at $4 - 2x = 0$, which gives $(2, 2)$. This is on the graph $y = |x|$. Hence, since the second graph is an inverted mod graph with gradient ± 2, this will be the only point of intersection.



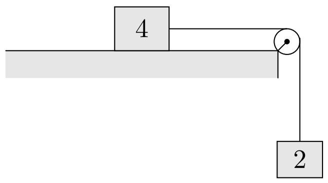
1661. You are given that $\frac{d}{dx}(x + y^2 - 3) = x + 1$. Find $\frac{dy}{dx}$ in terms of x and y.

Differentiate implicitly, using the chain rule.

Treating y as a function of x, we can differentiate implicitly by the chain rule:

$$\begin{aligned} & \frac{d}{dx}(x + y^2 - 3) = x + 1 \\ \Longrightarrow\ & 1 + 2y\frac{dy}{dx} = x + 1 \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{x}{2y}. \end{aligned}$$

1662. A smooth pulley system of two blocks is set up on a table as depicted below. Masses are given in kg. The hanging block accelerates at a.



(a) State two assumptions beyond smoothness which are necessary to find the acceleration of the system with the information given.

(b) Making these assumptions, find a.

(c) Find the force the pulley exerts on the string.

In (b), the string is taut, so you can resolve along it for the whole system.

(a) We need to assume that the string is i) light and ii) inextensible. [We are also ignoring air resistance.]

(b) The string is taut, so we can resolve along it for the system. This gives $2g = 6a$, so $a = \frac{1}{3}g$.

(c) NII for the hanging block is $2g - T = 2\frac{1}{3}g$, so the tension in the string is $T = \frac{4}{3}g$ N. This tension acts horizontally and vertically on the pulley. Pythagoras gives the overall contact force of the rope on the pulley as $C = \frac{4\sqrt{2}}{3}g$. By NIII, this is also the magnitude of the force exerted on the string by the pulley.

1663. Show that the range of the function $h : x \mapsto \sin x$ is the same over any domain of the form $[k, k+2\pi]$.

Consider the periodicity of the sine function.

The sine function is periodic, with period 2π radians. Hence, a domain of the form $[k, k + 2\pi]$ constitutes one entire period. So, regardless of the value of k, the range is $[-1, 1]$.

1664. One of the following statements is true; the other is not. Prove the one and disprove the other.

(a) $x^3 = x \implies x = 0$,

(b) $x^3 = -x \implies x = 0$.

The first statement is false, the second true.

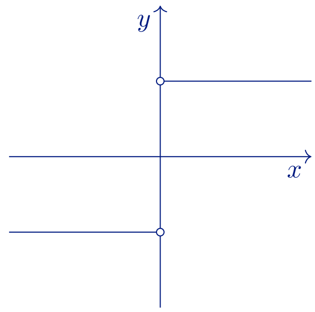
(a) This is false; counterexamples are $x = \pm 1$.

(b) This is true. Factorising gives $x(x^2 + 1) = 0$. But the quadratic factor has no roots, so $x = 0$ is implied.

1665. Sketch the graph $y = \dfrac{|x|}{x}$.

Consider the cases $x < 0$, $x = 0$ and $x > 0$.

The graph is undefined at $x = 0$. For negative x, it is $y = -1$, for positive x, it is $y = 1$.



This is also known, especially in coding, as the *sign function.*

1666. Prove that the quadratic $(x^2 + 4)$ is not a factor of $x^4 + x^3 + 5x^2 + 4x - 2$.

You can't use the real factor theorem here. So, instead, attempt the factorisation explicitly with $(x^2 + 4)(ax^2 + bx + c)$.

[Alternatively, you could venture into the complex numbers with $(x^2 + 4) = (x + 2i)(x - 2i)$.]

If there is a factorisation, it will be

$$(x^2 + 4)(ax^2 + bx + c).$$

Equating coefficients of: x^4 gives $a = 1$, x^3 gives $b = 1$, x^2 gives $c = 1$. But this gives the constant term as $4 \neq -2$. So, there is no such factorisation.

1667. Four cards are pulled from a standard deck of 52. Find the probability that

(a) all four cards are red.

(b) two cards are red and two are black.

Pick the cards one by one, counting outcomes.

(a) $p = \frac{26}{52} \times \frac{25}{51} \times \frac{24}{50} \times \frac{23}{49} = \frac{46}{833}$.

(b) There are ${}^4C_2 = 6$ successful outcomes here. So, $p = 6 \times \frac{26}{52} \times \frac{25}{51} \times \frac{26}{50} \times \frac{25}{49} = \frac{325}{833}$.

1668. By completing the square, or otherwise, find the range of the following function over $\mathbb{R}$:

$$f(x) = \frac{2}{8x^4 + 24x^2 + 28}.$$

Find the range of $8x^4 + 24x^2 + 28$ first.

Completing the square,

$$8x^4 + 24x^2 + 28 = 8(x^2 + \tfrac{3}{2})^2 + 10.$$

So, the denominator has range $[10, \infty)$. Taking the reciprocal, $f(x)$ has range $(0, \frac{1}{10}]$.

1669. Two functions f and g are such that $f'(x) - g'(x)$ is linear in x. Prove that the equation $f(x) = g(x)$ has at most two roots.

Translate into algebra, and integrate.

Algebraically, we have $f'(x) - g'(x) = 2ax + b$, where a, b are real constants. Integrating, and combining the constants of integration onto the RHS, we get $f(x) - g(x) = ax^2 + bx + c$.

The given equation $f(x) = g(x)$ is satisfied iff $f(x) - g(x) = 0$, which we now know occurs iff $ax^2 + bx + c = 0$. This is a quadratic equation, so has a maximum of two roots. □

1670. The usual projectile model assumes negligible air resistance. A particle is projected at $10\mathbf{i} - 2\mathbf{j}$ ms^{-1} from 5 metres above flat ground, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors.

(a) Determine the horizontal range, if

i. air resistance is neglected as usual,

ii. constant horizontal and vertical resistances of $\frac{1}{5}mg$ are instead assumed.

(b) Comment on these values.

In (a) ii, calculate the components of acceleration separately, and don't combine them.

(a) i. Resolving vertically to find the time of flight, $-5 = -2t - \frac{1}{2}gt^2$ gives $t = 0.82648...$. Horizontally, then, $x = 10t = 8.26$ m (3sf).

ii. Vertical acceleration is $-\frac{4}{5}mg$. So, time of flight satisfies $-5 = -2t - \frac{1}{2} \cdot \frac{4}{5}gt^2$, giving $t = 0.90273...$. Horizontal acceleration is $-\frac{1}{5}g$. So, $x = 10t - \frac{1}{2} \cdot \frac{1}{5}gt^2 = 9.83$ m (3sf).

(b) The range with air resistance is bigger than the range without air resistance. This implies that the assumption of constant horizontal and vertical resistances is not realistic.

1671. The monic quadratic function f is invertible over either of the domains $(-\infty, 2]$ or $[2, \infty)$ with the codomain $[3, \infty)$. By considering the coordinates of the vertex of $y = f(x)$, find $f(0)$.

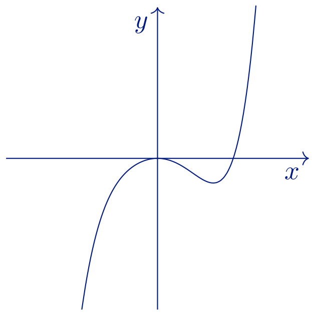
A function is only invertible if it is one-to-one. Hence, $(-\infty, 2]$ or $[2, \infty)$ must give the two halves of the parabola.

A function is only invertible if it is one-to-one. Hence, $(-\infty, 2]$ or $[2, \infty)$ must give the two halves of the parabola, and $[3, \infty)$ must be its range. So, the vertex is at $(2, 3)$. Hence, since it is a monic quadratic, its equation is $f(x) = (x-2)^2 + 3$. This gives $f(0) = 7$.

1672. Solve the inequality $x^7 - x^2 \geq 0$.

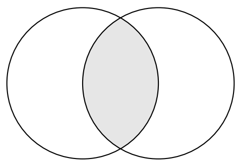
Solve the boundary equation $x^7 - x^2 = 0$, then, using your factorisation, sketch $y = x^7 - x^2$.

The expression $x^7 - x^2$ factorises as $x^2(x^5 - 1)$. The latter factor has one root at $x = 1$, giving $x^2(x-1)$(quartic with no real roots). Hence, the graph $y = x^7 - x^2$ is positive polynomial of odd degree, with a double root at $x = 0$ and a single root at $x = 1$:



We are looking for x values such that the graph is above or on the x axis. The solution set, therefore, is $\{0\} \cup [1, \infty)$.

1673. In the figure below, the circles, both of which have radius r, pass through each other's centres.

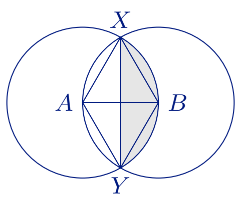


Show that the shaded region has area

$$A = r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right).$$

Split the shaded area into symmetrical segments, and calculate the area of one as the area of a sector minus the area of a triangle.

We split the shaded region into two segments:



Angle XAY is $120°$, so sector XAY has area $\frac{1}{3}\pi r^2$. Triangle XAY, then, has area $\frac{1}{2}r^2 \sin 120° = \frac{\sqrt{3}}{4}r^2$. Segment XAY has area $r^2\left(\frac{1}{3}\pi - \frac{\sqrt{3}}{4}\right)$. Doubling this, we simplify:

$$2r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) = r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right).$$

1674. A graph, whose equation is $f(x) = g(y)$ for some functions f and g, is reflected in $y = x$, then in $x = 0$, then in $y = 0$. Write down the equation of the transformed graph.

In each case, replace input variables. Reflection in $y = x$ is a switch $x \leftrightarrow y$; reflection in $x = 0$ is a replacement of x with $-x$.

Reflection in $y = x$ switches x and y. This gives $f(y) = g(x)$. Then reflection in both $x = 0$ and $y = 0$ is a replacement of (x, y) by $(-x, -y)$, giving $f(-y) = g(-x)$.

1675. A quintic function is defined as $f(x) = x^5 - x$. At each of its roots, find, as a function of x, the best linear approximation to $f(x) = x^5 - x$.

The "best linear approximation" is a tangent line.

$x^5 - x = 0$ factorises to $x(x+1)(x-1)(x^2+1) = 0$. Its roots are $x = 0, \pm 1$. Then $f'(x) = 5x^4 - 1$, so $f'(0) = -1$, $f'(\pm 1) = 4$. Substituting coordinates $(0, 0), (\pm 1, 0)$, we have three tangent lines $y = -x$, $y = 4x \mp 4$. Hence, the required approximations are the functions $g_1(x) = -x$, $g_2(x) = 4x - 4$ and $g_3(x) = 4x + 4$.

1676. A rigid object is in equilibrium under the action of the following forces in an (x, y) plane:

$$\begin{aligned} &2\mathbf{i} \text{ N at } (0, 1), \\ &2\mathbf{j} \text{ N at } (-1, 0), \\ &a\mathbf{i} + b\mathbf{j} \text{ N at some point} P. \end{aligned}$$

(a) Explain why the line of action of the third force must pass through $(-1, 1)$.

(b) Find a and b.

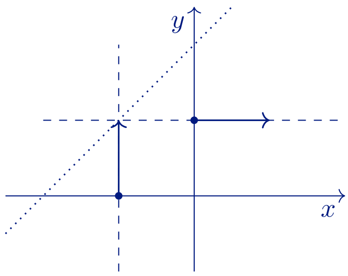
(c) Sketch the possible locations of P.

In (a), consider the intersection of the lines of action of the first two forces. In (b), use $\mathbf{F}_{\text{resultant}} = 0$. In (c), use (a) and (b).

(a) If an object is in equilibrium under the action of exactly three forces, then the lines of action of those forces must either all three be parallel, or be concurrent. Otherwise there would be a moment around the point of intersection of two of the lines of action. Hence, since two of the lines of action intersect at $(-1, 1)$, so must the third.

(b) The resultant force is zero, hence $a, b = -2$.

(c) P could lie anywhere on the dotted line:



1677. Prove rigorously that $\lim_{x\to\infty} \dfrac{x+1}{2x^2+1} = 0$.

Divide top and bottom by x^2 before taking the limit.

To do this rigorously, we divide top and bottom by x^2 before taking the limit. This gives

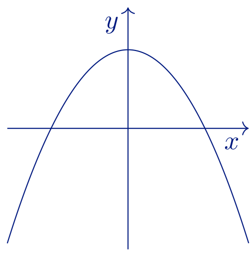
$$\lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{2 + \frac{1}{x^2}}.$$

As $x \to \infty$, the three inlaid fractions all tend to zero. The denominator, however, is well defined, tending to 2. So, the limit is $\frac{0}{2} = 0$.

1678. A parabola $y = ax^2 + bx + c$ is increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$, and has distinct roots. Sketch the curve and give conditions on the constants a, b, c.

The parabola must be negative.

The vertex must be on the y axis, so $b = 0$. The parabola must be negative, so $a < 0$. And there are distinct roots, so $c > 0$. Such parabolae are of the form



1679. Solve for a in $(2a + 1)^4 + (2a - 1)^4 = 82$.

Use the binomial expansion.

Expanding binomially,

$$(2a \pm 1)^4 = 16a^4 \pm 32a^3 + 24a^2 \pm 8a + 1.$$

The $\pm$ terms cancel, leaving a biquadratic:

$$\begin{aligned} 32a^4 + 48a^2 - 80 &= 0 \\ \Longrightarrow 16(a^2-1)(2a^2+5) &= 0. \end{aligned}$$

The quadratic factor is never zero, so $a = \pm 1$.

1680. State that the following holds, or explain why not: "Knowing that an object is in equilibrium is both necessary and sufficient for the resultant moment acting on it to be zero."

This is not true. Equilibrium isn't necessary.

The statement doesn't hold. Equilibrium isn't necessary for the resultant moment to be zero. Consider, as a counterexample, an object in freefall: the resultant moment (of the weight) is zero, yet the object is not in equilibrium.

1681. Prove that the curves $4y = x^2+1$ and $4x = y^2+1$ are not tangent.

Either solve simultaneously and use $\Delta = b^2 - 4ac$, or else consider the fact that these curves are reflections in $y = x$.

These curves are reflections in $y = x$. Hence, any intersections must be on $y = x$. So we need only solve $4x = x^2+1$. This has discriminant $\Delta = 17 \neq 0$. Hence, $y = x$ is not tangent to $4y = x^2+1$. Therefore, the two curves are not tangent to each other.

1682. A quadratic sequence has three consecutive terms $12, 16, 22$. Find the values of the two consecutive terms of the sequence which differ by sixty.

Find an ordinal formula $u_n = an^2 + bn + c$, using the fact that the second difference is $2a$.

The second difference of a quadratic sequence $an^2 + bn + c$ is $2a$. Here, the first differences are $4, 6, \ldots$, so the second difference is 2 and $a = 1$. Solving $1+b+c = 12$ and $4+2b+c = 16$ gives $b = 1$, $c = 10$. So the ordinal formula is $u_n = n^2+n+10$. We now want the difference to be equal to 60. So

$$\begin{aligned} (n+1)^2 + (n+1) + 10 - (n^2+n+10) &= 60 \\ \Longrightarrow \quad 2n + 2 &= 60 \\ \Longrightarrow \quad n &= 29. \end{aligned}$$

The terms are $u_{29} = 880$ and $u_{30} = 940$.

1683. Make x the subject of $y = \dfrac{x^3-1}{x^3+1}$.

Multiply up, gather terms in x^3, and factorise.

Multiplying up and gathering terms,

$$\begin{aligned} y(x^3+1) &= x^3 - 1 \\ \Longrightarrow \quad x^3 - x^3 y &= 1 + y \\ \Longrightarrow \quad x^3(1-y) &= 1+y \\ \Longrightarrow \quad x &= \sqrt[3]{\frac{1+y}{1-y}}. \end{aligned}$$

1684. True or false?

(a) $x \in A \implies x \in A \cup B$,
(b) $x \notin A \implies x \notin A \cap B$,
(c) $x \notin A \implies x \notin A \cup B$.

A Venn diagram might help.

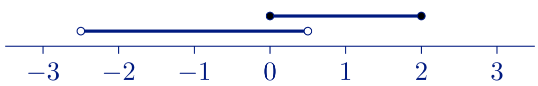
(a) True.
(b) True.
(c) False; elements of $B \cap A'$ are counterexamples.

1685. Write the following in simplified interval notation:

$$\{x \in \mathbb{R} : |x-1| \leq 1\} \cup \{x \in \mathbb{R} : |x+1| < \tfrac{3}{2}\}.$$

Sketch the regions on a number line.

On a number line, the individual sets are



Hence, the union is $(-\frac{5}{2}, 2]$

1686. Prove that a polynomial of order n can have at most $n-2$ points of inflection.

A point of inflection requires that the second derivative be zero.

If a polynomial is of order 0 or 1, then it trivially has no points of inflection. If it has order $n \geq 2$, then the first derivative is a polynomial of order $n-1$, and the second derivative is a polynomial of order $n-2$. So, setting the second derivative to zero gives a polynomial equation of order $n-2$, which can therefore have a maximum of $n-2$ roots. Q.E.D.

1687. Find the values of the constants A, B, C, D such that the following is an identity:

$$\frac{8x^3 - 6x^2 - x + A}{2x-1} \equiv Bx^2 + Cx + D.$$

Use the factor theorem.

If $2x-1$ is to be a factor of $x^3 + 3x^2 + 16x - A$, then $x = \frac{1}{2}$ must be a root. Substituting, we get $A = 1$. Taking out the factor $(2x-1)$ gives $B = 4$, $C = -1$, $D = -1$.

1688. Show that the following curve has no asymptotes:

$$y = \frac{1}{\cos^2 x + \cos x + 1}.$$

Show that the denominator is never zero.

The curve will have a vertical asymptote wherever the denominator is zero. $\cos^2 x + \cos x + 1 = 0$ is a quadratic in $\cos\theta$, with discriminant $-3 < 0$. Hence, it has no real roots, and thus the curve has no vertical asymptotes.

[It also has no horizontal asymptotes, as the value of the function is periodic as $x \to \pm\infty$.]

1689. By finding equations of perpendicular bisectors, or otherwise, show that a quadrilateral with vertices at $(8, 2)$, $(6, 4)$, $(12, 10)$ and $(14, 4)$ is cyclic.

You only need to find two perpendicular bisectors. From these you can find the centre, and hence test distances with Pythagoras.

The perpendicular bisector of points 1 and 2 is $y = x-4$, and that of points 2 and 3 is $y = -x+16$. Solving these simultaneously gives $P : (10, 6)$, which must be the centre of any circle passing through all four points. Testing the distances, each point is $\sqrt{20}$ from P. Hence, the quadrilateral is cyclic.

1690. Find $\displaystyle\int \cos\left(4x + \frac{\pi}{2}\right) dx$.

This is the reverse chain rule. Since the inside function is linear, of the form $ax+b$, a factor of $\frac{1}{a}$ is required.

By the reverse chain rule, a factor of $\frac{1}{4}$ is required:

$$\int \cos\left(4x + \frac{\pi}{2}\right) dx = \frac{1}{4}\sin\left(4x + \frac{\pi}{2}\right) + c.$$

1691. A hand of five cards is dealt from a standard deck, with ace counting high. Show that the probability p that the hand is a straight flush, i.e. a set of five consecutive cards all of the same suit, is given by

$$p = \frac{36 \times 5! \times 47!}{52!}.$$

Consider, in any given suit, the number of sets of the form $\{8, 9, 10, \text{J}, \text{Q}\}$.

In each suit, there are nine sets of numbers: $\{2, 3, 4, 5, 6\}, ..., \{10, \text{J}, \text{Q}, \text{K}, \text{A}\}$. Hence, there are $4 \times 9 = 36$ successful outcomes, out of a total of ${}^{52}C_5$. This gives

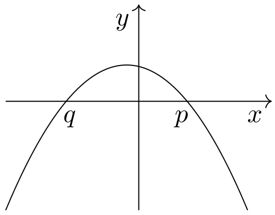
$$p = \frac{36}{\frac{52!}{5!\times 47!}} = \frac{36 \times 5! \times 47!}{52!}.$$

1692. The two lines $2x + 3y = 5$ and $ax + by = c$ do not intersect. Determine the value of $2bc - 3ac$.

Since the two lines do not intersect, they must be parallel.

Since the two lines do not intersect, they must be parallel. Hence, $\frac{b}{a} = \frac{3}{2}$. This gives $2b = 3a$, so $2b - 3a = 0$. Hence, $2bc - 3ac$ is also zero.

1693. A quadratic graph $y = ax^2+bx+c$ is shown below. It is given that $q < 0 < p$, and $|p| < |q|$.



State, with a reason in (b), whether the following facts are necessarily true:

(a) "a is negative",

(b) "b is negative",

(c) "c is positive".

You can consider any of: factorisation, completing the square, or differentiation to find the vertex.

(a) This is clearly true.

(b) This is true. Since $|p| < |q|$, the vertex of the graph is at $x < 0$. Hence, in completed square form, the graph must be $y = -r(x + s)^2 + t$, where r, s, t are positive. Multiplying this out gives $b = -2rs$, which is negative.

(c) This is clearly true.

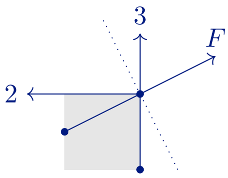
1694. A square of side length 1 lies in a horizontal (x, y) plane. Forces act on it: at $(1, 1)$, a force of $-2\mathbf{i}$ N; at $(1, 0)$, a force of $3\mathbf{j}$ N; at $(0, \frac{1}{2})$, a force $\mathbf{F} = a\mathbf{i} + b\mathbf{j}$ N.

(a) Draw a force diagram.

(b) Show that equilibrium cannot be maintained.

Using concurrence, place $\mathbf{F}$ in the direction necessary for zero resultant moment. In (b), show that this is incompatible with the resultant force being zero.

(a) Let us place $\mathbf{F}$ in the direction necessary for no resultant moment, in which its line of action is concurrent with the other two lines of action.



(b) Having directed force $\mathbf{F}$ to achieve rotational equilibrium, we consider forces perpendicular to F, along the dotted line. Since $\mathbf{F}$ has no component in this direction, there must be a resultant from the other two forces. Hence, either rotational or translational equilibrium must be broken.

1695. State, with a reason, whether the following hold:

(a) $\displaystyle\sum_{r=1}^{n} ku_r \equiv k\sum_{r=1}^{n} u_r$,

(b) $\displaystyle\sum_{k=1}^{n} ku_k \equiv k\sum_{k=1}^{n} u_k$,

(c) $\displaystyle\sum_{r=1}^{n} nu_r \equiv n\sum_{r=1}^{n} u_r$.

Think carefully about which values change across the different terms of the sum and which don't, i.e. which letters represents constants and which represent variables.

(a) This is true; k is a common factor of every term, so can be taken out of the sum.

(b) This is not true; k varies term by term in the sum, so cannot be taken out as a factor.

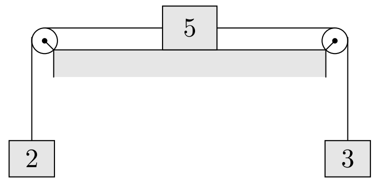
(c) This is true; while n is mentioned in the sum, it is, as far as the sum is concerned, a constant. Hence, it can be taken out exactly as k could be in (a).

1696. Show that the graph $y = 10^x$ is transformed into that of $y = e^{x+2}$ by a stretch in each of the x and y directions. Give the exact value of the scale factors.

Split the exponential, and then write e as 10^k.

Firstly, e^{x+2} can be written as $e^x \cdot e^2$, which gives a stretch in the y direction, scale factor e^2. Then, writing e as $10^{\ln 10}$, we can express the graph $y = e^x$ as $y = (10^{\ln 10})^x = 10^{x \ln 10}$, which, since we have replaced x by $x \ln 10$, is a stretch, scale factor $\frac{1}{\ln 10}$, in the x direction.

1697. A smooth pulley system of three masses, given in kg, is set up on a table as depicted below.



(a) Explain the assumptions necessary to model

i. the accelerations as equal in magnitude,

ii. the tension as the same in the vertical and horizontal sections of each string.

(b) Making all necessary assumptions, draw force diagrams for the masses.

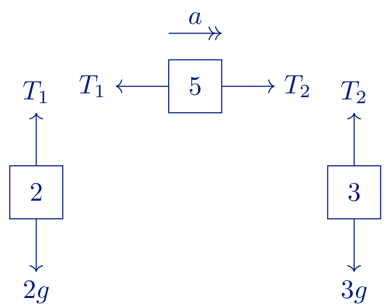
(c) Hence, find the acceleration of the system.

In (c), add the three equations of motion. This will cancel the internal NIII pairs, which are the tensions. This is equivalent to resolving along the (taut) strings for the entire system.

(a) i. Model the strings as inextensible.

ii. Model the pulleys as smooth and the strings as light.

(b) Ignoring the vertical forces on the 5 kg mass:



(c) The equations of motion are

$$\begin{aligned} T_1 - 2g &= 2a \\ T_2 - T_1 &= 5a \\ 3g - T_2 &= 3a \end{aligned}$$

Adding these three, the tensions cancel. This is equivalent to calculating an equation of motion for the entire system along the taut strings. We get $g = 10a$, so $a = \frac{1}{10}g$.

1698. State, with a reason, whether the following curves intersect the line $x = \frac{\pi}{2}$:

(a) $y = \operatorname{cosec} x$,

(b) $y = \sec x$,

(c) $y = \cot x$.

A vertical line such as $x = \frac{\pi}{2}$ must intersect any graph $y = f(x)$ if the value $x = \frac{\pi}{2}$ is in the domain of the function f.

The question is whether $x = \frac{\pi}{2}$ is in the domain of the function in question, i.e. whether the original function to be reciprocated is non-zero.

(a) $\operatorname{cosec} \frac{\pi}{2} = 1$, so they intersect.

(b) $\sec \frac{\pi}{2}$ is undefined, so they do not intersect.

(c) $\cot \frac{\pi}{2} = 0$, so they intersect.

1699. Two vertices of an equilateral triangle lie at $(0, 0)$ and $(2, 1)$. Show that the other vertex lies on the line $2y + 4x = 5$.

Find the perpendicular bisector.

We are not required to find the coordinates of the third vertex, so we need only note that the line passing through its two possible positions is the perpendicular bisector of the other two vertices. This has gradient $m = -2$ and passes through $(1, \frac{1}{2})$, so it has equation $y = -2x + \frac{5}{2}$, which may be rearranged to $2y + 4x = 5$ as required.

1700. Separate the variables in the following differential equation, writing it in the form $f(y)\frac{dy}{dx} = g(x)$ for some functions f and g:

$$x^2\frac{dy}{dx} - 2\sin x \cos y = \sin x.$$

Rearrange so as to take out a common factor of $\sin x$ on the RHS.

We rearrange as follows:

$$\begin{aligned} & x^2\frac{dy}{dx} - 2\sin x\cos y = \sin x \\ \Longrightarrow\; & x^2\frac{dy}{dx} = \sin x(1 + 2\cos y) \\ \Longrightarrow\; & \frac{1}{1 + 2\cos y}\frac{dy}{dx} = \frac{\sin x}{x^2}. \end{aligned}$$

1701. A quadratic function g has $g(a) = g(b)$, for some $a \neq b$. Prove that $g'(a) + g'(b) = 0$.

Use a symmetry argument.

Consider the graph $y = g(x)$. This has a line of symmetry of the form $x = k$, through its vertex. Since $g(a) = g(b)$, the points $x = a$ and $x = b$ must be symmetrical in this line. Hence, the tangents to $y = g(x)$ at these two points must also be reflections of one another, hence $g'(a) = -g'(b)$. This gives $g'(a) + g'(b) = 0$, as required.

1702. A student writes: "When you step on the pedals of a bicycle, a frictional force is generated, which acts forwards on the back wheel of the bicycle." Explain whether this is correct.

It is correct.

This is a correct description of the driving force on the bicycle. Stepping on the pedals is an attempt to make the part of the wheel in contact with the road travel backwards. Friction resists this. Equal and opposite frictional forces act backwards on the road and forwards on the bicycle wheel.

1703. For distinct and non-zero $a, b, c, d \in \mathbb{R}$, give the roots of the equation

$$\frac{(x^4 - a^4)(x^4 + b^4)}{(x^4 - c^4)(x^4 + d^4)} = 0.$$

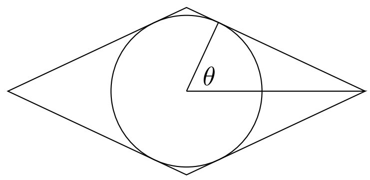
Set the numerator to zero.

A fraction can be zero only when its numerator is zero. The first factor is a difference of two squares, giving

$$\begin{aligned} & (x^4 - a^4)(x^4 + b^4) = 0 \\ \Longrightarrow\; & (x^2 - a^2)(x^2 + a^2)(x^4 + b^4) = 0 \\ \Longrightarrow\; & (x - a)(x + a)(x^2 + a^2)(x^4 + b^4) = 0. \end{aligned}$$

Since $a, b \neq 0$, only the first two factors have roots $x = \pm a$. Neither makes the denominator zero, as a, b, c, d are distinct, so the solution set is $\{a, -a\}$.

1704. A rhombus, with perimeter 16, is tangent to a unit circle at four points. The radius shown below is to one of those points. It splits a side of the rhombus into two parts.



(a) Show that $\tan^2\theta - 4\tan\theta + 1 = 0$.

(b) Hence, find the interior angles of the rhombus.

In (a), first show that the two parts of the side of the rhombus are $\tan\theta$ and $\cot\theta$. In (b), solve the quadratic to find $\tan\theta$.

(a) The radius is 1, so the longer part of the side has length $\tan\theta$. Then, by symmetry, the shorter part has length $\cot\theta$, because we have switched θ for $90 - \theta$, switching the values of $\sin\theta$ and $\cos\theta$. Hence, $\tan\theta + \cot\theta = \frac{16}{4} = 4$. Multiplying by $\tan\theta$ and rearranging gives $\tan^2\theta - 4\tan\theta + 1 = 0$ as required.

(b) This is a quadratic in $\tan\theta$. Solving with the formula, $\tan\theta = 2 + \sqrt{3}$, which gives $\theta = 75°$. So the interior angles are 150° and 30°.

1705. In the equation $16^x - 8^x + 4^x - 2^x = 0$, factorise the left-hand side fully, and hence show that the solution is $x = 0$.

This is a quartic in 2^x.

This is a quartic in 2^x. Setting $z = 2^x$, we have

$$\begin{aligned} & z^4 - z^3 + z^2 - z = 0 \\ \Longrightarrow\ & z(z^3 - z^2 + z - 1) = 0 \\ \Longrightarrow\ & z(z-1)(z^2+1) = 0. \end{aligned}$$

The quadratic factor is irreducible, so $2^x = 0, 1$. The former has no roots, so $x = 0$.

1706. It is given that $\displaystyle\int_0^1 f(x)\,dx = 2$. Evaluate

(a) $\displaystyle\int_0^1 1 + f(x)\,dx$,

(b) $\displaystyle\int_0^1 3(x^2 + f(x))\,dx$.

In each case, split the integral up.

In each case, we can split the integral:

(a)
$$\begin{aligned} & \int_0^1 1 + f(x)\,dx \\ &= \big[x\big]_0^1 + \int_0^1 f(x)\,dx \\ &= (1) - (0) + 2 \\ &= 3. \end{aligned}$$

(b)
$$\begin{aligned} & \int_0^1 3(x^2 + f(x))\,dx \\ &= \big[x^3\big]_0^1 + 3\int_0^1 f(x)\,dx \\ &= (1) - (0) + 3 \\ &= 4. \end{aligned}$$

1707. Find y in simplified terms of x, if

(a) $\ln y = 2\ln x - \ln\sqrt{x}$.

(b) $\log_2 y = 3\log_2 x + \log_4 x$,

In each case, write the RHS as a single logarithm, then write $y = ...$

(a) Using log rules,

$$\begin{aligned} & \ln y = 2\ln x - \ln\sqrt{x} \\ \Longrightarrow\ & \ln y = \ln\frac{x^2}{\sqrt{x}} \\ \Longrightarrow\ & y = x^{\frac{3}{2}}. \end{aligned}$$

(b) Using the fact that $\log_a b \equiv \log_{\sqrt{a}} \sqrt{b}$,

$$\begin{aligned} & \log_2 y = 3\log_2 x + \log_4 x \\ \Longrightarrow\ & \log_2 y = \log_2 x^3 + \log_2\sqrt{x} \\ \Longrightarrow\ & \log_2 y = \log_2 x^{\frac{7}{2}} \\ \Longrightarrow\ & y = x^{\frac{7}{2}}. \end{aligned}$$

1708. Show that $y = 2^x$ may be transformed to $y = -\frac{1}{2^x}$ by a rotation.

A rotation can be expressed as the composition of two reflections.

We can write $y = -\frac{1}{2^x}$ as $y = -2^{-x}$. Hence, the transformation may be seen as two reflections, one in the x axis and one in the y axis. These two reflections, switching (x, y) for $(-x, -y)$, combine to give rotation by $180°$ around the origin.

1709. Three sequences are given as follows:

$$\begin{aligned} a_n &= 1, 3, 5, 7, ... \\ b_n &= 2, 4, 6, 8, ... \\ c_n &= 2, 12, 30, 56, ... \end{aligned}$$

(a) Give ordinal formulae for a_n and b_n.

(b) Express c_n in terms of a_n and b_n.

(c) Hence, find the first term c_n to exceed 1000.

Sequences a_n and b_n are arithmetic, and c_n is the product of them.

(a) These are two APs, with ordinal formulae $a_n = 2n - 1$ and $b_n = 2n$.

(b) $c_n = a_n b_n$.

(c) Combining the first two parts of the question, $c_n = 2n(2n - 1)$. So, we require

$$\begin{aligned} & 2n(2n-1) = 1000 \\ \Longrightarrow\ & 4n^2 - 2n - 1000 = 0 \\ \Longrightarrow\ & n \approx -15.6, 16.1. \end{aligned}$$

So, the first term to exceed 1000 is $c_{17} = 1122$.

1710. A function f, defined over $\mathbb{R}$, has the property that, for all $x \in \mathbb{R}$, $f(x + 1) > f(x)$. State, with a reason, whether this implies that f is increasing for all $x \in \mathbb{R}$.

It doesn't imply that f is increasing everywhere. A counterexample might involve a sinusoidal wave.

This is not a valid implication. The fact that $f(x + 1) > f(x)$ for all x implies that $f(x)$ “gets bigger" globally, but that isn't the same as the technical term *increasing*, which requires that the local gradient $f'(x)$ is positive everywhere. We could construct a counterexample as follows. Set up the function $\sin(2\pi x)$, which has a period of 1, and add a linear function with a small positive gradient: $f(x) = \frac{1}{2}x + \sin(2\pi x)$.

1711. Prove that the sum of the cubes of a set of three consecutive integers is divisible by 3 and also by one of those integers.

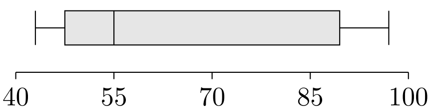
Use the binomial expansion.

Using the binomial expansion,

$$\begin{aligned} & n^3 + (n+1)^3 + (n+2)^3 \\ = \; & n^3 + n^3 + 3n^2 + 3n + 1 + n^3 + 6n^2 + 12n + 8 \\ = \; & 3n^3 + 9n^2 + 15n + 9. \end{aligned}$$

This is obviously divisible by 3. Also, substituting $n = -1$ gives $-3 + 9 - 15 + 9 = 0$. Hence, by the factor theorem, the central integer $(n+1)$ is also a factor.

1712. The marks for a test taken by a year group of 200 pupils are summarised in the following box-and-whisker diagram:



State, with a reason, if the following are definitely true of this set of data:

(a) Exactly half the scores are greater than or equal to the median.

(b) The mode is lower than the median.

(c) The mean is higher than the median.

Neither (a) nor (b) is necessarily true. The hard part here is explaining why (c) must be true for this particular set of data; you need to consider the locations of the various quartiles.

(a) This doesn't have to be true. There could be many pupils who scored exactly the median mark. In that case, more than half the scores will be greater than or equal to the median.

(b) There is no particular reason why this should be true. The mode could potentially be anywhere in the data set.

(c) This must be true. The median is 55. Consider the five marks $40, 55, ..., 100$ as four classes. The lower half of the data is all less than 15 from 55. The highest quarter of the data is all more than 30 from 55, which is enough on its own to raise the mean above the median 55.

1713. "The line $y = x$ is normal to the cubic curve $y = x^3 - x$." True or false?

Solve simultaneously and test the gradient.

Solving simultaneously, $x = x^3 - x$, which gives $x = 0, \pm\sqrt{2}$. The gradient of the curve is given by $\frac{dy}{dx} = 3x^2 - 1$. Testing this at our x values, we have $m_{\text{tangent}} = -1$ at $x = 0$. Hence, $y = x$ is normal to the curve at the origin.

1714. A rectangle has dimensions 4×6 cm. These lengths are changing at rates of -2 and 3 cm per second respectively. Find the instantaneous rate of change of the area.

Differentiate $A = xy$ by the product rule.

The area is given by $A = xy$. Differentiating both sides with respect to t by the product rule, we get

$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}.$$

Substituting values,

$$\frac{dA}{dx} = -2 \times 6 + 4 \times 3 = 0.$$

1715. Points $(1, -3)$, $(-3, 1)$, $(2, 6)$ lie on a parabola. Find its equation, in the form $y = ax^2 + bx + c$.

This just needs a bit of a brute force. Substitute values and solve a set of three equations in a, b, c.

We require that a, b, c satisfy

$$\begin{aligned} -3 &= a + b + c \\ 1 &= 9a - 3b + c \\ 6 &= 4a + 2b + c. \end{aligned}$$

We can easily eliminate c. The first two equations give $4 = 8a - 4b$; the second two give $-5 = 5a - 5b$. Hence, $a = 2$, $b = 3$. Substituting back in, $c = -8$. So, the required parabola is $y = 2x^2 + 3x - 8$.

1716. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $\sqrt[3]{xy} = z$,
- $xy = z^3$.

Cubing preserves sign.

Since cubing preserves sign, the implication goes both ways: $\Longleftrightarrow$.

1717. Show that the stationary value of the function $f(x) = 8e^x(1 - 2e^x)$ is $f(x) = 1$.

To find the stationary value, first find the input x at which the stationary value occurs.

Multiplying out and differentiating,

$$\begin{aligned} & f(x) = 8e^x - 16e^{2x} \\ \Longrightarrow\; & f'(x) = 8e^x - 32e^{2x}. \end{aligned}$$

So, $f(x)$ is stationary when $e^x(1 - 4e^x) = 0$. This occurs when $e^x = \frac{1}{4}$. We can substitute this into $f(x)$ directly, giving $8 \cdot \frac{1}{4}(1 - 2 \cdot \frac{1}{4}) = 1$ as required.

1718. Three dice are rolled. State, with a reason, which, if either, of the following events has the greater probability:

- two fives and a six, in any order,
- a four, a five and a six, in any order.

Four, five, six is more probable. Explain why, in terms of number of successful outcomes.

Four, five, six is more probable. Out of $6^3 = 216$, ${}^3C_1 = 3$ outcomes yield $\{5, 5, 6\}$, while $3! = 6$, i.e. twice as many outcomes yield $\{4, 5, 6\}$.

1719. A projectile is launched at 10 ms^{-1}, at $30°$ above the horizontal. Show that raising the launch point 1 metre above ground level will add around 1.5 metres to the range.

Perform the same calculation twice, with $s_y = 0$ and $s_y = -1$.

Projecting from ground level, the time of flight is given by $0 = 10\sin(30°)t - \frac{1}{2}gt^2$, so $t = 1.02...$ seconds. The range is $10\cos(30°) \times 1.02 \approx 8.84$ m.

From 1 m above ground level, time of flight is given by $-1 = 10\sin(30°)t - \frac{1}{2}gt^2$, so $t = 1.19...$ seconds. This exceeds the previous range by $10\cos(30°) \times (1.19 - 1.02) = 1.48... \approx 1.5$ m.

1720. A positive, monic, cubic function g has $x = 2$ as a fixed point, a stationary point, and a point of inflection.

(a) Explain which of the above facts are needed to show that $g(x) = (x - 2)^3 + c$.

(b) Determine the value of c.

In (a), consider the fact that the curve has a stationary point of inflection; only one type of cubic has such a point. In (b), use the fixed point information to set up an equation in c.

(a) The cubic has a stationary point of inflection, so it must be a translation of the basic cubic $y = x^3$: all other cubics have either no stationary points or else two which are not points of inflection. The point $x = 2$ is the image of the origin under this translation, which is by vector $\binom{2}{c}$ for some constant c. This gives $g(x) = (x - 2)^3 + c$.

(b) We can now use the information that $x = 2$ is a fixed point. This says that $g(2) = 2$, which means $c = 2$.

1721. Solve $(\cos x - 2)(2\cos 2x - 1) = 0$, for $0 \le x \le 2\pi$.

Solve each factor separately. You are looking for four roots.

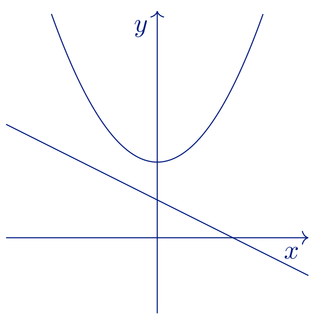
The first factor yields no roots, as the range of the cosine function is $[-1, 1]$. The second factor yields four:

$$\begin{aligned} & 2\cos 2x - 1 = 0 \\ \Longrightarrow\; & \cos 2x = \tfrac{1}{2} \\ \Longrightarrow\; & 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ \Longrightarrow\; & x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}. \end{aligned}$$

1722. You are given that, for variables p and q, $q = p^2 + 1$. Using a graphical method, or otherwise, show that $p + 2q > 1$.

If you find it easier, replace (p, q) with (x, y). Plot a quadratic, and show that it is always above a line.

Let $(p, q) = (x, y)$. Then we are restricted to points on the parabola $y = x^2 + 1$ below:



The line is $x + 2y = 1$. Solving for intersections, we have $\frac{1}{2}(1 - x) = x^2 + 1$, which has discriminant $\Delta = -\frac{7}{4} < 0$. Hence, the parabola is always above the line. So, since we are restricted to points on the parabola, we know that $x + 2y > 1$. Using the original variables, $p + 2q > 1$.

1723. Prove that every hexagon must have at least one exterior angle β satisfying $\beta \le \frac{\pi}{3}$ radians.

Prove this by contradiction. Begin "Assume, for a contradiction, that every exterior angle of a hexagon satisfies $\beta > \frac{\pi}{3}$ radians."

Assume, for a contradiction, that every exterior angle of a hexagon satisfies $\beta > \frac{\pi}{3}$ radians. Then the sum of the exterior angles must satisfy $S > 6 \times \frac{\pi}{3} = 2\pi$. But the exterior angles must sum to exactly $S = 2\pi$. This is a contradiction. Hence, every hexagon must have at least one exterior angle β satisfying $\beta \le \frac{\pi}{3}$ radians.

1724. A student is performing a binomial hypothesis test concerning a potentially biased selection process. The student writes "$H_0 : x = 10$, where x is the number of jobs out of the twenty in the sample which went to women." Explain why this cannot be a correct formulation of a statistical hypothesis, and give a corrected version.

Use the terms "sample" and "population".

This is incorrectly formulated because it refers to the sample, rather than the population. In order for a hypothesis test (indeed, in order for virtually all of statistics) to be meaningful, it must *use* a sample to *analyse* a population. The hypotheses must refer to probabilities in the population. In this case: "$H_0 : p = 0.5$, where p is the probability that any particular job in the population (of jobs) goes to a woman."

1725. A sequence is given, for constants $a, b > 0$, by $u_{n+1} = au_n + b, \quad u_1 = 1$. Prove that this sequence is neither arithmetic nor geometric.

Show algebraically that neither the differences nor the ratios are constant.

Firstly, since $u_1 = 1$ and $a, b > 0$, we know that the sequence is increasing: each term exceeds the last. The differences are given by

$$\begin{aligned} u_{n+1} - u_n &= au_n + b - u_n \\ &= (a-1)u_n + b. \end{aligned}$$

Since u_n increases, so does this difference, so the sequence cannot be arithmetic. The ratios, then, are given by

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{au_n + b}{u_n} \\ &= a + \frac{b}{u_n}. \end{aligned}$$

Since u_n increases, this ratio must decrease, tending towards a. Hence, the sequence tends towards a geometric progression, but nevertheless it isn't one.

1726. Solve $\dfrac{5\sqrt{x}}{2\sqrt{x}+1} - \dfrac{2\sqrt{x}-1}{\sqrt{x}} = \dfrac{1}{2}$.

Multiply up, and take care with minus signs. You might want to set $z = \sqrt{x}$ before you begin.

Setting $z = \sqrt{x}$, we have

$$\begin{aligned} & \frac{5z}{2z+1} - \frac{2z-1}{z} = \frac{1}{2} \\ \implies & 10z^2 - 2(4z^2 - 1) = z(2z+1) \\ \implies & z = 2 \end{aligned}$$

So $\sqrt{x} = 2$, giving $x = 4$.

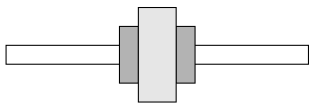
1727. Give the range of each of the following functions, over the domain $[-2, 2]$.

(a) $x \mapsto x^2 + 1$,
(b) $x \mapsto x^3 + 1$,
(c) $x \mapsto x^4 + 1$.

If in doubt, sketch graphs.

(a) $[1, 5]$,
(b) $[-7, 9]$,
(c) $[1, 17]$.

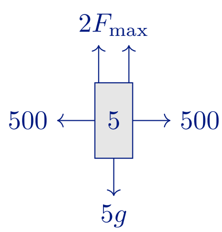
1728. A carpenter's vice, attached to a workbench, consists of two adjustable jaws, which are clamped around a block of wood, as follows:



The block of wood has mass 5 kg, and each jaw of the vice can exert a maximum horizontal force of 500 N. Determine the range of possible values of μ, the coefficient of friction between the jaws and the wood.

Draw a force diagram for the wood.

The force diagram for the wood, when on the point of slipping despite maximal horizontal force, is



Using $F_{\text{max}} = \mu R$, we get $2\mu \times 500 - 5g = 0$. Therefore, we require $\mu \ge \frac{5g}{1000} = 0.049$.

1729. A set of four lines forms a square. The first three are $y = \sqrt{3}x$, $\sqrt{3}y = -x$ and $\sqrt{3}x + 3y = 5$. Find the two possible equations of the last line.

Sketch the lines, determining which is parallel to which.

The first two lines $y = \sqrt{3}x$ and $\sqrt{3}y = -x$ are perpendicular and meet at the origin. The third line meets the first at $(\frac{5\sqrt{3}}{12}, \frac{5}{4})$. Using vectors to find the remaining vertices, the possible equations are $y = \sqrt{3}x \pm 3$.

1730. Prove the following statement, in which f and g are polynomial functions:

$$\int_0^x f(t)\,dt \equiv \int_0^x g(t)\,dt \implies f(x) \equiv g(x).$$

To prove this rigorously, name the integral of f as F, so that $F'(x) = f(x)$.

Suppose $F(x)$ and $G(x)$ are polynomials such that $F'(x) = f(x)$ and $G'(x) = g(x)$. Then, we have

$$\begin{aligned}
&\int_0^x f(t)\,dt \equiv \int_0^x g(t)\,dt \\
\implies &\Big[F(t)\Big]_0^x \equiv \Big[G(t)\Big]_0^x \\
\implies &F(x) - F(0) \equiv G(x) - G(0) \\
\implies &\frac{d}{dx}\big(F(x) - F(0)\big) \equiv \frac{d}{dx}\big(G(x) - G(0)\big) \\
\implies &f(x) \equiv g(x).
\end{aligned}$$

1731. The graphs $y = x^2 - ax + b$ and $y = -x^2 + cx + d$ may be transformed onto each other by a rotation of 180° around point P. Show that the coordinates of point P are

$$\left(\frac{a+c}{4}, \frac{4(b+d) - a^2 - c^2}{8}\right).$$

Consider the fact that the point of rotation must be the midpoint of the vertices of the two graphs.

The rotation must map one vertex onto the other, so P is the midpoint of the vertices. Completing the square, these are $(\frac{1}{2}a, b - \frac{1}{4}a^2)$ and $(\frac{1}{2}c, d - \frac{1}{4}c^2)$. Taking the mean of these, P must be at

$$\left(\frac{\frac{a}{2} + \frac{c}{2}}{2}, \frac{b - \frac{1}{4}a^2 + d - \frac{1}{4}c^2}{2}\right),$$

which can be rearranged to the required result.

1732. A function f is $f(x) = 8x^3 + 14x^2 - 553x - 1224$.

(a) Explain why any fixed point of the function

$$g(x) = \frac{8x^3 + 14x^2 - 1224}{553}$$

must be a root of $f(x) = 0$.

(b) Run the iteration $x_{n+1} = g(x_n)$, with $x_0 = 0$, to find a root of $f(x) = 0$.

(c) Using this root, factorise $f(x)$ fully.

In (a), fixed points satisfy $g(x) = x$. In (c), use the factor theorem.

(a) A fixed point of g satisfies the equation $g(x) = x$. Rearranging this,

$$\begin{aligned}
&\frac{8x^3 + 14x^2 - 1224}{553} = x \\
\implies &8x^3 + 14x^2 - 1224 = 553x \\
\implies &8x^3 + 14x^2 - 553x - 1224 = 0.
\end{aligned}$$

(b) Running the iteration, $x_n \to -2.25$.

(c) Since $x = -\frac{9}{4}$ is a root, we know that $(4x+9)$ is a factor. This gives

$$\begin{aligned}
&8x^3 + 14x^2 - 553x - 1224 \\
\equiv\ &(4x+9)(2x^2 - x - 136) \\
\equiv\ &(4x+9)(2x-17)(x+8).
\end{aligned}$$

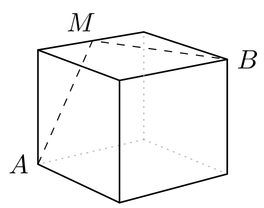
1733. Express the following set as a list of elements, in the form $\{a_1, a_2, ..., a_n\}$.

$$\{x \in \mathbb{N} : \log_2 x < 5\} \cap \{x \in \mathbb{N} : \log_4 x > \tfrac{12}{5}\}.$$

Solve the inequalities separately, and then combine them.

The inequalities are $\log_2 x < 5 \implies x < 32$, and $\log_4 x > \frac{12}{5} \implies x > 27.8...$. Over the naturals, then, we have $\{28, 29, 30, 31\}$.

1734. The diagram shows a solid cube, with the shortest path between two vertices marked.



Determine angle AMB.

Find the space diagonal AB using 3D Pythagoras, then use the cosine rule.

Assuming the cube has unit side length, the space diagonal AB has length $\sqrt{3}$, by Pythagoras. The lengths AM and MB are both $\frac{\sqrt{5}}{2}$. In triangle ABC, we can use the cosine rule. This gives

$$\begin{aligned}\cos\theta &= \frac{\frac{5}{4}+\frac{5}{4}-3}{2\cdot\frac{5}{4}} \\ &= -\tfrac{1}{5}.\end{aligned}$$

Therefore, $\theta = 101.5°$ (1dp).

1735. A jar contains m black and n white counters. From the jar, two counters are taken out at random. Find, in terms of m and n, the probability that these are one of each.

Draw a tree diagram if you need it.

There are two successful outcomes BW and WB, which are equally likely. This gives

$$\begin{aligned}p &= 2\times\frac{m}{m+n}\times\frac{n}{m+n-1} \\ &= \frac{2mn}{(m+n)(m+n-1)}.\end{aligned}$$

1736. Consider $(y-|x|+1)(y+|x|-1)\le 0$.

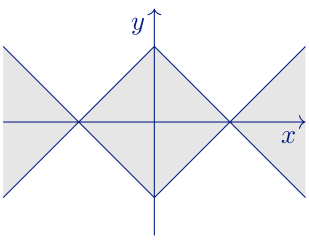
(a) Explain, by sketching the graphs $y-|x|+1=0$ and $y+|x|-1=0$, why the boundary of the (x,y) region defined by this inequality consists of four straight line segments.

(b) On your sketch, shade the (x,y) region which is satisfied by the inequality.

In (a), consider the signs of the two factors, if they are to multiply to a non-positive number.

(a) The boundaries of the region are given by the boundary equation $(y-|x|+1)(y+|x|-1)=0$, which is satisfied when either $y-|x|+1=0$ or $y+|x|-1=0$. These are the usual $\vee$-shaped modulus graphs, so we have a boundary graph consisting of four line segments.

(b) The inequality is satisfied either on one or more of the boundary lines, or when exactly one factor is negative. This gives all regions vertically between the two mod graphs:



1737. The interior angles of a pentagon form an AP. Give, in radians, the upper bound on the value of

(a) the largest angle,

(b) the common difference.

Consider the sum of the angles.

The interior angles of a pentagon sum to $(5-2)\pi = 3\pi$ radians. Hence, the middle angle of the AP is $\frac{3}{5}\pi$ radians. And the smallest angle must be greater than zero. This gives upper bounds (which are not attained) of

(a) $\frac{6}{5}\pi$ for the largest angle,

(b) $\frac{3}{10}\pi$ for the common difference.

1738. Projectiles are launched on Earth and on the Moon, from ground level, with the same initial speed u and angle of inclination θ. On the Moon, the acceleration due to gravity is around $\frac{1}{6}g$. Prove that, if the range of the projectile on Earth is d, then the range on the Moon is around $6d$.

Find the time of flight on Earth in terms of the u

On Earth, we have a time of flight satisfying $0 = u\sin\theta t - \frac{1}{2}gt^2$, so $t = \frac{2u\sin\theta}{g}$. The range, as per the standard formula, is then

$$d = \frac{2u^2\sin\theta\cos\theta}{g}.$$

Hence, scaling the gravitational acceleration g down by a factor $\frac{1}{6}$ produces the reciprocal scaling in the range. Therefore, the range on the moon is around $6d$.

1739. Express $12x^2-2x-13$ in terms of $X = 2x-1$, giving your answer in the form aX^2+bX+c.

Start with a, matching the coefficient 12, and then work your way down the terms.

The coefficient of x^2 requires $a=3$, giving $3X^2 = 12a^2-12a+1$. The coefficient of x, then, requires $b=5$. To match the constant terms, $c=-11$. So, we have $3X^2+5X-11$.

1740. A variable X has distribution $B(10, 0.5)$.

(a) Explain, without doing any calculations, why $P(X=a) = P(X=10-a)$ for any a.

(b) Hence, write down $P(X=2 \mid X\in\{2,8\})$.

In (a), use symmetry. In (b), likewise!

(a) Since the binomial distribution $B(10, 0.5)$ has $p=q=0.5$, it is symmetrical about $x=5$. Therefore, the probability of values either side of that, i.e. a and $10-a$, must be equal.

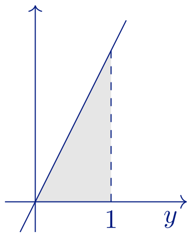
(b) The values 2 and 8 are equally likely. So, the probability of either, having restricted the possibility space to $\{2, 8\}$, is $\frac{1}{2}$.

1741. Find the linear function g for which

$$\int_0^1 g(y)\,dy = 1, \quad g(0) = 0.$$

This can be done graphically, by considering the first statement as one about area, or algebraically, using $g(y) = ay + b$.

Thinking graphically, the integral statement says that the triangle below has area 1:



The triangle's height must be 2, which gives the required function g as $g(y) = 2y$.

1742. An isosceles triangle, whose sides have integer length, has perimeter 18 and area 12.

(a) By defining suitable variables on a diagram, show that this information may be expressed in the simultaneous equations

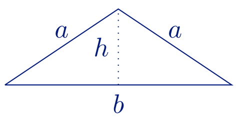
$$\begin{aligned} 2a + b &= 18 \\ b\sqrt{4a^2 - b^2} &= 48. \end{aligned}$$

(b) Show that $b^3 - 9b^2 + 64 = 0$.

(c) Using a numerical method, find the side lengths of the triangle.

In (a), split the triangle in half. In (c), use N-R, or else a calculator solver.

(a) We define a and b as follows:



The perimeter gives $2a + b = 18$. Splitting the base, the height is given by $h^2 = a^2 - \frac{1}{4}b^2$, so

$$\begin{aligned} &\tfrac{1}{2}b\sqrt{a^2 - \tfrac{1}{4}b^2} = 12 \\ \Longrightarrow\ &b\sqrt{4a^2 - b^2} = 48. \end{aligned}$$

(b) The second equation squares to $4a^2 - b^2 = \frac{48^2}{b^2}$. Substituting for a gives

$$\begin{aligned} &4(9 - \tfrac{1}{2}b)^2 - b^2 = \frac{48^2}{b^2} \\ \Longrightarrow\ &324 - 36b = \frac{2304}{b^2} \\ \Longrightarrow\ &324b^2 - 36b^3 = 2304 \\ \Longrightarrow\ &9b^2 - b^3 = 64 \\ \Longrightarrow\ &b^3 - 9b^2 + 64 = 0, \quad \text{as required.} \end{aligned}$$

(c) The Newton-Raphson iteration is

$$b_{n+1} = b_n - \frac{b_n^3 - 9b_n^2 + 64}{3b_n^2 - 18b_n}.$$

Running this iteration with $b_0 = 1$ or $b_0 = 5$, we get a non-integer root. Running it with $b_0 = 10$, we get $b_n \to 8$. This gives $(5, 5, 8)$ as the lengths of the triangle.

1743. If $y = (1 + \cos x)(2 - \cos x)$, find and simplify $\dfrac{dy}{dx}$.

Multiply out before differentiating, or use the product rule.

Using the product rule directly,

$$\begin{aligned} &y = (1 + \cos x)(2 - \cos x) \\ \Longrightarrow\ &\frac{dy}{dx} = -\sin x(2 - \cos x) + (1 + \cos x)\sin x \\ &\quad\;\; = \sin x(2\cos x - 1). \end{aligned}$$

1744. Write the following in terms of a^x:

(a) a^{2x},

(b) a^{2x-1},

(c) a^{1-2x}.

Use index laws.

(a) $a^{2x} = (a^x)^2$,

(b) $a^{2x-1} = \dfrac{(a^x)^2}{a}$,

(c) $a^{1-2x} = \dfrac{a}{(a^x)^2}$.

1745. Factorise $a^3 - b^3$, using the fact that setting $a = b$ makes the expression zero.

Use the factor theorem.

Since $a = b$ is a root of the expression, $(a - b)$ is a factor. This gives

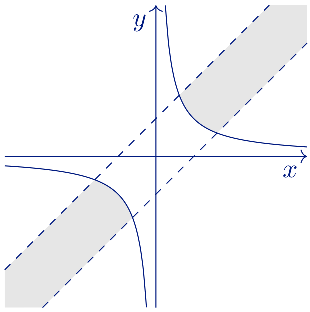
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

1746. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$xy \geq 1, \qquad -1 < x - y < 1.$$

Draw the boundary lines first, and then consider the relevant inequalities.

The boundary equations are $xy = 1$, which is a hyperbola, and $x - y = \pm 1$, which is a pair of straight lines. The region we want is between the straight lines and outside the hyperbola, in either the positive or negative quadrants:



1747. Prove that, if tangent lines are drawn to the parabola $y = x^2 + x$ at $x = \pm k$, then they meet on the y axis.

Find the equations in the form $y = mx + c$, where m and c both depend on k. Show that the two equations have the same value for c.

Differentiating, $\frac{dy}{dx} = 2x + 1$. Hence, the gradient of the tangent at $\pm k$ is $m = \pm 2k + 1$. The first tangent line has gradient $2k+1$ and passes through (k, k^2+k), so, using the formula $y - y_0 = m(x - x_0)$, it has equation

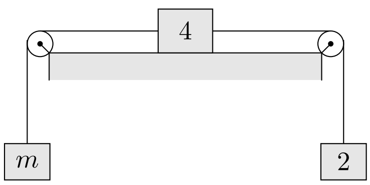
$$y - (k^2 + k) = (2k + 1)(x - k).$$

The y intercept is at $-k^2$. The second tangent line has gradient $-2k + 1$ and passes through $(-k, k^2 - k)$, so it has equation

$$y - (k^2 - k) = (-2k + 1)(x + k).$$

Its y axis intercept is also $-k^2$. Therefore, the two tangents meet on the y axis.

1748. A pulley system of three blocks is set up on a table as shown. Masses are given in kg. The coefficient of friction between the 4 kg block and the table is $\frac{1}{2}$. The system is in equilibrium.



(a) State two assumptions which are necessary to find the range of possible values of m.

(b) Explain we don't need to assume that the strings are inextensible.

(c) Making the necessary assumptions, find, in set notation, all possible values of m.

In (b), consider the fact that the system, being as it in equilibrium, has $a = 0$ throughout.

(a) That the pulleys are smooth and the strings are light.

(b) The system is in equilibrium, so $a = 0$ for all masses. Hence, it doesn't matter whether the strings are potentially extensible or not: we know they are not extending in this scenario.

(c) Vertically, $R = 4g$ for the block on the table, so $F_{\text{max}} = 2g$ in limiting equilibrium. There are two possible cases: equilibrium on the point of sliding rightwards or leftwards. We deal with these case by case, resolving along the taut strings for the whole system. On the point of sliding leftwards:

$$mg - 2g - 2g = 0 \implies m = 4.$$

On the point of sliding rightwards:

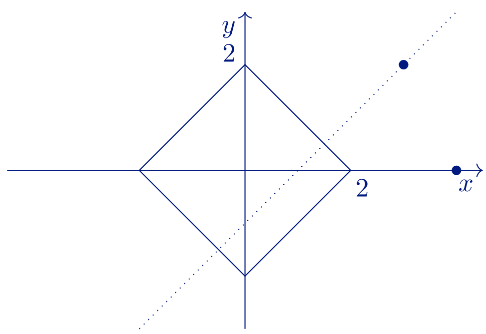
$$2g - 2g - mg = 0 \implies m = 0.$$

Hence, since it would require $m < 0$ to make the system slide rightwards, the set of possible values of m is $[0, 4]$ kg.

1749. Determine which of the points $(3, 2)$ and $(4, 0)$ is closer to the square $|x| + |y| = 2$.

Sketch the square carefully, considering first the quadrant with $x, y > 0$.

The square consists of $x + y = 2$ in the positive quadrant, and then similar lines in the other three quadrants. So, the scenario is

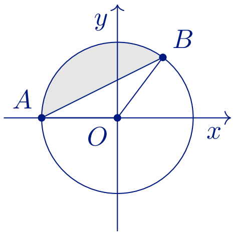


The distance from $(4, 0)$ to the square is clearly 2. The distance from $(3, 2)$ is along a perpendicular, which is the line $y = x+1$. Solving simultaneously, the closest point is $(\frac{3}{2}, \frac{1}{2})$. This gives the distance as $\sqrt{4.5} > 2$. Hence, $(4, 0)$ is closer.

1750. A chord is drawn on the unit circle between the points $(\frac{3}{5}, \frac{4}{5})$ and $(-1, 0)$, dividing the circle into major and minor segments. Show that the area of the minor segment is approximately 0.71.

Find the angle subtended at the centre, using the cosine rule. Then calculate the area of the sector and subtract the area of the triangle.

The scenario is as follows:



Triangle AOB has lengths $(1, 1, \sqrt{3.2})$. The angle subtended at the centre is 2.214 radians, and the area of the triangle is $\frac{1}{2}\sin 2.214 = 0.4$. The area of sector AOB, then, is $\frac{1}{2}\cdot 1^2 \cdot 2.214 = 1.107$. So, the area of the segment is $1.107 - 0.4 = 0.707 \approx 0.71$.

1751. Without a calculator, evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 4x - 4\sin x\, dx$.

Integrate and use exact trig values.

Using the standard trig values,

$$\begin{aligned}
&\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 4x - 4\sin x\, dx \\
&= \Big[2x^2 + 4\cos x\Big]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\
&= \left(\frac{8\pi^2}{9} + 4\cos\frac{2\pi}{3}\right) - \left(\frac{2\pi^2}{9} + 4\cos\frac{\pi}{3}\right) \\
&= \frac{6\pi^2}{9} - 2 - 2 \\
&= \frac{2\pi^2}{3} - 4.
\end{aligned}$$

1752. You are given that the variables x, y, z satisfy

$$z = ax + by + c,$$

for constants a, b. Determine whether the following statements are true:

(a) z and x are linearly related,

(b) if x and y are linearly related, then z and x are linearly related.

In (b), consider a linear relationship $y = px + q$.

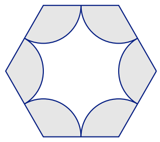
(a) This isn't true. Variable y may depend on variable x in a non-linear way. A counterexample is $y = x^2$, which gives $z = ax + bx^2$, which is quadratic in x.

(b) This is true. If x and y are linearly related, then $y = px + q$ for some constants p and q. This gives $z = ax + b(px + q)$, which we can write as $z = (a + bp)x + bq$. Since $(a + bp)$ and bq are constants, this is a linear relationship.

1753. A regular hexagon of side length 2 cm is drawn. Find, in exact form, the fraction of its area which is within 1 cm of a vertex.

Sketch the scenario, and notice that the areas of the six sectors generated add up to a whole number of circles.

The hexagon has area $6\sqrt{3}$. We need to find the shaded area:



Three of the shaded sectors sums to one circle. The radius is 1, so this gives a total of 2π shaded. Hence, the fraction is

$$\frac{2\pi}{6\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

1754. At meetings of a friendly society, everyone shakes hands with everyone.

(a) Find the total number of handshakes when 10 people are present.

(b) Find the total number of people present if there are 1485 handshakes.

Use ${}^nC_2 = \frac{1}{2}n(n-1)$.

(a) The number of handshakes is the number of ways of choosing two people to shake hands. This gives ${}^{10}C_2 = 45$.

(b) With n people, there are ${}^nC_2$ handshakes. Using the factorial formula for ${}^nC_r$, this is $\frac{1}{2}n(n-1)$. So, we solve

$$\begin{aligned}
&\tfrac{1}{2}n(n-1) = 1485 \\
\Longrightarrow\ & n^2 - n - 2970 = 0 \\
\Longrightarrow\ & n = -54, 55
\end{aligned}$$

Therefore, $n = 55$.

1755. The line $2x+3y+5=0$ is closest to $(\frac{3}{2},6)$ at point Q. Find the coordinates of point Q.

Use the fact that the shortest path must be along a perpendicular to the line. Alternatively, minimise the squared distance using calculus.

The shortest path is perpendicular to the line. So, we need the perpendicular to $2x+3y+5=0$ through $(\frac{3}{2},6)$. This is $3x-2y+\frac{15}{2}=0$. Solving simultaneously, the lines intersect at $Q:(-\frac{5}{2},0)$.

1756. Show that, if the relationship between x and y is polynomial, of the form $y=bx^n$, then $\ln y$ and $\ln x$ are related linearly.

Take natural logs of both sides, and use log laws.

Taking natural logs of both sides,

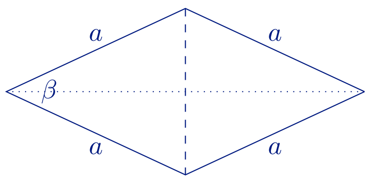
$$\begin{aligned} & y = bx^n \\ \Longrightarrow\ & \ln y = \ln(bx^n) \\ \Longrightarrow\ & \ln y = \ln b + \ln x^n \\ \Longrightarrow\ & \ln y = \ln b + n\ln x. \end{aligned}$$

This is a linear relationship between $\ln y$ and $\ln x$, as required.

1757. Prove that the diagonals of a rhombus have lengths d_1, d_2 given by $d_1, d_2 = a\sqrt{2\pm 2\cos\beta}$, where a is the side length and β is any interior angle.

Use the fact that $\cos(180° - \beta) \equiv -\cos\beta$.

The rhombus is



Using the cosine rule, the vertical diagonal has length given by

$$\begin{aligned} & d_1^2 = a^2 + a^2 - 2a^2\cos\beta \\ \Longrightarrow\ & d_1 = a\sqrt{2-2\cos\beta}. \end{aligned}$$

The other interior angle is given, then, by $180° - \beta$. Using the identity $\cos(180° - \beta) \equiv -\cos\beta$ (easily proved on the unit circle), the horizontal diagonal has length given by

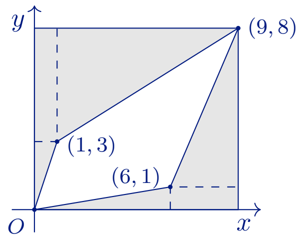
$$\begin{aligned} & d_2^2 = a^2 + a^2 - 2a^2\cos(180° - \beta) \\ \Longrightarrow\ & d_2 = a\sqrt{2-2\cos(180° - \beta)} \\ \Longrightarrow\ & d_2 = a\sqrt{2+2\cos\beta}, \end{aligned}$$

giving $d_1, d_2 = a\sqrt{2\pm 2\cos\beta}$ overall.

1758. A convex quadrilateral has vertices at $(0,0)$, $(1,3)$, $(9,8)$, and $(6,1)$. Find its area.

There are many ways of finding the area. Perhaps easiest is to bound the quadrilateral in a rectangle whose sides are parallel to the x and y axes.

We bound the quadrilateral in a rectangle whose sides are parallel to the x and y axes:



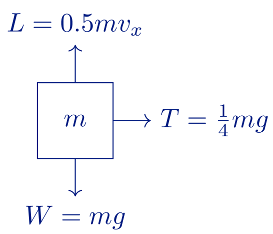
The bounding rectangle has area $8\times 9 = 72$. We then have two unwanted rectangles with areas 5 and 3, and four unwanted triangles with areas (clockwise from left) $\frac{3}{2}, 20, \frac{21}{2}, 3$. Hence, the area of the quadrilateral is $72-5-3-\frac{3}{2}-20-\frac{21}{2}-3=29$ square units.

1759. Take $g=10$ ms^{-2} in this question.

A piloted glider of mass m is being accelerated by a cable attached to a winch. The tension in the cable is modelled as a constant $\frac{1}{4}mg$. The glider starts from rest, 100 metres from the winch, and experiences lift proportional to its horizontal speed v_x, given by $L=0.5mv_x$. Making any necessary assumptions, show that the glider takes off at 20 ms^{-1}, and determine the distance from the winch at which this happens.

Draw a force diagram and consider both horizontal and vertical $F=ma$.

At the last instant for which the wheels are in contact with the ground, we model both the reaction force and the vertical acceleration as being zero. At that point, a force diagram for the glider is



Vertically, resultant force is zero (the plane is on the point of leaving the ground, but hasn't yet left it), so $0.5mv_x - mg = 0$. Solving this gives $v_x = 20$ ms^{-1}. Horizontally, the acceleration is $\frac{1}{4}g = 2.5$

ms^{-2}, $u = 0$ and $v = 20$. So, *suvat* gives

$$\begin{aligned} 20^2 &= 0^2 + 2 \cdot 2.5 s_x \\ \Longrightarrow \quad s_x &= 80\text{m}. \end{aligned}$$

Hence, the glider takes off 20 m from the winch.

1760. Solve for x in $\displaystyle\sum_{r=1}^{\infty} x^r = \frac{1}{2}$.

This is an infinite geometric series.

The sum is an infinite geometric series, with first term x and common ratio x.

$$\begin{aligned} & \sum_{r=1}^{\infty} x^r = \frac{1}{2} \\ \Longrightarrow \quad & \frac{x}{1-x} = \frac{1}{2} \\ \Longrightarrow \quad & x = \frac{1}{3}. \end{aligned}$$

1761. State, with a reason, whether the following holds: "In any two-tail test for the mean of a normal distribution, the acceptance region is symmetrical about the hypothesised population mean μ."

Sketch the normal distribution.

This is true. A normal distribution is symmetrical about μ, and a two-tail test must be symmetrical by definition. This means that both the critical and acceptance regions must be symmetrical about μ. This isn't true for other distributions, e.g. the binomial, nor for one-tail tests.

1762. You are given that $f'(x) = g(x)$, and that the equation $f(x) = 0$ has roots at $x = \alpha$ and $x = \beta$. Evaluate the following:

(a) $\displaystyle\int_{\alpha}^{\beta} g(x)\, dx$.

(b) $\displaystyle\int_{\alpha}^{\beta} \frac{2x + g(x)}{3}\, dx$.

In (b), integrate term by term. You can take the factor of $\frac{1}{3}$, or leave it where it is.

(a) Since $f(x) = 0$ has roots at $x = \alpha$ and $b = \beta$, we know that $f(\alpha) = f(\beta) = 0$. Hence,

$$\begin{aligned} \int_{\alpha}^{\beta} g(x)\, dx &= \Big[f(x) \Big]_{\alpha}^{\beta} \\ &= f(\alpha) - f(\beta) \\ &= 0. \end{aligned}$$

(b) Likewise,

$$\begin{aligned} \int_{\alpha}^{\beta} \frac{2x + g(x)}{3}\, dx &= \left[\frac{x^2 + f(x)}{3} \right]_{\alpha}^{\beta} \\ &= \frac{\alpha^2 + f(\alpha)}{3} - \frac{\beta^2 + f(\beta)}{3} \\ &= \tfrac{1}{3}(\alpha^2 - \beta^2). \end{aligned}$$

1763. The numbers $1, 2, 3, 4, 5, 6$ are randomly assigned, each to a different vertex of a regular hexagon. Find the probability that the numbers appear in either ascending or descending order around the perimeter.

Place number 1 first, without loss of generality. Then consider the placement of the number 2. Once the 1 and 2 are placed, there is only one successful outcome remaining.

The number 1 can be placed somewhere without loss of generality. Consider, then, the number 2. The probability that its placement allows for an ascending or descending sequence is $\frac{2}{5}$, as it can occupy either of the spaces next to the 1. Once these two are placed, there is only one successful location for the other numbers. This gives

$$p = \tfrac{2}{5} \times \tfrac{1}{4} \times \tfrac{1}{3} \times \tfrac{1}{2} \times \tfrac{1}{1} = \tfrac{1}{60}.$$

1764. The positions of two particles, for $t \geq 0$, are given by $x_1 = t^2 - 3t$ and $x_2 = 4 - t^2$.

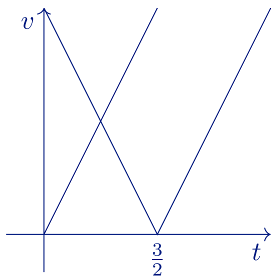
(a) Explain why $|2t - 3|$ gives the speed of the first particle, and find a similar expression for the speed of the second.

(b) Determine the t values for which the first particle is moving faster than the second.

In (a), the mod function calculates the magnitude. In (b), consider sketching two mod graphs.

(a) Differentiating gives the velocity as $v_1 = 2t - 3$. The speed is the magnitude of this quantity, which is encoded algebraically as $v_1 = |2t - 3|$. The speed of the second particle is given by $v_2 = |-2t|$.

(b) Sketching the speed-time graphs:



By symmetry, the point of intersection is halfway to $\frac{3}{2}$, at $t = \frac{3}{4}$. Hence, the first particle is moving faster for $t \in [0, \frac{3}{4})$.

1765. A quartic is defined by $y - k = kx^4 - kx^2$, for some constant $k \neq 0$. Show that this curve has stationary points, two of which have the same y coordinate.

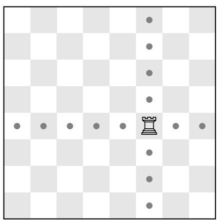
Rearrange to $y = ...$ and take out a factor of k. Then set the first derivative to zero and solve.

The curve is $y = k(x^4 - x^2 + 1)$. For stationary points, we set the first derivative to zero, giving

$$\begin{aligned} & k(4x^3 - 2x) = 0 \\ \implies & x(2x^2 - 1) = 0 \\ \implies & x = 0, \pm\frac{1}{\sqrt{2}}. \end{aligned}$$

Substituting these back in, we have stationary points at $(0, k)$ and $(\pm\frac{1}{\sqrt{2}}, \frac{3k}{4})$. The latter pair have the same y coordinate. This is due to the even symmetry of the graph, and would be true of any curve of the form $y = ax^4 + bx^2 + c$ with a stationary point at $x \neq 0$.

1766. In chess, a rook threatens squares as shown. Find the probability that, if two rooks are placed on a chessboard at random, they threaten each other.



The rows and columns are, as far as the rook is concerned, symmetrical. So, the first rook can be placed anywhere, without loss of generality.

The rows and columns are, as far as the rook is concerned, symmetrical. So, the first rook can be placed anywhere, without loss of generality. The first rook threatens 14 squares. Hence, with 63 squares remaining, the probability that the second rook is placed on one of the threatened squares is $\frac{14}{63} = \frac{2}{9}$.

1767. Show that the area of the region enclosed by the curves $y = x^4 + x^2$ and $y = x^3 + x$ is $\frac{13}{60}$.

Find the intersections of the curves, then set up a single definite integral with integrand $y_1 - y_2$.

Solving to find intersections, we have

$$\begin{aligned} & x^4 + x^2 = x^3 + x \\ \implies & x^4 - x^3 + x^2 - x = 0 \\ \implies & x(x^3 - x^2 + x - 1) = 0 \\ \implies & x(x-1)(x^2+1) = 0. \end{aligned}$$

The quadratic factor has no roots, so the curves intersect exactly twice. Hence, the area between them is given, up to a sign, by

$$\begin{aligned} & \int_0^1 x^4 + x^2 - x^3 - x \, dx \\ = & \left[\tfrac{1}{5}x^5 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_0^1 \\ = & \left(\tfrac{1}{5} + \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{2}\right) - (0) \\ = & -\tfrac{13}{60}. \end{aligned}$$

The signed area has come out negative, signifying that, in the previous calculation, we subtracted (not knowing which was which) the upper curve from the lower. The area of the region enclosed by the curves, therefore, is $\frac{13}{60}$.

1768. On any given day in winter, the probability that I wear a jacket is 83%, and the probability that it rains at some stage is 40%. On days when it rains at some stage, the probability that I wear a jacket is 95%.

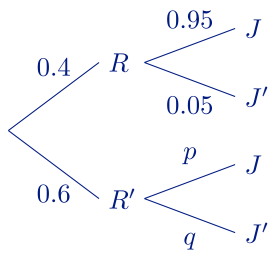
(a) Draw a tree diagram, conditioned on rain/no rain, to represent this information.

(b) Find the probability that it doesn't rain, but I still wear a jacket.

(c) Determine the probability that it rains at some stage, given that it is a day on which I am wearing a jacket.

The probability in (b) is absolute, the probability in (c) is conditional. Hence, in (c), restrict the possibility space accordingly.

(a) The tree diagram is



(b) Since $P(J) = 0.83$, we have

$$\begin{aligned} & 0.83 = 0.4 \times 0.95 + 0.6p \\ \implies & p = 0.75. \end{aligned}$$

(c) To calculate $P(R \mid J)$, we restrict the possibility space to the first and third branches, and calculate the probability of the first:

$$P(R \mid J) = \frac{0.4 \times 0.95}{0.4 \times 0.95 + 0.6 \times 0.75} = \frac{38}{83}.$$

1769. Prove that no Pythagorean triple is isosceles. N.B. a Pythagorean triple consists of integers.

Prove this by contradiction. Start with "Assume, for a contradiction, that (a, a, b) is a Pythagorean triple."

Assume, for a contradiction, that (a, a, b) is a Pythagorean triple. So, $a, b \in \mathbb{N}$ and, since the hypotenuse must always be longer than the other two sides, $a^2 + a^2 = b^2$, i.e. $2a^2 = b^2$. The RHS is a square, so must have an even number of prime factors of 2. By the same logic, the LHS must have an odd number of prime factors of 2, since it has one more than a square. This is a contradiction. Hence, no Pythagorean triple is isosceles. □

1770. A slate of mass m is attached to a roof inclined at $\theta°$ to the horizontal. Give the magnitudes of the following forces on the slate, in terms of m, g, θ:

(a) the contact force parallel to the roof,

(b) the contact force perpendicular to the roof,

(c) the total contact force.

Parts (a) and (b) are the components of the total force in (c).

Parts (a) and (b) are the components of the total force in (c):

(a) $mg \sin\theta$,

(b) $mg \cos\theta$,

(c) mg.

1771. The pair of simultaneous equations $2x + ay = 3$ and $4x = 2y + b$ has infinitely many points (x, y) which satisfy it. Find a and b.

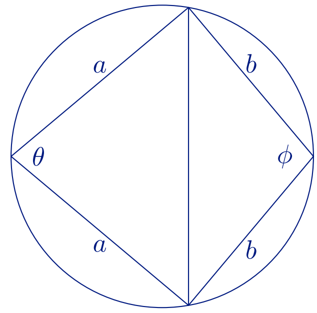
Since there are infinitely many points satisfying the equations, the two lines must be the same.

Since there are infinitely many points satisfying the equations, the two lines must be the same. The scale factor is 2, giving $a = -1$ and $b = 6$.

1772. Quadrilateral K is a kite, and it is cyclic. It has side lengths a and b, and the angle between the two sides of length a is θ. Prove that the area of K is given by $A = \frac{1}{2}(a^2 + b^2)\sin\theta$.

Use the cyclic quadilateral circle theorem, and the fact that $\sin(180° - \theta) \equiv \sin\theta$.

We can split the kite into two triangles as follows:



Using the sine area formula on the two triangles, the area of the kite is given by

$$A = \tfrac{1}{2}a^2 \sin\theta + \tfrac{1}{2}b^2 \sin\phi.$$

And, since the kite is cyclic, $\phi = 180° - \theta$. This gives $\sin\phi = \sin\theta$. Therefore, the area of the kite simplifies to

$$A = \tfrac{1}{2}(a^2 + b^2)\sin\theta,$$

as required.

1773. The ellipse $3x^2 + y^2 = 1$ is transformed to a second ellipse $(x + 2)^2 + 3(y - 2)^2 = 1$ by reflection in a straight line. Find the equation of the line.

You don't need to think about the ellipses here; determining the image of a single point is enough. You might want to consider $x^2 + 3y^2 = 1$ as a stepping stone.

To begin with, we can see this as two successive transformations: reflection in $y = x$, which switches the coordinates to $x^2 + 3y^2 = 1$, followed by translation by vector $\binom{-2}{2}$. Analysis of any particular point is enough to determine the equation of the line. For example, $(0, 1)$ transforms to $(1, 0)$, then to $(-1, 2)$. Hence, we have reflection in the perpendicular bisector of $(0, 1)$ and $(-1, 2)$, which is $y = x + 2$.

1774. A researcher is analysing historical data, looking at correlation between extreme weather events and incidences of civil unrest. She logs the numbers w_i and c_i of each in five randomly selected years.

w_i	73	30	19	84	66
c_i	6	4	4	12	11

She then sets up a hypothesis test, using critical values from the following entry in the usual table:

1-tail	5%	2.5%	1%	0.5%
2-tail	10%	5%	2%	1%
$n = 5$	0.8054	0.8783	0.9343	0.9587.

(a) Verify, using the statistical functions on a calculator, that the sample correlation coefficient is approximately 0.82.

(b) Defining the variable you use, write down suitable hypotheses for the researcher's test.

(c) Carry out the test, at the 5% level, to ascertain whether or not there is evidence for correlation between the two variables.

(d) Such a test requires the assumption that the underlying population has a bivariate normal distribution. Comment on whether this assumption is appropriate or not.

In (b), make sure that the hypotheses refer to ρ, the correlation coefficient in the population, not r. In (c), remember not to be too certain in your conclusion.

(a) Using a calculator, $r = 0.8154... \approx 0.82$.

(b) Let ρ be the correlation coefficient between the variables w and c in the population of years. The hypotheses are

$$\mathrm{H}_0 : \rho = 0 \qquad \mathrm{H}_1 : \rho \neq 0.$$

(c) This is a two-tail test at the 5% significance level. The critical value is 0.8783. Since $0.8154 < 0.8783$, there is insufficient evidence to reject H_0. The researcher cannot claim to have found significant evidence for correlation between extreme weather and civil unrest.

(d) This is certainly not a reasonable application of such a test. By their one-off nature, it is certain that neither extreme weather events nor civil unrest follow normal distributions. Hence, the assumption that the underlying population in this test is bivariate normal is not appropriate. The result of such a test will not be useful to the researcher.

1775. The points on a straight line are described, in terms of a parameter $t \in \mathbb{R}$, by a position vector

$$\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

Find the Cartesian equation of the line.

Use the vector $\binom{3}{6}$ to calculate the gradient, then sub $(x_0, y_0) = (1, 5)$ into $y - y_0 = m(x - x_0)$.

The direction vector of the line is $\binom{3}{6}$. Hence, the gradient is $\frac{6}{3} = 2$. When $t = 0$, the line passes through $(1, 5)$. Hence, the equation of the line is $y - 5 = 2(x - 1)$, which simplifies to $y = 2x + 3$.

1776. Functions f and g are given, for constants a, b, by

$$\begin{aligned} f(x) &= 4x^2 - 10x + a, \\ g(x) &= x^2 + 6x + b. \end{aligned}$$

The solution set of the inequality $f(x) \geq g(x)$ is $(-\infty, -\frac{2}{3}] \cup [6, \infty)$. Determine the value of $b - a$.

Using the solution set, determine the boundary equation, in the form $3x^2 + px + q = 0$. Use the value of q to find $b - a$.

The boundary equation $f(x) - g(x) = 0$ is a quadratic with roots at $x = -\frac{2}{3}$ and $x = 6$. Hence, $f(x) - g(x) = 0$ may be written as

$$\begin{aligned} & (3x + 2)(x - 6) = 0 \\ \implies & 3x^2 - 16x - 12 = 0. \end{aligned}$$

We can also, by subtracting the given functions, write this as $3x^2 - 16x + a - b = 0$. Hence, $a - b = -12$, giving $b - a = 12$.

1777. Prove that, if two tangent lines are drawn to the parabola $y = 3x^2 - 5x + 2$ at $x = \pm a$, then they meet on the y axis.

Find, in terms of a, the gradient $m_\pm$ at the generic points $(\pm a, 3a^2 \mp 5a + 2)$. Substituting these points into $y = m_\pm x + c_\pm$, find $c_\pm$ in terms of a. You need to show that the $c_\pm$ values for each tangent line are the same.

Differentiating, $\frac{dy}{dx} = 6x - 5$, so the gradients are $\pm 6a - 5$. The relevant points are $(\pm a, 3a^2 \mp 5a + 2)$. Substituting these into $y = m_\pm x + c_\pm$, we get

$$\begin{aligned} & 3a^2 \mp 5a + 2 = (\pm 6a - 5)(\pm a) + c_\pm \\ \implies & c_\pm = 3a^2 \mp 5a + 2 - (\pm 6a - 5)(\pm a) \\ \implies & c_\pm = 3a^2 \mp 5a + 2 - 6a^2 \pm 5a \\ \implies & c_\pm = -3a^2 + 2. \end{aligned}$$

Since the y intercept is the same for both tangent lines, they meet on the y axis.

1778. Here, use the $\mu \pm 2\sigma$ definition of an outlier.

A random sample of ten data is taken from a large population, which is normally distributed. Show that, in such a sample, the probability of there being at least one outlier is around 37%.

Using a cumulative normal distribution, calculate the probability that any one datum is not an outlier. Then consider $P(\text{at least one outlier}) = 1 - P(\text{no outliers})$.

The probability that a datum is not classified as an outlier is, according to the 2 standard deviation definition, $p = 0.9545$. Hence, the probability that a sample of 10 contains at least one outlier is

$$\begin{aligned} P(\text{at least one outlier}) &= 1 - P(\text{no outliers}) \\ &= 1 - 0.9545^{10} \\ &= 0.372289... \\ &\approx 37\%. \end{aligned}$$

1779. Solve the equation $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x^2}} = x$.

The best way of dealing with inlaid fractions is to multiply top and bottom of the main fractions by the denominator(s) of the inlaid fractions. [Here, you might alternatively spot the difference of two squares immediately.]

Multiplying top and bottom by x^2, we have

$$\begin{aligned} & \frac{x^2+x}{x^2-1} = x \\ \implies & \frac{x(x+1)}{(x+1)(x-1)} = x \\ \implies & \frac{x}{x-1} = x \\ \implies & 0 = x(x-2) \\ \implies & x = 0, 2. \end{aligned}$$

However, $x = 0$ is not a root; we introduced it by top and bottom by x^2. So, the solution is $x = 2$.

1780. The curve $y = \dfrac{ax+b}{cx}$ is defined, with $c \neq 0$.

(a) Show that the curve may be written $y = p + \frac{q}{x}$, where p and q are rational constants to be determined.

(b) Hence, write down the two transformations that turn the graph $y = \frac{1}{x}$ into this curve.

(c) Write down the equations of the horizontal and vertical asymptotes of the curve.

In (b), the transformations are a stretch and a translation, both in the y direction.

(a) Splitting the fraction up, we have

$$y = \frac{a}{c} + \frac{b}{cx}.$$

So $p = \frac{a}{c}$, $q = \frac{b}{c}$.

(b) The transformations are a stretch, scale factor $\frac{b}{c}$, in the y direction, followed by a translation by vector $\frac{a}{c}\mathbf{j}$.

(c) The untransformed graph $y = x$ has the axes $x = 0$ and $y = 0$ as its asymptotes. The axis $x = 0$ is unaffected by the transformations, while $y = 0$ is transformed to $y = \frac{a}{c}$.

1781. Solve $3^{-2x} + 3^{-x} = 30$, giving your answer(s) in the form $y = \log_a b$, for $a, b \in \mathbb{N}$.

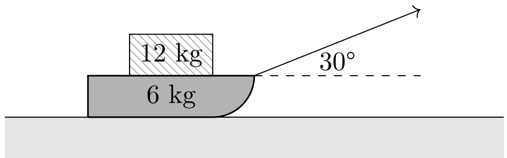
This is a quadratic in 3^{-x}.

This is a quadratic in 3^{-x}:

$$\begin{aligned} & 3^{-2x} + 3^{-x} - 30 = 0 \\ \implies & (3^{-x} - 5)(3^{-x} + 6) = 0 \\ \implies & 3^{-x} = 5, -6. \end{aligned}$$

The latter has no roots, so we have $3^{-x} = 5$. This gives $x = -\log_3 5$, which can be rewritten as $x = \log_5 3$.

1782. A sledge, carrying a 12 kg load of equipment, is being pulled along at constant velocity by means of a rope angled at 30° above the horizontal. The ground may be modelled as smooth; air resistance of magnitude 25 N is acting on the load.



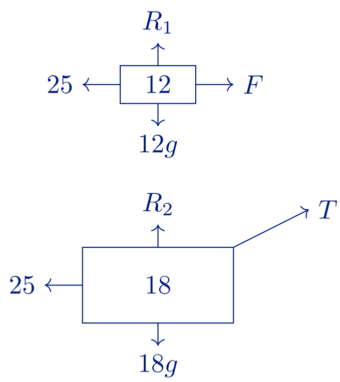
(a) Draw force diagrams for the load and for the combined sledge and load.

(b) Find the tension in the rope.

(c) Find the least possible value of the coefficient of friction between the load and the sledge.

In (b), resolve horizontally for the combined sledge and load. In (c), consider the load alone.

(a) The force diagrams are:



(b) Resolving horizontally for the combined sledge and load, $T\cos 30° - 25 = 0$, giving $T = 28.9$ N (3sf).

(c) Resolving vertically for the load, $R_1 = 12g$. When the load is just about to slip, friction is at $F_{\text{max}} = \mu R_1$. This gives $\mu \times 12g - 25 = 0$, so $\mu_{\text{min}} = 0.213$ (3sf).

1783. Show that the following is an identity, identifying all values of x for which the identity cannot be defined:

$$\frac{x^2+3x+2}{3x^3+4x^2+x} = \frac{2}{x} - \frac{5}{3x+1}.$$

Start with the LHS, and factorise.

Starting with the LHS,

$$\begin{aligned}&\frac{x^2+3x+2}{3x^3+4x^2+x}\\ \equiv\ &\frac{(x+1)(x+2)}{x(3x+1)(x+1)}\\ \equiv\ &\frac{x+2}{x(3x+1)}, \quad \text{assuming } x \neq -1.\end{aligned}$$

Putting the RHS over a common denominator,

$$\begin{aligned}&\frac{2}{x} - \frac{5}{3x+1}\\ \equiv\ &\frac{2(3x+1)-5x}{x(3x+1)}\\ \equiv\ &\frac{x+2}{x(3x+1)}.\end{aligned}$$

This proves the identity for all real numbers except $x = 0$ and $x = -\frac{1}{3}$, where both sides are undefined, and $x = -1$, where the LHS is undefined.

1784. Find a rational number in the interval $(\pi, \frac{101}{100}\pi)$.

There are infinitely many to choose from! To find one, take the decimal expansions of the boundaries to e.g. 2dp, and convert them into fractions.

To 2dp, these bounds of this interval are 3.14 and 3.17. As fractions, these are $\frac{314}{100}$ and $\frac{317}{100}$. Hence, the rationals $\frac{315}{100}$ and $\frac{316}{100}$ fulfil our requirements.

1785. In each case, a list of quantities is given, which refers to a finite geometric series. State, with a reason, whether knowing the relevant quantities would allow you to calculate the sum of the series.

(a) First term; last term; number of terms.

(b) Number of terms; mean.

(c) First term; last term; mean.

The main indicator here is the number of pieces of information you are given.

(a) Yes. This combination gives you the common ratio, and thence the sum.

(b) No. While this combination would give you to sum for an AP, for a GP it doesn't. The common ratio isn't fixed by this information.

(c) Yes. Any three pieces of information, so long as they are not redundant, is sufficient to calculate all of the terms, and therefore the sum, of a geometric series.

1786. Simplify $\dfrac{d}{dt}(\sqrt{x} + \sqrt[3]{y})$.

Both terms must be differentiated using the chain rule, as per implicit differentiation. In this question, both x and y should be thought of as implicitly depending on t.

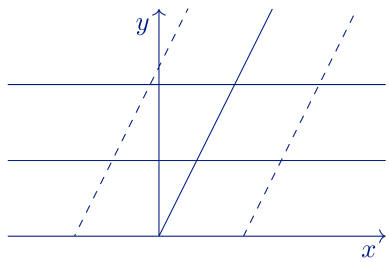
Differentiating implicitly by the chain rule,

$$\begin{aligned}&\frac{d}{dt}(\sqrt{x} + \sqrt[3]{y})\\ =\ &\frac{d}{dt}(x^{\frac{1}{2}} + y^{\frac{1}{3}})\\ =\ &\tfrac{1}{2}x^{-\frac{1}{2}}\frac{dx}{dt} + \tfrac{1}{3}y^{-\frac{2}{3}}\frac{dy}{dt}.\end{aligned}$$

1787. Three lines have equations $y = 2x$, $y = 1$ and $y = 2$. Find the possible equations of a fourth line, if it is to form a rhombus with the first three.

Sketch the scenario carefully. The consider the fourth line in the form $y = 2x + c$. Find c to make the lengths match.

The two possible positions for the fourth line are the dashed lines below:



The distance along $y = 2x$ between $y = 1$ and $y = 2$ is given by $d = \sqrt{0.5^2 + 1^2} = \frac{1}{2}\sqrt{5}$. Hence, we must translate $y = 2x$ by vector $\pm\sqrt{5}\mathbf{i}$. This gives $y = 2(x \mp \sqrt{5})$. The two lines, therefore, are $y = 2x + \sqrt{5}$ and $y = 2x - \sqrt{5}$.

1788. Given the distribution $X \sim N(0, 4)$, find

(a) $P(X \in (1, 2))$,

(b) $P(X \in \{1,2\})$,

(c) $P(X \in [1,2])$.

Two of the answers are the same; the other can be written down immediately.

Since this is a continuous distribution, the answer to (b) is zero automatically. Hence, the answers to (a) and (c) are the same. Using a calculator, to 3sf,

(a) $P(X \in (1,2)) = 0.150$,

(b) $P(X \in \{1,2\}) = 0$,

(c) $P(X \in [1,2]) = 0.150$.

1789. A triangle has perimeter 9 and two angles $\arccos \frac{7}{8}$ and $\arccos \frac{11}{16}$. Solve to find the side lengths.

Define x to be the side opposite $\arccos \frac{7}{8}$. Then calculate the other lengths in terms of x, using the sine rule. Equate the sum to 9 and solve.

Defining x to be the side opposite $\arccos \frac{7}{8}$, we can use the sine rule to calculate the other two sides. The side opposite $\arccos \frac{11}{16}$ is given by

$$y = \frac{\sin(\arccos \frac{11}{16})}{\sin(\arccos \frac{7}{8})} x = \frac{3}{2}x.$$

And, using the fact that the angles in a triangle add up to 180°,

$$z = \frac{\sin(180° - \arccos \frac{7}{8} - \arccos \frac{11}{16})}{\sin(\arccos \frac{7}{8})} x = 2x.$$

Using the perimeter, $x + \frac{3}{2}x + 2x = 9$, giving the side lengths as $(2, 3, 4)$.

1790. Two unit vectors $\mathbf{a}$ and $\mathbf{b}$ can be written in terms of the standard perpendicular unit vectors as

$$\begin{aligned} \mathbf{a} &= p\mathbf{i} + q\mathbf{j}, \\ \mathbf{b} &= q\mathbf{i} - p\mathbf{j}. \end{aligned}$$

(a) Write down the value of $p^2 + q^2$.

(b) Express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{a}, \mathbf{b}$ and p, q.

(c) Show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

(d) Sketch $\mathbf{a}$ and $\mathbf{b}$, for $p = \frac{1}{2}$ and positive q.

In (b), solve simultaneously for $\mathbf{i}$ and $\mathbf{j}$. In (c), calculate the gradients.

(a) Since a and b have unit length, $p^2 + q^2 = 1$.

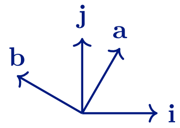
(b) Solving for $\mathbf{j}$ by elimination,

$$\begin{aligned} q\mathbf{a} &= pq\mathbf{i} + q^2\mathbf{j}, \\ p\mathbf{b} &= pq\mathbf{i} - p^2\mathbf{j}. \end{aligned}$$

This gives $q\mathbf{a} - p\mathbf{b} + (q^2 + p^2)\mathbf{j}$. Hence, using part (a), $\mathbf{j} = p\mathbf{b} - q\mathbf{a}$. Substituting back in gives $\mathbf{i} = p\mathbf{a} + q\mathbf{b}$.

(c) The gradients of vectors $\mathbf{a}$ and $\mathbf{b}$ are $\frac{q}{p}$ and $-\frac{p}{q}$, which are negative reciprocals.

(d) This is a rotation by angle 60°:



1791. Evaluate $\displaystyle\lim_{x \to \frac{1}{3}} \frac{6x^2 + 13x - 5}{12x^2 - 7x + 1}$.

Put the fraction in its lowest terms before taking the limit.

Numerator and denominator are zero at $x = \frac{1}{3}$, so we must cancel factors of $(3x - 1)$ before taking the limit:

$$\begin{aligned} &\lim_{x \to \frac{1}{3}} \frac{6x^2 + 13x - 5}{12x^2 - 7x + 1} \\ &= \lim_{x \to \frac{1}{3}} \frac{(3x-1)(2x+5)}{(3x-1)(4x-1)} \\ &= \lim_{x \to \frac{1}{3}} \frac{2x+5}{4x-1} \\ &= \frac{2 \cdot \frac{1}{3} + 5}{4 \cdot \frac{1}{3} - 1} \\ &= 17. \end{aligned}$$

1792. From a large population, a sample of five is taken. Find the probability that all five lie between the quartiles of the population.

Since the population is large, the probability that any particular datum sampled from it lies between the quartiles is 50%.

Since the population is large, the probability that any particular datum sampled from it lies between the quartiles is 50%. So, we have a binomial distribution $X \sim B(5, 0.5)$. The required probability is $0.5^5 = \frac{1}{32}$.

1793. Prove that, if a, b, c, d are positive numbers in arithmetic progression, then

$$\frac{d^2 - a^2}{c^2 - b^2} = 3.$$

Use a difference of two squares. Then consider the symmetry of an AP around its mean.

Factorising the LHS, we get

$$\frac{d^2 - a^2}{c^2 - b^2} = \frac{(d+a)(d-a)}{(c+b)(c-b)}.$$

Since an AP is symmetrically distributed about its mean, $(d+a)=(c+b)$. Furthermore, $(d-a)$ is three common differences, while $(c-b)$ is one; therefore $(d-a)=3(c-b)$. Cancelling the algebraic factors yields the required result.

1794. State whether each of the following implications hold, for $a,b\in\mathbb{R}$. If not, give a counterexample.

(a) $a>b \implies a^2>b^2$,

(b) $a>b \implies -1+2a>-1+2b$,

(c) $a>b \implies \dfrac{1}{a}>\dfrac{1}{b}$.

The implications in (a) and (c) aren't true.

(a) False: $a=-1$, $b=-2$ is a counterexample.

(b) True: an inequality is preserved under linear transformations $mx+c$ with $m>0$.

(c) False: $a=3$, $b=2$ is a counterexample.

1795. The graph $y=x^2-6x+k$, where k is a constant, crosses the x axis twice. The distance between these points of intersection is 2. Find k.

Use the quadratic formula, and consider the symmetry around $x=\frac{-b}{2a}$.

The quadratic formula gives

$$x=\frac{6\pm\sqrt{36-4k}}{2}=3\pm\sqrt{9-k}.$$

Hence, the distance between the intersections is $2\sqrt{9-k}$. Equating this to 2, we get

$$\begin{aligned} & 2\sqrt{9-k}=2 \\ \implies\ & 9-k=1 \\ \implies\ & k=8. \end{aligned}$$

1796. Give a counterexample to the following claim: "A pentagon cannot have two pairs of parallel sides."

Consider a truncated square.

A square with one corner cut off is a counterexample to the claim.

1797. True or false?

(a) $\displaystyle\sum_{i=a}^{c}u_i=\sum_{i=a}^{b}u_i+\sum_{i=b}^{c}u_i$,

(b) $\displaystyle\int_a^c y\,dx=\int_a^b y\,dx+\int_b^c y\,dx$.

Consider any possible overlap at b.

(a) False. These are discrete sums, so the value u_b contributes twice to the RHS, but only once on the LHS.

(b) True. Since integrals are continuous in x, the overlapping of the limits at $x=b$ contributes nothing to the integral.

1798. A six-sided die and an n-sided die, where $n>6$, are rolled together. Find, in terms of n, the probability that the score on the six-sided die is larger.

Visualise the possibility space as an $n\times6$ rectangle.

The possibility space is an $n\times6$ rectangle. Of these $6n$ outcomes, there are 15 in which the six-sided die has a higher score, forming a triangle. Hence, the probability is $\frac{15}{6n}=\frac{5}{2n}$.

1799. Find $6+8+10+...+2p$, in simplified terms of p, where p is a natural number greater than 2.

Find the common difference. Then find number of terms in the sequence, in terms of p. Then use the arithmetic partial sum formula.

The common difference is $d=2$, so the number of terms in the sequence is $n=\frac{2p-6}{2}+1=p-2$. Therefore, the sum is

$$S_n=\tfrac{1}{2}(p-2)(12+2(p-3))=(p-2)(p+3).$$

1800. Simplify $(\sqrt{x}+\sqrt{x+1})^3+(\sqrt{x}-\sqrt{x+1})^3$.

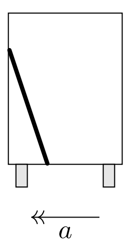
Use the binomial expansion.

Using the binomial expansion, the terms are

$$x^{\frac{3}{2}}\pm3x(x+1)^{\frac{1}{2}}+3x^{\frac{1}{2}}(x+1)\pm(x+1)^{\frac{3}{2}}.$$

Upon adding, the second and fourth terms cancel, leaving $x^{\frac{3}{2}}+3x^{\frac{1}{2}}(x+1)$. This has a common factor of $\sqrt{x}$, which gives $\sqrt{x}(4x+3)$.

1801. A rigid, rectangular sheet of plywood of mass 10 kg is being moved in a van. The diagram shows the view forwards, inside the van, as it turns left. The sheet of plywood is at 75° to the vertical, and the floor is rough enough to ensure that it does not slide. The wall of the van is modelled as smooth.



(a) Find the horizontal reaction force at the van wall before the turn, i.e. when $a = 0$.

(b) The van turns sharply. The sheet of plywood leaves the wall, and stands vertically for a moment, at rest relative to the van. Show that, if the van continues with $a > 0$, then the sheet will fall towards the opposite wall.

In (b), consider the turning effect of the horizontal frictional force that the van floor must exert on the board during the turn.

(a) Taking moments around the contact with the floor, we have

$$R_{\text{wall}} l \sin 75° - 10g \cdot \tfrac{1}{2} l \cos 75° = 0,$$

where l is the slant height of the plywood. This gives

$$R_{\text{wall}} = 49(2 - \sqrt{3}) = 13.1 \text{ N (3sf)}.$$

(b) The van is accelerating to the left. So, since the floor is rough enough to rule out sliding, the base of the plywood must be accelerating to the left also. There must be a frictional force acting leftwards on the base of the sheet. And, since the sheet is momentarily vertical, neither the reaction force from the floor nor the weight can have a turning effect. So, there must be a resultant moment clockwise. This will cause the plywood to topple towards the opposite wall.

1802. Disprove the following statement: "If u_n is a GP and f is a linear function, then $f(u_n)$ is also a GP."

Find a counterexample. Any non-constant GP and any linear function involving addition of a constant will do the job.

Any non-constant GP, together with any linear function involving addition of a constant, provides a counterexample to this statement. Using $1, 2, 4, ...$ as the GP, consider $f(x) = x + 1$. The sequence $2, 3, 5, ...$ is not a GP.

1803. If $u = \sec x$, determine the exact value of $\frac{du}{dx}\Big|_{x=\frac{\pi}{6}}$.

Use the chain rule to differentiate $(\cos x)^{-1}$, or quote a standard result.

The derivative is a standard result, which comes directly from applying the chain rule to $(\cos x)^{-1}$. We get $\frac{du}{dx} = \sec x \tan x$. The relevant exact values are $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ and $\tan(\frac{\pi}{6}) = \frac{\sqrt{3}}{3}$. Combining these, we end up with

$$\frac{du}{dx}\bigg|_{x=\frac{\pi}{6}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{3} = \frac{2}{3}.$$

1804. Show that no (x, y) points satisfy both of

$$\begin{aligned} x^2 + (y+4)^2 &< 4, \\ (x-2)^2 + y^2 &< 6. \end{aligned}$$

This is a coordinate geometry problem involving circles. It boils down to showing that the relevant circles don't intersect.

The regions are the interiors of two circles: one with radius 2, centred on $(0, -4)$, the other with radius $\sqrt{6}$, centred on $(2, 0)$. The distance between the centres is $\sqrt{4^2 + 2^2} = \sqrt{20}$. The sum of the two radii is $2 + \sqrt{6} < \sqrt{20}$. The circles, therefore, do not intersect. Hence, there are no points which are simultaneously in both interiors.

1805. Make y the subject of $x = \dfrac{y - \sqrt{y^2 + 16y}}{8}$.

Make the square root the subject of the equation, and then square both sides.

We make the square root the subject, enabling us to square both sides and get rid of it:

$$\begin{aligned} x &= \frac{y - \sqrt{y^2 + 16y}}{8} \\ \Longrightarrow\ & \sqrt{y^2 + 16y} = y - 8x \\ \Longrightarrow\ & y^2 + 16y = y^2 - 16xy + 64x^2 \\ \Longrightarrow\ & y = -xy + 4x^2 \\ \Longrightarrow\ & y(x+1) = 4x^2 \\ \Longrightarrow\ & y == \frac{4x^2}{x+1}. \end{aligned}$$

1806. A student writes the following

$$\begin{aligned} & \ln x + \ln(x-1) = \ln 6 \\ \Longrightarrow\ & x + (x-1) = 6 \\ \Longrightarrow\ & x = \tfrac{7}{2}. \end{aligned}$$

Explain the error, and correct it.

In algebra, the same operation/function must be applied to both sides, as in op(LHS) = op(RHS). Only in certain cases, such as multiplication by a constant, does this then distribute over the individual terms.

The exponential function, which is the inverse of the natural logarithm function, cannot be applied to the individual terms, as this student has done.

Exponentiation doesn't distribute over addition. Instead, log (or index) rules must be used:

$$\begin{aligned} & \ln x + \ln(x-1) = \ln 6 \\ \Longrightarrow\ & \ln(x^2 - x) = \ln 6 \\ \Longrightarrow\ & x^2 - x = 6 \\ \Longrightarrow\ & x = -2, 3. \end{aligned}$$

Since the natural logarithm function is undefined for $x = -2$, the solution is $x = 3$.

1807. "The y axis is tangent to the curve $x = y^3 - y$." True or false?

Solve $y^3 - y = 0$.

This is false. Solving $y^3 - y = 0$ gives $y = -1, 0, 1$, each of which is a single root. Hence, the y axis intersects the curve three times, but is not tangent to it.

1808. A sample x_i has mean $\bar{x}$ and standard deviation s_x. Explain why a coded sample $y_i = ax_i + b$ has mean $\bar{y} = a\bar{x} + b$ and variance $s_y^2 = b^2 s_x^2$.

Consider the fact that the mean is a measure of central tendency, while the variance is a squared measure of spread.

The mean is a measure of central tendency, and is therefore affected by both scaling and translation; it is transformed in the obvious way. The variance, however, is a squared measure of spread, and is unaffected by translation; it is only scaled by a^2.

1809. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $\sqrt[4]{x} = 2$,
- $x = 16$.

The implication is one-way.

The implication $\sqrt[4]{x} = 2 \implies x = 16$ holds, as $2^4 = 16$. The reverse implication doesn't hold, however, because it is also true that $(-2)^4 = 16$.

1810. Solve $\tan^2 2x - \sqrt{3}\tan 2x = 0$ for $x \in [0, \pi]$.

Factorise the quadratic in $\tan 2x$.

We factorise as a quadratic in $\tan 2x$:

$$\begin{aligned} & \tan^2 2x - \sqrt{3}\tan 2x = 0 \\ \Longrightarrow\ & \tan 2x(\tan 2x - \sqrt{3}) = 0 \\ \Longrightarrow\ & \tan 2x = 0, \sqrt{3}. \end{aligned}$$

Firstly, $\tan 2x = 0$ gives $2x = 0, \pi, 2\pi$. Secondly, $\tan 2x = \sqrt{3}$ gives $2x = \frac{\pi}{3}, \frac{4\pi}{3}$. Dividing by two, the solution set is $\{0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi\}$.

1811. Consider the curve $y = x^4 - x^3 - x^2 + x - 1$. You are given that $(1, -1)$ is a stationary point.

(a) Determine, without using a calculator, the x coordinates of all stationary points.

(b) Hence, give an accurate sketch of the curve. You don't need to calculate axis intercepts.

In (a), differentiate and use the factor theorem.

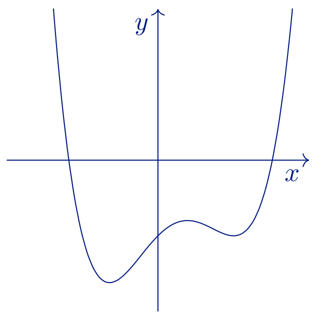
(a) The derivative is

$$\frac{dy}{dx} = 4x^3 - 3x^2 - 2x + 1.$$

For stationary points, we set this to zero and solve. Since $x = 1$ is a stationary point, we know, using the factor theorem, that $(x - 1)$ is a root.

$$\begin{aligned} & 4x^3 - 3x^2 - 2x + 1 = 0 \\ \Longrightarrow\ & (x-1)(4x^2 + x - 1) = 0 \\ \Longrightarrow\ & x = 1, \frac{-1 \pm \sqrt{17}}{8}. \end{aligned}$$

(b) Since the equation is a positive quartic, the three stationary points must be two minima at $x = \frac{-1-\sqrt{17}}{8}$ and $x = 1$ and a maximum in between then at $x = \frac{-1+\sqrt{17}}{8}$. So, the graph is



1812. From a committee of ten people, a chairperson and a secretary are to be chosen. Find the number of different ways in which this can be done.

Since chairperson and secretary are different roles, this isn't about ${}^nC_r$. Simply pick the people for the roles one after another.

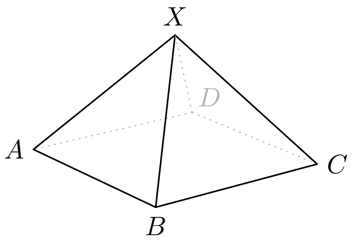
Since chairperson and secretary are different roles, we can simply pick them one after the other; we don't need to worry about the order. There are 10 choices for chairperson, and then 9 for secretary. This gives 90 ways in total.

1813. Show that, for all $q \in \mathbb{R}$, $x^2 + qx + 1$ leaves a remainder when divided by $(qx - 1)$.

Use the factor theorem.

We need to show that $(qx - 1)$ is never a factor of $x^2 + qx + 1$. By the factor theorem, we need to show that $x = \frac{1}{q}$ is not a root of $x^2 + qx + 1 = 0$. Substituting in, $\frac{1}{q^2} + 2 = 0$. But this has no roots, as squares are positive. Hence, $x^2 + qx + 1$ leaves a remainder when divided by $(qx - 1)$.

1814. The square-based pyramid shown below is formed of eight edges of unit length.



Determine the length of the shortest path from A to C, on the surface of the pyramid,

(a) if travel on the base is possible,

(b) if travel on the base is not possible.

For (b), consider unfolding the surface to make a flat rhombus $AXCB$.

(a) Travelling on the base, then the shortest length is directly from A to C, length $\sqrt{2}$.

(b) If travel on the base is not possible, then the shortest length is via the midpoint of BX (or equivalently DX). Using standard trig values for the sloped faces, which are equilateral, this distance is $2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$.

1815. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $(x-1)e^{2x-3} = 0 \implies x = 1$,

(b) $(x-1)\ln(2x-3) = 0 \implies x = 1$.

The second statement is false.

The first statement is true, as e^{2x-3} can never be zero. The second is false, as $\ln 1 = 0$. The counterexample is $x = 2$.

1816. When chimneying, a rock-climber wedges him or herself between two parallel rock faces, back to one of them, feet against the other. In this question, take the faces to be vertical, the contact forces to act at the same horizontal level, and the climber's centre of mass to lie at a point dividing the width of the chimney in the ratio $1:4$.

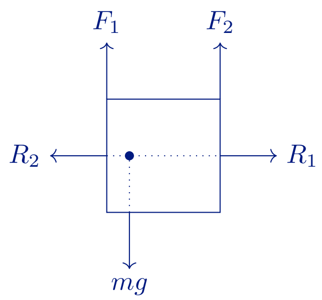
(a) Explain why the reaction forces exerted on the climber by the two faces must have the same magnitude.

(b) Explain why, if the coefficients of friction are different at the two rock faces, the climber should face away from the rougher wall.

(c) If the walls have coefficients of friction $\frac{1}{3}$ and $\frac{1}{5}$, find the minimum reaction force required to maintain equilibrium.

Draw a force diagram (as ever!) In (b), consider moments around the point at which the lines of action of the reaction forces and weight are concurrent. In (c), do the same, only numerically.

(a) The force diagram for the rock climber is



Since the reaction forces are the only forces acting horizontally, $R_1 = R_2$.

(b) Around the point marked above, the reaction forces and weight have no moment. Hence, the frictional forces must be in the ratio $4:1$, with F_1, as drawn, larger than F_2. The centre of mass of the climber is closer to the wall away from which he or she is facing, because his or her torso is there; hence, a greater frictional force is required there.

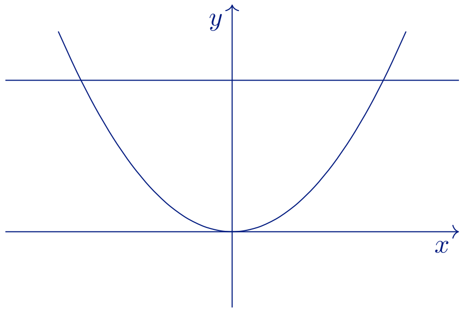
(c) Oriented as above, the maximum frictional forces are $\frac{1}{3}R$ at the back and $\frac{1}{5}R$ at the feet. Since the frictional force at the back is 4 times larger, that is where slipping will first occur. Vertical equilibrium gives

$$
\begin{aligned}
& F_{\max} + \tfrac{1}{4}F_{\max} - mg = 0 \\
\implies & \tfrac{1}{3}R + \tfrac{1}{12}R - mg = 0 \\
\implies & R = \tfrac{12}{5}mg.
\end{aligned}
$$

1817. By considering the signs of the factors in a sketch, show that the inequality $(y - x^2)(y - 4) \le 0$ defines three distinct regions in the (x, y) plane.

Sketch the graph $(y-x^2)(y-4)=0$, and then consider what happens as you cross one of the lines.

The equation $(y-x^2)(y-4)=0$ is satisfied when either $y=x^2$ or $y=4$. These two curves divide the plane up into five distinct regions:



The region enclosed by the two curves satisfies the inequality, because $y-x^2$ and $y-4$ have opposite signs. Furthermore, since crossing any line negates the sign of one of the factors, the top-left and top-right regions must also satisfy the inequality. This gives three distinct regions.

1818. Describe all functions f for which f'' is quadratic.

Such functions f are polynomial of degree 4.

If f'' is quadratic, then f must be quartic, of the form $f(x)=ax^4+bx^3+cx^2+dx+e$, where $a,b,c,d,e\in\mathbb{R}$ and $a\neq 0$.

1819. Find the equations of the two straight lines all of whose constituent points P are equidistant from the lines $y=2x-6$ and $y=-2x+6$.

Sketch the scenario and use its symmetry. Every pair of lines has two angle bisectors, one in the acute angle between them, and one in the obtuse angle.

The given lines meet at $(3,0)$, and, since they have gradients ± 2, are symmetrical in $x=3$ and $y=0$. These, therefore, are the lines we require.

1820. For small θ, $\cos\theta\approx 1-\frac{1}{2}\theta^2$.

(a) Use $\theta=\frac{\pi}{6}$ to show that $\pi\approx 6\sqrt{2-\sqrt{3}}$.

(b) Calculate the percentage error, and comment.

Substitute the exact trig value and rearrange.

(a) Substituting the exact value for $\cos\frac{\pi}{6}$, we get

$$\begin{aligned}\frac{\sqrt{3}}{2}&\approx 1-\frac{\pi^2}{72}\\ \implies \pi^2&\approx 36(2-\sqrt{3})\end{aligned}$$

So, $\pi\approx 6\sqrt{2-\sqrt{3}}$.

(b) The percentage error is

$$\frac{\pi-6\sqrt{2-\sqrt{3}}}{\pi}=1.14\%\text{ (2dp)}.$$

Even though $\frac{\pi}{6}$ radians is not a particularly small angle, the approximation still holds well, with only a 1% error.

1821. In this question, the function p is a polynomial. State, with a reason, whether the curve $y=p(x)$ necessarily intersects the following curves:

(a) $y=p(-x)$,

(b) $y=-p(x)$,

(c) $y=-p(-x)$.

Consider each graph as one or two reflections of the original graph. Then consider whether the graph must intersect the line of symmetry.

(a) Yes, the graphs intersect at $(0,p(0))$.

(b) No, a counterexample is $p(x)=x^2+1$.

(c) No, with the same counterexample.

1822. A coin is tossed 6 times. Find the probability that no two consecutive tosses yield the same result.

The possibility space is 64 equally likely outcomes. List the succcessful ones.

Out of 64 outcomes in the possibility space, only HTHTHT and THTHTH are successful. Hence, the probability is $\frac{1}{32}$.

1823. A function is defined as $f(x)=\ln(4-x)-\ln x$.

(a) Find the largest real domain over which $f(x)$ can be defined.

(b) Show that $y=f(x)$ has a point of inflection on the x axis.

(c) By considering values close to the boundaries of the domain of $f(x)$, sketch $y=f(x)$.

In (a), logarithms are only defined for positive numbers. In (b), set the second derivative to zero.

(a) Both logarithms must be well defined, with positive inputs, so the domain is $(0,4)$.

(b) Differentiating twice, we get

$$\begin{aligned} f'(x) &= -\frac{1}{4-x} - \frac{1}{x} \\ \implies f''(x) &= -\frac{1}{(4-x)^2} + \frac{1}{x^2}. \end{aligned}$$

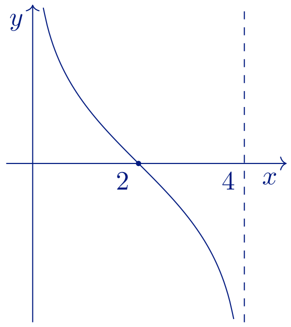
At a point of inflection, $f''(x) = 0$, so

$$\begin{aligned} &-\frac{1}{(4-x)^2} + \frac{1}{x^2} = 0 \\ \implies &-x^2 + (4-x)^2 = 0 \\ \implies &16 - 8x = 0 \\ \implies &x = 2. \end{aligned}$$

Substituting this into the original equation gives $y = 0$. We can then establish that $f''(x)$ changes sign at $x = 2$ by simplifying to

$$f''(x) = \frac{-8(x-2)}{(4-x)^2x^2}.$$

(c) The curve has vertical asymptotes at $x = 0$ and $x = 4$. At the former, $y \to \infty$; at the latter, $y \to -\infty$. So, the curve is



1824. Show that the area enclosed by the curve $y = x^2$ and the line $y = kx$ is given by $\left|\frac{1}{6}k^3\right|$.

Find the intersections in terms of k, and calculate a definite integral. Assume $k > 0$ to begin with, and then consider the $k < 0$ case.

The intersections are at $kx = x^2$, so $x = 0$ and $x = k$. Assuming positive k, the line is above the parabola, so the area is given by

$$\begin{aligned} &\int_0^k kx - x^2\, dx \\ &= \left[\tfrac{1}{2}kx^2 - \tfrac{1}{3}x^3\right]_0^k \\ &= \left(\tfrac{1}{2}k^3 - \tfrac{1}{3}k^3\right) - (0) \\ &= \tfrac{1}{6}k^3. \end{aligned}$$

If k is negative, the calculation is the same, but its result $\frac{1}{6}k^3$ is negative. Hence, to find the area, which is always positive, we need to apply the modulus function, giving $\left|\frac{1}{6}k^3\right|$ as required.

1825. In each case, find all possible values of the area of the given shape. Write your answers in interval set notation.

(a) parallelogram, side lengths a, b, a, b;

(b) kite, side lengths a, a, b, b.

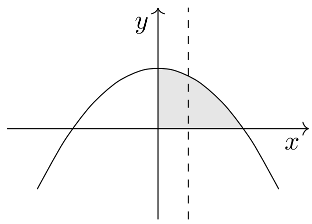
Consider the boundary cases: what shape should a parallelogram/kite be to ensure maximal area?

Let us assume that the shapes aren't degenerate, i.e. that no three vertices are collinear.

(a) The lower bound is 0 (degenerate), which isn't attainable; the upper bound is ab (rectangle), which is attainable. So, the set of possible areas is $(0, ab]$.

(b) The lower bound is 0 (degenerate), which isn't attainable; the upper bound is $2 \times \frac{1}{2}ab$ (right-angled kite), which is attainable. So, again, the set of possible areas is $(0, ab]$.

1826. The axes below show the parabola $y = 1 - x^2$, and a line $x = k$, for some $k \in (0, 1)$.



(a) Show that, if the dashed line is to divide the shaded area in half, k must satisfy the equation $k^3 - 3k + 1 = 0$.

(b) Solve this equation using fixed-point iteration.

In (a), first find the shaded area. Then equate a definite integral to half of that.

(a) The shaded area is

$$\int_0^1 1 - x^2\, dx = \left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}.$$

Hence, we need to solve for k in

$$\begin{aligned} &\int_0^k 1 - x^2\, dx = \tfrac{1}{3} \\ \implies &\left[x - \tfrac{1}{3}x^3\right]_0^k = \tfrac{1}{3} \\ \implies &k - \tfrac{1}{3}k^3 = \tfrac{1}{3} \\ \implies &k^3 - 3k + 1 = 0, \text{ as required.} \end{aligned}$$

(b) Since the root we are looking for has $|k| < 1$, the most sensible rearrangement $k = g(k)$ of the equation is $k = \frac{1}{3}(k^3 + 1)$: the cube will keep the numbers small, giving convergence. The iteration is $k_{n+1} = \frac{1}{3}(k_n^3 + 1)$. Starting with $k_0 = 0.5$, we get $k_n \to 0.34729...$. So, $k = 0.347$ (3dp).

1827. Solve $\log_{e^3} x + \ln x = 4$.

Use the fact that $\log_{a^k} b^k \equiv \log_a b$.

Using $\log_{a^k} b^k \equiv \log_a b$, we can replace $\log_{e^3} x$ with $\log_e x^{\frac{1}{3}}$. This gives

$$\begin{aligned} & \ln x^{\frac{1}{3}} + \ln x = 4 \\ \Longrightarrow\ & \tfrac{1}{3}\ln x + \ln x = 4 \\ \Longrightarrow\ & \ln x = 3 \\ \Longrightarrow\ & x = e^3. \end{aligned}$$

1828. Two particles, projected at the same time from two points on horizontal ground, have trajectories described by the Cartesian equations $y = 8x - 2x^2$ and $2y = -x^2 + 14x - 33$.

(a) By calculating the vertices of the parabolae, or otherwise, show that the particles are always at the same height as each other.

(b) The parabolae intersect at $(\frac{11}{3}, \frac{22}{9})$. Explain whether it is guaranteed that the particles will collide at this point.

In (b), it is not guaranteed.

(a) The vertices of the parabolae are at $(2, 8)$ and $(7, 8)$. Hence, since they reach the same vertical height, they must have had the same initial vertical velocity, and so will remain at the same height as each other.

(b) It is not guaranteed, because we don't know which direction each particle is travelling in. When one particle is at $(\frac{11}{3}, \frac{22}{9})$, the other particle could be at the same height but on the other side of its parabolic trajectory.

1829. Show that no (x, y) pairs satisfy both of

$$\begin{aligned} & x^2 - y^2 = 1, \\ & 16x^2 + y^2 < 15. \end{aligned}$$

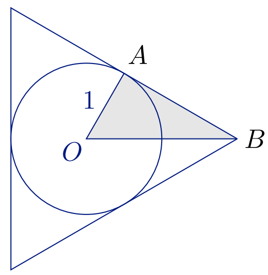
The equation describes a hyperbola. The inequality describes the interior of a ellipse; its boundary equation is a ellipse. Show algebraically that the hyperbola doesn't intersect the ellipse, and therefore lies outside it everywhere.

The boundary ellipse has equation $16x^2 + y^2 = 15$. Solving for any intersections, we can eliminate y^2 by adding. This gives $17x^2 = 16$, so $x^2 = \frac{16}{17}$. Substituting back in gives $y^2 = -\frac{1}{17}$, which has no real roots. Hence, the hyperbola doesn't intersect the boundary ellipse. Furthermore, the hyperbola is outside the ellipse, as can be verified by e.g. $(\pm\sqrt{100}, \pm\sqrt{99})$. This gives the required result.

1830. An equilateral triangle is circumscribed around a circle. Prove that the ratio of the areas of the two shapes is $3\sqrt{3} : \pi$.

On a diagram, draw in a radius, and set its length to 1. Then, using exact trig values, find the area of the circumscribed triangle.

Setting the radius to 1, we have



Using the exact value of $\tan 60°$, length $AB = \sqrt{3}$. Therefore, the area of triangle AOB is $\frac{\sqrt{3}}{2}$. The equilateral triangle consists of 6 such triangles, so it has area $3\sqrt{3}$. So, since the area of the circle is π, the ratio of areas is $3\sqrt{3} : \pi$.

1831. Write down $\displaystyle\int e^{2x+1}\,dx$.

Using the inverse chain rule, multiply by the reciprocal of the derivative of the linear inside function. And don't forget the...

According to the inverse chain rule, we multiply by the reciprocal of the derivative of the linear inside function:

$$\int e^{2x+1}\,dx = \tfrac{1}{2}e^{2x+1} + c.$$

1832. The curve $y = x^2 - x$ has two tangents that pass through the point $(6, 14)$.

(a) Show that the tangent at a general point $x = p$ has equation $y = (2p - 1)x - p^2$.

(b) Hence, determine the equation of the tangents that pass through $(6, 14)$.

In (a), differentiate to and show that $m = 2p - 1$ at $(p, p^2 - p)$. Then use $y - y_0 = m(x - x_0)$. In (b), substitute $(6, 14)$ into your generic tangent line.

(a) First, we differentiate to get $\frac{dy}{dx} = 2x - 1$. At $x = p$, this is $m = 2p - 1$. The tangent line passes through $(p, p^2 - p)$, so it has equation $y - (p^2 - p) = (2p - 1)(x - p)$, which we can simplify to $y = (2p - 1)x - p^2$.

(b) The point $(6, 14)$ must satisfy this equation, so

$$\begin{aligned} & 14 = 6(2p-1) - p^2 \\ \implies & p = 2, 10. \end{aligned}$$

Hence, the tangents have equations $y = 3x - 4$ and $y = 19x - 100$.

1833. Express $4z^2 + 10z + 19$ in terms of $(2z + 1)$.

Begin with $(2z + 1)^2$, and work from there.

We require $(2z+1)^2$ to match the term in z^2. This provides $4z$, which means we need $3(2z+1)$ for the term in z. The constant term is now $1 + 3 = 4$, so we need $+15$. This gives

$$4z^2 + 10z + 19 \equiv (2z + 1)^2 + 3(2z + 1) + 15.$$

1834. A game involves two dice: one is regular cube, the other a regular dodecahedron. Both dice are numbered $1, 2, \ldots$. The dice are rolled together, and the total S is noted.

(a) Show that $P(S = 3) = \frac{1}{36}$.

(b) Over a period of time, there are 10 instances when $S = 3$. Determine the expected number of these in which the dodecahedron showed 2.

The possibility space has $6 \times 12 = 72$ outcomes in it. In (b), this is simply a calculation of conditional probability, combined with the expectation of a binomial distribution.

(a) The possibility space consists of $6 \times 12 = 72$ outcomes. There are two outcomes which give $S = 3$, namely $(1, 2)$ and $(2, 1)$. Hence,

$$P(S = 3) = \tfrac{2}{72} = \tfrac{1}{36}.$$

(b) Since $S = 3$, the restricted possibility space consists of two outcomes. Of these, one has the dodecahedron showing a 2. Hence,

$$P(\text{dodecahedron shows } 2 \mid S = 3) = \tfrac{1}{2}.$$

There are 10 trials, so the expected number is $np = 10 \times \frac{1}{2} = 5$.

1835. Show that the curves $y = \ln ax$ and $y = \ln bx$, for constants $a, b > 0$, are translations of one another, and give the translation vector.

Use log rules.

Using log rules, the curves are $y = \ln a + \ln x$ and $y = \ln b + \ln x$. These as both translations, parallel to the y axis, of the curve $y = \ln x$. Hence, they are translations of each other. The relevant vector is $\binom{0}{\ln b - \ln a}$.

1836. Simplify $\dfrac{d}{dx}(1 + y)^2 + \dfrac{d}{dx}(1 - y)^2$.

Differentiate implicitly using the chain rule.

Using the chain rule,

$$\begin{aligned} & \frac{d}{dx}(1+y)^2 + \frac{d}{dx}(1-y)^2 \\ \equiv\ & 2(1+y) \cdot \frac{dy}{dx} + 2(1-y) \cdot -\frac{dy}{dx} \\ \equiv\ & 4y\frac{dy}{dx}. \end{aligned}$$

1837. In this question, S is the sum of the integers from 1 to 100 which are not multiples of 4.

(a) Explain why the sum of the integers from 1 to 100 which **are** multiples of 4 is given by

$$4 \times \tfrac{1}{2}n(n+1)\big|_{n=25}$$

(b) Hence, find S.

In (b), subtract the sum of the multiples of four from the overall sum.

(a) We can use the fact that the sum of the first n integers is $\frac{1}{2}n(n+1)$. Proceeding algebraically,

$$\sum_{0}^{20} 4n = 4\sum_{0}^{25} n = 4 \times \tfrac{1}{2}n(n+1)\big|_{n=25}.$$

(b) Subtracting the above from the sum of all 100 integers, we have

$$\begin{aligned} S &= \tfrac{1}{2}n(n+1)\big|_{n=100} - 4 \times \tfrac{1}{2}n(n+1)\big|_{n=25} \\ &= 5050 - 1300 \\ &= 3750. \end{aligned}$$

1838. Two dice have been rolled. Determine whether the fact “The individual scores differ by two” increases, decreases or doesn’t change the probability that the combined score is ten.

Restrict the possibility space.

Without this fact, $P(10) = \frac{3}{36} = \frac{1}{12}$. With it, the possibility space is

$$\{(0,2),(1,3),(2,4),(3,5),(4,6), \text{ and vice versa}\}.$$

Of these ten outcomes, two give a total of 10. So $P(10 \mid \text{this fact}) = \frac{2}{10} = \frac{1}{5}$. Since $\frac{1}{5} > \frac{1}{12}$, this fact increases the probability of a total of 10.

1839. A triangle has side lengths in GP, with shortest side l and common ratio $\frac{3}{2}$. Show that the area of the triangle is given by

$$A_\triangle = \frac{\sqrt{1463}}{64} l^2.$$

Find the side lengths, and then use the cosine rule, followed by the sine area formula. [Or use Heron's formula, if you know it.]

We can set aside the scale factor l, reintroducing it as an area scale factor l^2 at the end. The lengths in GP are $1, \frac{3}{2}, \frac{9}{4}$. Using the cosine rule, the angle between the two longer sides is

$$\cos\theta = \frac{\frac{3}{2}^2 + \frac{9}{4}^2 - 1^2}{2 \times \frac{3}{2} \times \frac{9}{4}} = \frac{101}{108}.$$

Hence, using $\sin^2\theta + \cos^2\theta \equiv 1$,

$$\sin\theta = \sqrt{1 - \frac{101^2}{108^2}} = \frac{\sqrt{1463}}{108}.$$

The sine area formula gives

$$A_\triangle = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{9}{4} \cdot \frac{\sqrt{1463}}{108}.$$

Reinstating the area scale factor, we get

$$A_\triangle = \frac{\sqrt{1463}}{64} l^2.$$

1840. For a general quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, the quadratic formula is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In terms of a, b, c, find simplified expressions for

(a) the sum of the roots of a quadratic,

(b) the product of the roots of a quadratic.

In (a), the square roots cancel. In (b), the expression is a difference of two squares.

(a) Summing the roots, the $\pm\sqrt{\Delta}$ cancels, and we are left with $-\frac{b}{a}$.

(b) Multiplying the roots produces a difference of two squares. This simplifies to

$$\frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

1841. Prove that the diameter, from vertex to vertex, of a regular $2n$-gon of side length l is given by the formula $d = l \operatorname{cosec} \frac{90°}{n}$.

Consider the isosceles triangle formed of two radii and a side. Split this in two, then use trigonometry.

Two radii, length r, and one side form an isosceles triangle. Its central angle is $\frac{360°}{2n} = \frac{180°}{n}$. Splitting this triangle into two right-angled triangles, they have angle $\frac{90°}{n}$, opposite $\frac{1}{2}l$ and hypotenuse r. The diameter is given, then, by

$$d = 2r = 2\frac{\frac{1}{2}l}{\sin\frac{90°}{n}} = l \operatorname{cosec} \frac{90°}{n} \quad \text{QED.}$$

1842. Two sets of bivariate data have $r_1 = 0.386$ and $r_2 = 0.417$. In two individual hypothesis tests for correlation, each of these lies in the acceptance region. State, with a reason, whether a test on the combined set of data would necessarily yield a value for r_{com} lying in the new acceptance region.

It would not. Remember that a sample in the acceptance region signifies *insufficient* evidence to reject the null hypothesis.

It doesn't necessarily follow that r_{com} lies in the acceptance region. A sample in the acceptance region means that there is *insufficient* evidence to reject the null hypothesis. But it may well still provide *some* evidence. It is possible that the evidence contained in the combined sample is sufficient for rejection of H_0, even if the evidence in each individual sample isn't.

1843. Prove that the quadratic $(3x^2 + x + 3)$ is not a factor of $27x^4 + 5x^3 - x - 9$.

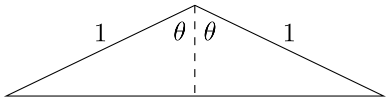
Assume, for a contradiction, that the other factor is $ax^2 + bx + c$. Then compare coefficients, starting with x^4.

Assume, for a contradiction, that there is a quadratic factorisation $(3x^2 + x + 3)(ax^2 + bx + c)$. Comparing coefficients:

$$\begin{aligned}
&x^4 : 27 = 3a \implies a = 9. \\
&x^3 : 5 = a + 3b \implies b = -\tfrac{4}{3} \\
&x^2 : 0 = 3a + b + 3c \implies c = -\tfrac{77}{9}.
\end{aligned}$$

But this gives the constant term as $-\frac{77}{3} \neq -9$. Hence, such a factorisation is not possible.

1844. By finding the area of the isosceles triangle shown below in two different ways, prove the double-angle formula $\sin 2\theta \equiv 2\sin\theta\cos\theta$.



Use the sine area formula, and the standard $\frac{1}{2}bh$.

Using the sine area formula, the total area is $\frac{1}{2}\sin 2\theta$. Using $\frac{1}{2}bh$, each right-angled triangle has area $\frac{1}{2}\sin\theta\cos\theta$. Equating the expressions for the total area gives

$$\begin{aligned}&\tfrac{1}{2}\sin 2\theta \equiv \sin\theta\cos\theta\\ \Longrightarrow\ &\sin 2\theta \equiv 2\sin\theta\cos\theta, \text{ as required.}\end{aligned}$$

1845. A differential equation is given as

$$\frac{dx}{dt} + \frac{x}{t} = t^2.$$

A quadratic solution $x = at^2 + bt + c$ is suggested.

(a) Find $\dfrac{dx}{dt}$ for the proposed solution.

(b) Hence, show that no such solution exists.

In (b), substitute for x and $\frac{dx}{dt}$. Remember that a solution (to a differential equation) requires that the two sides be *identical*.

(a) $\frac{dx}{dt} = 2at^2 + b$.

(b) Substituting in, we require

$$\begin{aligned}&2at^2 + \frac{at^2+bt+c}{t} \equiv t^2\\ \Longrightarrow\ &2at^3 + at^2 + bt + c \equiv t^3.\end{aligned}$$

The coefficient of t^2 requires $a = 0$, but this means the LHS has no term in t^3. Hence, there is no such solution to the differential equation.

1846. Prove that, if $P(A \mid B) > P(B \mid A)$ for events A and B, then $P(A) > P(B)$.

Expand the conditional probabilities using the standard formula.

Using the standard formula for conditional probability, we have

$$\begin{aligned}&P(A \mid B) > P(B \mid A)\\ \Longrightarrow\ &\frac{P(A\cap B)}{P(B)} > \frac{P(A\cap B)}{P(A)}\\ \Longrightarrow\ &\frac{1}{P(B)} > \frac{1}{P(A)}\\ \Longrightarrow\ &P(A) > P(B).\end{aligned}$$

The direction of the inequality is maintained in the last line because $P(A)$ and $P(B)$ must both be positive.

1847. A *Reuleaux triangle* is a shape constructed from an equilateral triangle of side length r, by drawing an arc of radius r centred on each vertex.

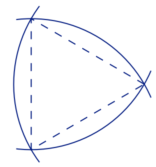
(a) Sketch a Reuleaux triangle.

(b) Show that, in every orientation, a Reuleaux triangle has the same width r.

(c) Show that the area is $\frac{1}{2}(\pi - \sqrt{3})r^2$.

In (c), calculate the area of three segments in the usual fashion, i.e. as sector minus triangle.

(a) A Reuleaux triangle is composed of three arcs drawn as follows:



(b) The width of the shape is always, regardless of orientation, from a vertex to the opposite side. Hence, it always lies along the radius of one of the arcs. So, the width is r.

(c) Each of the segments has area given by sector minus triangle, which is

$$\tfrac{1}{2}r^2 \times \tfrac{\pi}{3} - \tfrac{1}{2}r \times \tfrac{\sqrt{3}}{2}r.$$

Adding three of these to the area of the central triangle, we get

$$\begin{aligned}&3\big(\tfrac{1}{2}r^2 \times \tfrac{\pi}{3} - \tfrac{1}{2}r \times \tfrac{\sqrt{3}}{2}r\big) + \tfrac{1}{2}r \times \tfrac{\sqrt{3}}{2}r\\ =\ &\tfrac{1}{2}(\pi - \sqrt{3})r^2.\end{aligned}$$

1848. Two sets I and J are defined as subsets of $\mathbb{R}$ by the values that satisfy the inequalities $x^2 - 2x - 24 < 0$ and $5x - 4 \geq 1$ respectively. Find the set $I \cap J$, giving your answer in interval set notation.

Solve each inequality individually, then consider the intersection of the sets.

The individual sets are $I = (-4, 6)$ and $J = [1, \infty)$. The intersection, therefore, is $[1, 6)$.

1849. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

- $(x-a)(x-b)(x-c) = 0$,
- $x \in \{a, b\}$.

The implication goes backwards. Find a counterexample to the forwards implication.

The implication is $\Longleftarrow$. If $x \in \{a, b\}$, then either $(x-a)$ or $(x-b)$ is zero, so $(x-a)(x-b)(x-c) = 0$. The counterexample to the forwards implication is $x = c$.

1850. Describe the single transformation which takes the graph $y = 2e^x$ onto the graph $y = \ln \frac{1}{2}x$.

Rewrite the logarithmic equation in index form, and compare.

The graph $y = \ln \frac{1}{2}x$ can be rewritten as $2e^y = x$. Since this is the same as $y = 2e^x$, except with the roles of x and y reversed, the two graphs are reflections in the line $y = x$.

1851. A trapezium, which is not a parallelogram, has three of its sides described, in tip-to-tail order around the perimeter, by the vectors $\mathbf{a}$, $\mathbf{b}$ and $k\mathbf{a}$. Give the set of possible values of the scalar k.

Show that $k < 0$. Then eliminate the case in which the trapezium is also a parallelogram.

The shape must have one pair of parallel sides; these must be the sides with vectors $\mathbf{a}$ and $k\mathbf{a}$. And, since these vectors run tip-to-tail, $\mathbf{a}$ and $k\mathbf{a}$ must be antiparallel, i.e. k must be negative. However, this includes one possibility which would make the shape a parallelogram. Excluding this, we have $k \in (-\infty, 0) \setminus \{-1\}$.

1852. If $p = a^x$ and $q = b^x$, show that $p = q^{\log_b a}$.

Substitute the second equation into the RHS of the result you are trying to reach.

The RHS may be rewritten as follows:

$$\begin{aligned} & q^{\log_b a} \\ =\ & (b^x)^{\log_b a} \\ =\ & (b^{\log_b a})^x \\ =\ & a^x \\ =\ & p, \quad \text{as required.} \end{aligned}$$

1853. Show that the following implication holds:

$$\begin{aligned} & 12x = 3t^2 + \frac{1}{t^2} - 4t + 6 \\ \Longrightarrow\ & t\frac{dx}{dt} + 2x = t^2 - t + 1. \end{aligned}$$

Calculate the relevant derivative and substitute into the LHS.

Assuming the first equation of the implication, we know that $12\frac{dx}{dt} = 6t - 2t^{-3} - 4$. Substituting this and x into the LHS of the second equation, we get

$$\begin{aligned} & t\left(\tfrac{1}{2}t - \tfrac{1}{6}t^{-3} - \tfrac{1}{3}\right) + 2\left(\tfrac{1}{4}t^2 + \tfrac{1}{12}t^{-2} - \tfrac{1}{3}t + \tfrac{1}{2}\right) \\ =\ & \tfrac{1}{2}t^2 - \tfrac{1}{6}t^{-2} - \tfrac{1}{3}t + \tfrac{1}{2}t^2 + \tfrac{1}{6}t^{-2} - \tfrac{2}{3}t + 1 \\ =\ & t^2 - t + 1, \quad \text{as required.} \end{aligned}$$

1854. Prove that there is only one way of shading five squares of a three-by-three grid such that no two shaded squares share a border.

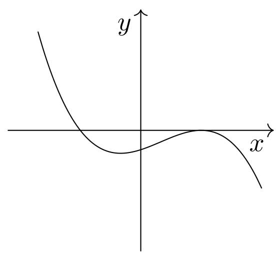
Consider the cases in which the central square is shaded or not.

There are two cases:

1. If the central square is shaded, then its four neighbours cannot be. Only the four corner squares remain, which must then all be shaded.
2. If the central square is not shaded, then, since the remaining squares form a ring, it is not possible to shade more than every other square. This gives a maximum of four shaded squares in total.

Hence, overall, there is only one way of shading five squares, as in case 1.

1855. Determine whether $y = (1-x)^2(1+x)$ could be the equation generating the following graph:



Consider the multiplicity of the roots/factors and the sign of the leading coefficient.

The roots do match, with a double root at $x = 1$ and a single root at $x = -1$. However, the graph is a negative cubic, whereas the equation suggested has leading coefficient $+1$. So, the given equation could not be that of the given graph.

1856. In this question, the random variable X has a generic normal distribution $N(\mu, \sigma^2)$, and probabilities are being numerically approximated using the trapezium rule.

(a) Explain whether this approximating procedure will over or underestimate probabilities, for values of X

i. close to the mean,

ii. in the tails of the distribution.

(b) Explain why the trapezium rule will give good approximations at around $|X - \mu| = \sigma$.

Sketch a normal distribution, and think about its curvature (second derivative). You can use the standard fact that the points of inflection on a normal distribution are at $|X - \mu| = \sigma$.

(a) The trapezium rule gives an overestimate of the area beneath a positive convex function, and an underestimate of the area beneath a positive concave function. In these case, therefore, it

i. underestimates: the curvature is negative.

ii. overestimates: the curvature is positive.

(b) Good approximations will occur at points where the normal distribution curve is close to being linear, which is where the second derivative is approximately zero. On a normal distribution, the second derivative is zero at the points of inflection at $|X - \mu| = \sigma$.

1857. Solve the simultaneous equations $2\sqrt{x} + 2\sqrt{y} = 3$ and $y = x^2$, giving your answer(s) in simplified exact form.

Substitute for y, and solve a quadratic in $\sqrt{x}$ using the formula. Consider the validity of the roots you get.

Substituting for y, we get a quadratic in $\sqrt{x}$:

$$\begin{aligned} & 2\sqrt{x} + 2\sqrt{x^2} = 3 \\ \implies & 2x + 2\sqrt{x} - 3 = 0 \\ \implies & \sqrt{x} = \frac{-2 \pm \sqrt{28}}{4} \end{aligned}$$

We take the positive root, as $\sqrt{x} \geq 0$, which gives

$$x = \tfrac{1}{2}(4 - \sqrt{7}), \quad y = \tfrac{1}{4}(23 - 8\sqrt{7}).$$

1858. Write down the area scale factor when $y = f(x)$ is transformed to $y = af(bx + c) + d$.

Translations do not affect areas; stretches do.

The translations have no effect on area. The stretches do. We have a stretch parallel to the y axis by scale factor a and a stretch parallel to the x axis by scale factor $\frac{1}{b}$. Hence, overall, the area scale factor is $\frac{a}{b}$.

1859. Using the compound-angle formulae, express the following separable differential equation in the form $f(y)\frac{dy}{dx} = g(x)$ for some functions f and g:

$$\frac{dy}{dx} = \cos(x - y) - \cos(x + y).$$

Use $\cos(x \pm y) \equiv \cos x \cos y \mp \sin x \sin y$. Note the $\mp$ sign on the RHS.

Using $\cos(x \pm y) \equiv \cos x \cos y \mp \sin x \sin y$, we get

$$\begin{aligned} & \frac{dy}{dx} = 2 \sin x \sin y \\ \implies & \operatorname{cosec} y \frac{dy}{dx} = 2 \sin x. \end{aligned}$$

1860. An iterative sequence is defined by

$$x_{n+1} = \frac{1}{x_n + 1}, \quad x_0 = a + b\sqrt{2}.$$

Find an ordinal formula for x_n in terms of a, b and n, of the form $x_n = 1 - \ldots$

Find x_1, x_2, x_3, simplifying as you go. You should see a pattern.

The second term x_1 is

$$x_1 = \frac{1}{a + 1 + b\sqrt{2}}.$$

Substituting this, we get an inlaid fraction. This is dealt with by multiplying top and bottom of the large fraction by the denominator of the small fraction:

$$\begin{aligned} x_2 &= \frac{1}{\frac{1}{a+1+b\sqrt{2}} + 1} \\ &= \frac{a + 1 + b\sqrt{2}}{a + 2 + b\sqrt{2}}. \end{aligned}$$

And the same again gives

$$\begin{aligned} x_3 &= \frac{1}{\frac{a+1+b\sqrt{2}}{a+2+b\sqrt{2}} + 1} \\ &= \frac{a + 2 + b\sqrt{2}}{a + 3 + b\sqrt{2}}. \end{aligned}$$

Spotting the pattern, the ordinal formula is

$$x_n = \frac{a + n - 1 + b\sqrt{2}}{a + n + b\sqrt{2}}.$$

Splitting the fraction up gives the required form:

$$x_n = 1 - \frac{1}{a + n + b\sqrt{2}}.$$

1861. Write the following in terms of 4^x:

(a) 2^{6+2x},

(b) 8^{4-2x}.

In each case, write the base as a power of 4, then use index laws.

(a) We first write the base as 4^k, then use index laws to manipulate the expression:

$$\begin{aligned} 2^{6-2x} &= \left(4^{\frac{1}{2}}\right)^{6+2x} \\ &= 4^{3+x} \\ &= 4^3 \times 4^x \\ &= 64 \times 4^x. \end{aligned}$$

(b) This time, we use $4^{\frac{3}{2}} = 8$:

$$\begin{aligned} 8^{4-2x} &= \left(4^{\frac{3}{2}}\right)^{4-2x} \\ &= 4^{6-3x} \\ &= \frac{4^6}{4^{3x}} \\ &= \frac{4096}{(4^x)^3}. \end{aligned}$$

1862. State, with a reason, whether the following hold, with regard to a hypothesis test:

(a) "Some samples fall in neither the critical or acceptance regions."

(b) "If a sample has a test statistic which falls in the critical region, then its p-value is less than the significance level."

One is true and the other is false.

(a) This is false. Critical and acceptance regions are mutually exclusive and exhaustive, i.e. they are complementary. In less formal terms, they are opposites. Every sample must lie in one or the other.

(b) This is true. The statements "the sample has a test statistic lying in the critical region" and "the sample's p-value is less than the significance level" are two ways of saying exactly the same thing, in the languages of test statistics or probabilities.

1863. Three distinct vertices are selected at random on a regular octagon. Determine the probability that all three vertices are adjacent.

Place the first vertex arbitrarily, without loss of generality. Then consider the ${}^7C_2$ equally likely ways in which the remaining two vertices may be chosen.

The octagon is symmetrical, so we can choose the first vertex arbitrarily without loss of generality. There are now ${}^7C_2 = 21$ equally likely ways ways of choosing the other two vertices. Of these, three (both one side, both the other, one either side) are successful. So, the probability is $\frac{3}{21} = \frac{1}{7}$.

1864. The iteration $x_{n+1} = x_n^2 + px_n + q$ has exactly one fixed point. Find p in terms of q.

Set up the standard equation for fixed points, and use the discriminant.

The standard equation for fixed points is $x = f(x)$. So, we require that $x = x^2 + px + q$ has exactly one root. Rearranging to $x^2 + (p-1)x + q = 0$, we need

$$\begin{aligned} & (p-1)^2 - 4q = 0 \\ \Longrightarrow\ & p = \pm 2\sqrt{q} + 1. \end{aligned}$$

1865. For $X \sim N(0,1)$, find $P(X^2 - 1 > X)$.

Solve the inequality algebraically before using the distribution of X to calculate probabilities.

A successful outcome is where

$$\begin{aligned} & X^2 - 1 > X \\ \Longrightarrow\ & X^2 - X - 1 > 0 \\ \Longrightarrow\ & X < \frac{1-\sqrt{5}}{2} \text{ or } X > \frac{1+\sqrt{5}}{2}. \end{aligned}$$

Using a cumulative distribution function, we get

$$\begin{aligned} P(X^2 - 1 > X) &= 1 - P\left(\frac{1-\sqrt{5}}{2} < X < \frac{1+\sqrt{5}}{2}\right) \\ &= 1 - 0.678896... \\ &= 0.321 \text{ (3sf)} \end{aligned}$$

1866. An operation $\star$, acting on two real numbers a, b, is defined as $a \star b = a^2 + ab + b^2$.

(a) Evaluate $\sqrt{3} \star \sqrt{27}$.

(b) Solve the equation $x \star (x+2) = 1$.

In (b), you should get a quadratic.

(a) $\sqrt{3} \star \sqrt{27} = 3 + \sqrt{3}\sqrt{27} + 27 = 39$.

(b) This yields a quadratic:

$$\begin{aligned} & x \star (x+2) = 1 \\ \Longrightarrow\ & x^2 + x(x+2) + (x+2)^2 = 1 \\ \Longrightarrow\ & 3x^2 + 6x + 3 = 0 \\ \Longrightarrow\ & (x+1)^2 = 0 \\ \Longrightarrow\ & x = -1. \end{aligned}$$

1867. Find the area enclosed by the parabolae $y = x^2$ and $y = 1 + x - x^2$.

Find intersections, and then set up a single definite integral for the (signed) area.

The intersections are where

$$\begin{aligned} & x^2 = 1 + x - x^2 \\ \Longrightarrow\ & 2x^2 - x - 1 = 0 \\ \Longrightarrow\ & (2x+1)(x-1) = 0 \\ \Longrightarrow\ & x = -\tfrac{1}{2}, 1. \end{aligned}$$

The negative parabola must be above the positive parabola between these intersections, so the relevant area is given by

$$\begin{aligned} & \int_{-\frac{1}{2}}^{1} 1 + x - 2x^2 \, dx \\ &= \left[x + \tfrac{1}{2}x^2 - \tfrac{2}{3}x^3 \right]_{-\frac{1}{2}}^{1} \\ &= \left(1 + \tfrac{1}{2} - \tfrac{2}{3} \right) - \left(-\tfrac{1}{2} + \tfrac{1}{8} + \tfrac{2}{24} \right) \\ &= \tfrac{9}{8}. \end{aligned}$$

1868. Prove the following identities:

(a) $\sin(90° - \theta) \equiv \cos(\theta)$,

(b) $\sin(45° + \theta) \equiv \cos(45° - \theta)$.

You can either use a compound-angle formula, or else (and rather slicker) consider the symmetry of the sine and cosine functions as defined on the unit circle.

Both of these results follow from the same result: that the definitions of sine and cosine on the unit circle are symmetrical in $y = x$, or equivalently in $\theta = 45°$.

(a) $(90° - \theta)$ and θ are symmetrical in $\theta = 45°$.

(b) $(45° + \theta)$ and $(45° - \theta)$ are also symmetrical in $\theta = 45°$.

1869. In music theory, the intervals known as an *octave* and a *fifth* correspond to frequency ratios of 2 : 1 and 3 : 2 respectively. Prove that no interval, i.e. frequency ratio can be expressed both as a whole number of octaves and as a whole number of fifths.

Prove this by contradiction. Assume there is such an interval, turn the information given into an equation, and use prime factors to show that something breaks.

Assume, for a contradiction, that frequency ratio $k : 1$, where $k > 1$ can be expressed as a whole number of octave and a whole number of fifths. This means that $k = 2^p$ for $p \in \mathbb{N}$ and $k = \frac{3}{2}^q$, where $q \in \mathbb{N}$. Equating these, we get

$$\begin{aligned} & 2^p = \tfrac{3}{2}^q \\ \Longrightarrow\ & 2^{p+q} = 3^q. \end{aligned}$$

But the LHS and RHS have prime factors of 2 and 3 respectively. Hence, this equation can only hold when $p = q = 0$. But $1 = 2^p = k > 1$. Hence, we have a contradiction. So, no interval can be expressed both as a whole number of octaves and as a whole number of fifths. □

1870. By proposing $p + q\sqrt{2}$ and setting up simultaneous equations, determine the positive square root of $33 + 8\sqrt{2}$.

Compare coefficients in $(p + q\sqrt{2})^2$.

We propose a root as $p + q\sqrt{2}$, where $p, q \in \mathbb{Q}$. So, we need

$$\begin{aligned} & (p + q\sqrt{2})^2 = 33 + 8\sqrt{2} \\ \Longrightarrow\ & p^2 + 2q^2 + 2pq\sqrt{2} = 33 + 8\sqrt{2}. \end{aligned}$$

Since $p, q \in \mathbb{Q}$, we can equate coefficients, giving

$$\begin{aligned} & p^2 + 2q^2 = 33 \\ & 2pq = 8. \end{aligned}$$

Substituting the latter into the former, we get a biquadratic

$$\begin{aligned} & p^2 + \frac{32}{p^2} = 33 \\ \Longrightarrow\ & p^4 - 33p^2 + 32 = 0 \\ \Longrightarrow\ & (p^2 - 32)(p^2 - 1) = 0 \\ \Longrightarrow\ & p = \pm\sqrt{32}, \pm 1. \end{aligned}$$

Since p is rational and positive, we need $p = 1$. This gives the required root as $1 + 4\sqrt{2}$.

1871. Variables x and y have constant rates of change

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b.$$

Find the rate of change of xy in terms of x, y, a, b.

Use the product rule.

Using the product rule,

$$\begin{aligned} \frac{d}{dt}(xy) &= \frac{dx}{dt}y + x\frac{dy}{dt} \\ &= ay + bx. \end{aligned}$$

1872. Show that, although the second derivative of the curve $y = \sqrt[3]{x}$ is undefined at the origin, the curve is nevertheless inflected there.

Sketch the curve, and consider its reflection in $y = x$.

Differentiating,

$$\begin{aligned} & y = x^{\frac{1}{3}} \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = -\frac{2}{9}x^{-\frac{5}{3}}. \end{aligned}$$

This is undefined as $0^{-\frac{5}{2}}$ involves division by zero. However, $y = \sqrt[3]{x}$ is a reflection of $y = x^3$ in the line $y = x$, and $y = x^3$ is inflected at the origin. Hence, so is $y = \sqrt[3]{x}$. The upshot is: the curve is inflected, but it is not possible to *show* that it is so using derivatives with respect to x.

1873. The equations $f(x) = 0$ and $g(x) = 0$, where f and g are distinct quadratic functions, have the same solution set S. The equation $f(x) = g(x)$ is denoted E. State, with a reason, whether the following claims hold:

(a) "E has solution set S",
(b) "S is a subset of the solution set of E,
(c) "the solution set of E is a subset of S".

Note that the empty set is a subset of every set.

Consider the case in which the solution set S is the empty set.

(a) This is not true. Both $f(x) = 10x^2 + 1$ and $g(x) = x^2 + 2$ have the same S, viz. the empty set, yet their graphs intersect.
(b) This is true. If S is not empty, then any value where $f(x) = 0$ and $g(x) = 0$ must satisfy $f(x) = g(x)$. Hence, anything in S is in the solution set of E. And if S is empty, then it is automatically a subset.
(c) This is not true, for the same reason as in (a).

1874. A sequence is given, for constant $k > 0$, by

$$\ln k, \ln k^2, \ln k^3, ..., \ln k^9.$$

(a) Show that the sequence is arithmetic.
(b) Find the mean and median of the sequence, giving your answer in terms of k.

In (a), use log laws. In (b), the mean and median of an AP are the same: the mean of the first and last terms.

(a) Using a log law, the sequence is

$$\ln k, 2\ln k, 3\ln k, ..., 9\ln k$$

Hence, it is an AP with first term $\ln k$ and last term $9\ln k$.

(b) The mean of an AP is the mean of first and last, which is $5\ln k$. This is also the median.

1875. Write down the largest real domains over which the following functions may be defined:

(a) $x \mapsto \dfrac{1}{x+1}$,
(b) $x \mapsto \dfrac{1}{x^2+1}$,
(c) $x \mapsto \dfrac{1}{x^3+1}$.

Consider whether the denominator can be zero.

In each case, the domain is $\mathbb{R}$, excluding any real numbers which make the denominator zero.

(a) $\mathbb{R} \setminus \{-1\}$,
(b) $\mathbb{R}$,
(c) $\mathbb{R} \setminus \{-1\}$.

1876. Show that the following algebraic fraction cannot be simplified to a polynomial:

$$\frac{36x^3 + 216x^2 + 233x - 48}{3x+5}.$$

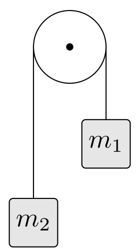
This is the factor theorem. Substitute a value to show that $(3x+5)$ is not a factor of the numerator.

We can use the factor theorem. For the algebraic fraction to simplify to a polynomial, $(3x+5)$ would have to be a factor of the numerator. But, when we substitute $x = -\frac{5}{3}$, we get

$$36\big(-\tfrac{5}{3}\big)^3 + 216\big(-\tfrac{5}{3}\big)^2 + 233\big(-\tfrac{5}{3}\big) - 48 = -3.$$

Since this is non-zero, the algebraic fraction can't be simplified to a polynomial.

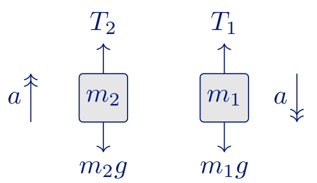
1877. Two masses $m_1 > m_2$ are attached to opposite ends of a light, inextensible string, which is passed over a pulley.



The masses accelerate, but friction in the pulley is non-negligible, generating different tensions T_1 on the right and T_2 on the left of the pulley. Give, with justification, the order of the magnitudes of the four quantities m_1g, m_2g, T_1, T_2. Write your answer in the form $Q_a < Q_b < Q_c < Q_d$.

Draw clear force diagrams. You might consider the extreme cases of no friction and total friction.

The string is inextensible, so, despite the friction in the pulley, the accelerations will be the same. The tensions, however, will not:



At one extreme, without any friction in the pulley, we would have $T_1 = T_2$. At the other, with very high friction, i.e. the pulley so stiff as to maintain equilibrium, we would have

$$T_1 = m_1g > m_2g = T_2.$$

This scenario is between the two, so $T_1 > T_2$. And, since the system accelerates, we know that $T_1 < m_1g$ and $T_2 > m_2g$. Hence, we have

$$m_2g < T_2 < T_1 < m_1g.$$

1878. Find the 10^{th} and 90^{th} percentiles of the normal distribution $X \sim N(40, 9)$.

Solve for x_1 in $P(X < x_1) = 0.1$, and set up a similar equation for x_2.

We need x_1, x_2 such that $P(X < x_1) = 0.1$ and $P(X < x_2) = 0.9$. Using a calculator inverse normal distribution function, this gives, to 3sf, $x_1 = 36.2$ and $x_2 = 43.8$.

1879. Show that $\displaystyle\int_1^2 \sum_{i=1}^{3} x^{-i}\,dx = \tfrac{7}{8} + \ln 2$.

The sum is simply a polynomial in x. Write it out longhand, and integrate as usual.

Writing the sum longhand, we can integrate:

$$\begin{aligned}
&\int_1^2 \sum_{i=1}^{3} x^{-i}\,dx \\
&= \int_1^2 x^{-1} + x^{-2} + x^{-3}\,dx \\
&= \Big[\ln|x| - x^{-1} - \tfrac{1}{2}x^{-2}\Big]_1^2 \\
&= \big(\ln 2 - \tfrac{1}{2} - \tfrac{1}{8}\big) - \big(\ln 1 - 1 - \tfrac{1}{2}\big) \\
&= \tfrac{7}{8} + \ln 2, \text{ as required.}
\end{aligned}$$

1880. The curve $y = x^6 + x$ has second derivative zero at the origin. Show that this is not a point of inflection.

At a point of inflection, the second derivative must change sign. It is not sufficient that the second derivative is zero.

For a point of inflection, it is not sufficient that the second derivative be zero; it must change sign. Here, the second derivative is $\frac{d^2y}{dx^2} = 30x^4$. This is zero at the origin but positive either side of it. Since the second derivative does not change sign, this is not a point of inflection.

1881. Find simplified expressions for the sets

(a) $(A \setminus B) \cap A$,

(b) $(A \cap B) \setminus B$,

(c) $(A \setminus B) \cup B$.

You might find a Venn diagram useful. The backslash is the "set minus" notation, so that $A \setminus B$ is the same as $A \cap B'$.

(a) $A \setminus B$,

(b) $\emptyset$,

(c) B.

1882. From the formulae $s = ut + \frac{1}{2}at^2$ and $v^2 = u^2 + 2as$, show algebraically that $s = \frac{1}{2}(u \pm v)t$.

Determine which variable you don't want, and eliminate it using substitution.

We want no acceleration in the formula. So, we can rearrange the second equation to the form $a = \ldots$ and equate:

$$\begin{aligned}
& s = ut + \tfrac{1}{2}\frac{v^2 - u^2}{2s}t^2 \\
\Longrightarrow\ & s^2 = sut + \tfrac{1}{4}(v^2 - u^2)t^2 \\
\Longrightarrow\ & 4s^2 - 4sut + u^2t^2 - v^2t^2 = 0 \\
\Longrightarrow\ & (2s - ut)^2 = v^2t^2 \\
\Longrightarrow\ & 2s - ut = \pm vt \\
\Longrightarrow\ & 2s = (u \pm v)t \\
\Longrightarrow\ & s = \tfrac{1}{2}(u \pm v)t.
\end{aligned}$$

1883. A set of two linear equations in three unknowns is given as follows:

$$\begin{aligned}
a_1x + b_1y + c_1z &= 0 \\
a_2x + b_2y + c_2z &= 0.
\end{aligned}$$

True or false: "There is no set of real numbers $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ for which these equations yield a unique point (x, y, z) as a solution."

Consider the same question with one linear equation in two unknowns.

This is true. In a lower-dimensional analogy, one linear equation in two unknowns can only produce a line, never a unique value. The same holds true in 3D: each equation restricts the solution space by one dimension, so 3D becomes 2D becomes 1D. The solution to a set of two linear equations in three unknowns must be a line, never a unique point.

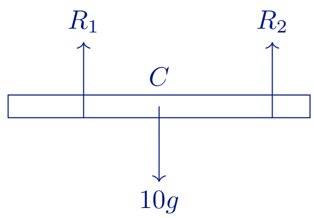
1884. Two workmen are carrying a long plank on their shoulders. The plank is 4 metres long and has a mass of 10 kg, and you may assume it is uniform, rigid and horizontal. The workmen are 0.5 m and 1 m from their respective ends. Find the contact forces they each exert on the plank when

(a) they are at rest on horizontal ground,

(b) they are moving at constant speed up a steady slope of 10°.

Draw a force diagram. In (b), consider the accleration of the system.

(a) The force diagram is



$$\begin{aligned} \updownarrow\ &: R_1 + R_2 - 10g = 0 \\ \overset{\frown}{C}\ &: R_2 \times 1.5 - R_1 \times 1 = 0 \end{aligned}$$

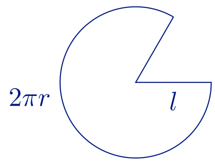
So, $R_1 = 6g$ and $R_2 = 4g$.

(b) Because the velocity is constant, the fact that the workmen are on a slope doesn't change anything. The acceleration is still zero, and the ratio between the distances from the centre of mass is as before. So $R_1 = 6g$ and $R_2 = 4g$ again.

1885. By unwrapping its curved surface to form a sector, prove that the total surface area of a right-circular cone is given, in terms of the radius r and slant height l, by $A = \pi r(r + l)$.

The sector formed when a cone is unwrapped has an arc length defined by the circumference of the base of the cone.

The unwrapped sector is as follows:



Of a total circumference $2\pi l$, we have $2\pi r$, so the sector above is $\frac{r}{l}$ of the circle. Hence, the area is $\frac{r}{l} \times \pi l^2 = \pi r l$. Combined with the base of the cone, which has area πr^2, we have $A = \pi r(r + l)$ as required.

1886. A function g has domain $[0, 1]$ and range $[0, 1]$. State, with a reason, whether the following are well-defined functions over $[0, 1]$:

(a) $x \mapsto g(1 + g(x))$,

(b) $x \mapsto g(1 - g(x))$.

Consider the range of the inside function $1 \pm g(x)$.

(a) This is not well defined. The inside function $1 + g(x)$ has range $[1, 2]$, which is outside the domain of g.

(b) This is well defined. The range of $1 - g(x)$ is $[0, 1]$, which is precisely the domain of g.

1887. An arithmetic series has first term 15, common difference 7, and partial sum $S_n = 29385$. Find n.

Use the standard formula for the partial sum of an AP, and solve a quadratic in n.

The standard formula gives

$$\begin{aligned} & 29385 = \tfrac{1}{2}n(30 + 7(n - 1)) \\ \Longrightarrow\ & 7n^2 + 23n - 58770 = 0 \\ \Longrightarrow\ & n = -\tfrac{653}{7}, 90. \end{aligned}$$

Since this is an arithmetic series with a first term at $n = 1$, we want $n = 90$.

1888. State the possible numbers of intersections of the graphs $y = f(x)$ and $y = g(x)$, when they are

(a) a cubic and a quadratic,

(b) two cubics,

(c) a cubic and a quartic.

In each case, consider the degree of the polynomial equation satisfied by the intersections.

(a) The intersections satisfy a cubic equation. So, there could be $\{1, 2, 3\}$ intersections.

(b) Generally, the intersections satisfy a cubic equation. However, $y = x^3$ and $y = x^3 + 1$ do not intersect at all. So, $\{0, 1, 2, 3\}$.

(c) The intersections satisfy a quartic. Hence, any number of roots $\{0, 1, 2, 3, 4\}$ is possible.

1889. A function has second derivative $f''(x) = 4x$, and $f'(2) = 6$. Show that the function is stationary at $x = \pm\sqrt{2}$.

Integrate the second derivative, and use the value of $f'(2)$ to find the $+c$. Then solve $f'(x) = 0$.

Integrating the second derivative $f''(x)$, we have $f'(x) = 2x^2 + c$. Substituting $f'(2) = 6$ gives $6 = 8 + c$, so $c = -2$. Therefore, the first derivative is $f'(x) = 2x^2 - 2$. For stationary points, then, we solve $f'(x) = 2x^2 - 2 = 0$ to get $x = \pm\sqrt{2}$.

1890. Samples are taken from a large population, whose distribution is modelled as approximately normal. The standard deviation of the means of samples of size k is 0.12. Give the standard deviation of the means of samples of size $4k$.

Consider that the standard deviation scales inversely with the square root of the sample size.

Scaling the sample size by 4 scales the standard deviation of sample means by $\frac{1}{2}$. So the standard deviation is 0.06.

1891. Show that the solution set of $3\sin^2 x - \cos^2 x = 0$ is $\{x : x = 180°n \pm 30°, n \in \mathbb{Z}\}$.

Either use $\sin^2 x + \cos^2 x \equiv 1$, or $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$.

Rearranging, we get

$$\begin{aligned} & 3\sin^2 x - \cos^2 x = 0 \\ \Longrightarrow\ & 3\sin^2 x = \cos^2 x \\ \Longrightarrow\ & \tan x = \pm\tfrac{1}{\sqrt{3}}. \end{aligned}$$

The primary values are $x = \pm 30°$. Other values are given by multiples of $180°$ from the primary values. Hence, the solution set is $\{x : x = 180°n \pm 30°, n \in \mathbb{Z}\}$ as required.

1892. In terms of d and g, find the minimum speed at which it is possible to throw a projectile to a friend who is d metres away horizontally.

The minimum speed will be attained at 45° above the horizontal.

The minimum speed will be attained at 45° above the horizontal. With initial speed u, this gives components of $\frac{\sqrt{2}}{2}u$. So, we have $0 = \frac{\sqrt{2}}{2}ut - \frac{1}{2}gt^2$ vertically and $d = \frac{\sqrt{2}}{2}ut$ horizontally. Rearranging the latter to make t the subject, we get $t = \frac{d\sqrt{2}}{u}$. Substituting gives

$$0 = d - \frac{d^2 g}{u^2}.$$

We can solve to get $u = \sqrt{dg}$.

1893. Show that it is impossible to find constants P, Q such that the following is an identity:

$$\frac{1}{x^3 + x} \equiv \frac{P}{x} + \frac{Q}{x^2 + 1}.$$

Assume that you do have such constants, and work to find a contradiction. Start by multiplying up by the denominators.

Multiplying up by the denominators,

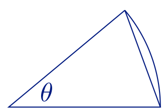
$$1 \equiv P(x^2 + 1) + Qx \equiv Px^2 + Qx + P.$$

Equating constant terms, $P = 1$. But coefficients of x^2 then give $0 = 1$. So, such constants cannot be found.

1894. The cubic approximation for $\sin\theta$, valid for small angles defined in radian measure, is $\sin\theta \approx \theta - \frac{1}{6}\theta^3$. Show that the area of a segment, subtending a small angle θ at the centre of a circle of radius r, may be approximated as $A \approx \frac{1}{12}r^2\theta^3$.

Calculate the area of the segment using the usual method of sector minus triangle. Each of these should be in terms of r and θ.

With θ exaggerated for clarity, the scenario is



The sector has area $\frac{1}{2}r^2\theta$. The triangle has area $\frac{1}{2}r^2\sin\theta$. Using the cubic approximation to the sine function, this gives the area of the segment as

$$\begin{aligned} A_{\text{seg}} &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta \\ &= \tfrac{1}{2}r^2(\theta - \sin\theta) \\ &\approx \tfrac{1}{2}r^2\big(\theta - (\theta - \tfrac{1}{6}\theta^3)\big) \\ &= \tfrac{1}{12}r^2\theta^3, \text{ as required.} \end{aligned}$$

1895. Assuming $\dfrac{d}{dx}e^x = e^x$, prove that $\dfrac{d}{dx}2^x = \ln 2 \cdot 2^x$.

Write 2 over base e, then use an index law and the chain rule.

Writing 2 over base e, we get

$$\begin{aligned}\frac{d}{dx}2^x &= \frac{d}{dx}(e^{\ln 2})^x \\ &= \frac{d}{dx}e^{x\ln 2} \\ &= \ln 2 \cdot e^{x \ln 2} \\ &= \ln 2 \cdot 2^x.\end{aligned}$$

1896. A sample $\{x_i\}$ is taken, and the sample variance s_x^2 is calculated. Afterwards, 20% of x_i values, chosen at random, are increased by a quarter. Find the expected percentage change in s_x^2.

Find a weighted average of the scale factors of the unchanged part and the changed part.

Since the x_i values to be increased are chosen at random, we can take a weighted average of the scale factors, noting that the variance is scaled by the square of the coding scale factor. Hence, the calculation is s.f. $= 1 \times 0.8 + 1.25 \times 0.2 = 1.05$. The squared scale factor is $1.05^2 = 1.1025$, so the variance scales by 10.25%.

1897. *Polar coordinates* (r, θ) are defined as the distance from O and the angle at O anticlockwise from the positive x axis.

(a) Express x and y in terms of r and θ.

(b) Express $\tan\theta$ and $\cot\theta$ in terms of x and y.

(c) Hence, by finding the Cartesian equation, show that the curve $r\sin^2\theta = \cos\theta$ is a parabola.

In (a), draw a sketch and use trigonometry. In (b), divide the equations from part (a). In (c), divide both sides by $\sin\theta$.

(a) Generalising the unit circle to any radius, $(x, y) = (r\cos\theta, r\sin\theta)$.

(b) Dividing, $\tan\theta = \frac{y}{x}$ and $\cot\theta = \frac{x}{y}$.

(c) Dividing both sides by $\sin\theta$,

$$\begin{aligned}& r\sin^2\theta = \cos\theta \\ \implies & r\sin\theta = \cot\theta \\ \implies & y = \tfrac{x}{y} \\ \implies & y^2 = x.\end{aligned}$$

This is a parabola.

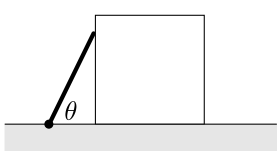
1898. Solve $\dfrac{2\ln x}{\ln(5x-4)} = 1$.

Multiply up, then use log rules to manipulate the equation into the form $\ln a = \ln b$. At that point, you can exponentiate both sides. Remember to check the validity of any roots you find.

$$\begin{aligned}& \frac{2\ln x}{\ln(5x-4)} = 1 \\ \implies & 2\ln x = \ln(5x-4) \\ \implies & \ln x^2 = \ln(5x-4) \\ \implies & x^2 = 5x - 4 \\ \implies & x = 1, 4.\end{aligned}$$

However, the root $x = 1$ gives division by $\ln 1 = 0$ in the initial equation. So, the solution is $x = 4$.

1899. A mechanical switch inside a machine is built as follows: a uniform rod of mass m is freely hinged to a horizontal surface, and leans up against a fixed block, in equilibrium. The contact between rod and block is modelled as smooth.



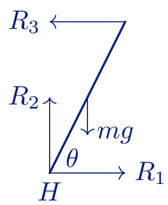
(a) Draw a force diagram for the rod.

(b) Show that, at the hinge, the rod experiences a contact force of magnitude

$$\tfrac{1}{2}mg\sqrt{\operatorname{cosec}^2\theta + 3}.$$

In (a), give the contact force on the rod at the hinge as horizontal and vertical components. Having done so, there should be four forces. In (b), use two NII equations and moments about the hinge.

(a) Expressing the contact force at the hinge in components, the forces on the rod are:



(b) Calling the length of the rod l,

$$\begin{aligned}\updownarrow &: R_2 = mg \\ \leftrightarrow &: R_1 = R_3 \\ \overset{\frown}{H} &: mgl\cos\theta - 2mgR_3 l\sin\theta = 0.\end{aligned}$$

From $\overset{\frown}{H}$, we get

$$R_3 = \tfrac{1}{2}mg\cot\theta = R_1.$$

So, the magnitude of the hinge force is

$$\begin{aligned} F &= \sqrt{R_1^2 + R_2^2} \\ &= mg\sqrt{\tfrac{1}{4}\cot^2\theta + 1} \\ &= \tfrac{1}{2}mg\sqrt{\cot^2\theta + 4} \\ &= \tfrac{1}{2}mg\sqrt{\operatorname{cosec}^2\theta + 3}. \end{aligned}$$

1900. Determine whether the curve $x = \sin t, y = \sin 2t$, for $t \in [0, \frac{\pi}{6}]$ intersects the circle $x^2 + y^2 = 1$.

Substitute the two parametric equations into the equation of the circle, and attempt to solve.

Substituting in, we have $\sin^2 t + \sin^2 2t = 1$. The double-angle formula $\sin 2t = 2\sin t\cos t$ gives

$$\begin{aligned} & \sin^2 t + 4\sin^2 t\cos^2 t = 1 \\ \Longrightarrow\ & \sin^2 t + 4\sin^2 t(1 - \sin^2 t) = 1 \\ \Longrightarrow\ & 4\sin^4 t - 5\sin^2 t + 1 = 0 \\ \Longrightarrow\ & (4\sin^2 t - 1)(\sin^2 t - 1) = 0 \\ \Longrightarrow\ & \sin t = \pm\tfrac{1}{2}, \pm 1. \end{aligned}$$

Since $t = \frac{\pi}{6}$ is a root of $\sin t = \frac{1}{2}$. This lies in $[0, \frac{\pi}{6}]$, so the curve does intersect the circle.

1901. Find the linear function g which best approximates $f(x) = x^3 - 4x$ at $x = \sqrt{8}$.

Find "the best linear approximation" is equivalent to finding the tangent line $y = mx + c$.

The best linear approximation to f at $x = \sqrt{8}$ is g such that $y = g(x)$ is tangent to $y = f(x)$. So, we differentiate to get $f'(x) = 3x^2 - 4$, which yields $f'(\sqrt{8}) = 20$. Furthermore, $f(\sqrt{8}) = 8\sqrt{2}$. Hence, the tangent line is $y - 8\sqrt{2} = 20(x - \sqrt{8})$. Simplifying, we have $g(x) = 20x - 32\sqrt{2}$.

1902. Explain why histograms are frequently set up with narrower classes around the centre of the data and wider classes in the tails.

Remember that the idea of a histogram is to try to represent the underlying population, not the particular sample you happen to have.

Statisticians want to represent the underlying population, using the sample. In distributions, there are generally more data around the centre, and fewer in the tails. Hence, it is possible to produce images with finer detail at the centre, which is achieved with narrow classes, while it is necessary to do more to mitigate sampling variation in the tails, which is achieved with broad classes.

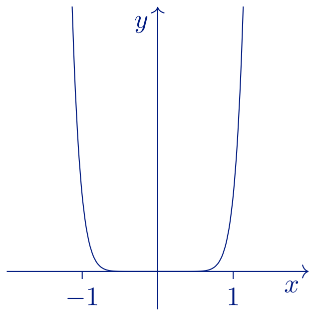
1903. Sketch the following graphs, for large $k \in \mathbb{N}$:

(a) $y = x^{2k}$,

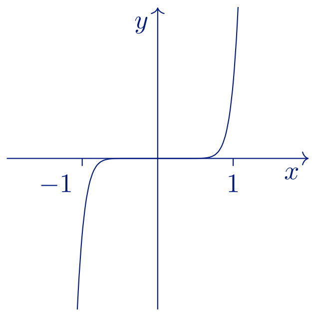
(b) $y = x^{2k+1}$.

In (a), consider the graphs $y = x^2$, $y = x^4$, etc. In (b), consider $y = x^3$, $y = x^5$, etc.

(a) The even-powered curves have a broadly parabolic shape, which becomes more and more "snub-nosed" as the degree increases. So, for large $k \in \mathbb{N}$, $y = x^{2k}$ looks like



(b) The odd-powered curves have a broadly cubic shape, which hugs the x axis more tightly on the interval $(-1, 1)$ as the degree increases. So, for large $k \in \mathbb{N}$, $y = x^{2k+1}$ looks like



1904. By integrating, determine, in the form $y = f(x)$, all solution curves satisfying the following equation, for constant k:

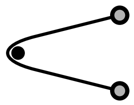
$$\frac{d^3y}{dx^3} = k.$$

At each stage of integration, introduce a constant. You will then need to integrate that constant in the next stage of integration.

Integrating, the second derivative is $kx + c_1$, the first derivative is $\frac{1}{2}kx^2 + c_1x + c_2$, and the original function is $\frac{1}{6}kx^3 + \frac{1}{2}c_1x^2 + c_2x + c_3$. Simplifying the constants, the set of solution curves is all cubics of the form

$$y = \tfrac{1}{6}kx^3 + bx^2 + cx + d.$$

1905. A catapult fires ball-bearings of mass 40 grams from a light elastic sling attached to two fixed prongs. The prongs are 10 cm apart.

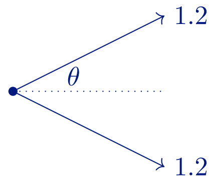


(a) Calculate the initial horizontal acceleration of the ball-bearing if it is drawn back a distance of 20 cm, generating a tension of 1.2 Newtons in the sling.

(b) State a simplifying assumption that you made to produce an answer to part (a).

Use triangle geometry to find the angle between the sling and the horizontal. Then use NII.

(a) Using triangle geometry, the two parts of the sling are at angle θ to the horizontal, where $\tan\theta = \frac{5}{20} = \frac{1}{4}$. Hence, our force diagram is



This gives NII horizontally as

$$\begin{aligned} & 2 \times 1.2\cos\theta = 0.04a \\ \implies & a = 58.2 \text{ ms}^{-2} \text{ (3sf)}. \end{aligned}$$

(b) In order to find the angle in part (a), we had to assume that the ball-bearing was a particle, i.e. that it had negligible size. We also assumed that the sling itself had negligible thickness.

1906. You are given that $\operatorname{cosec} 75° = \sqrt{6} - \sqrt{2}$. Using the Pythagorean trig identity $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$, or otherwise, show that $\tan 75° = 2 + \sqrt{3}$.

Substitute 75° into the identity, and then use the fact that $\cot 75° = \frac{1}{\tan 75°}$.

Substituting 75° into the identity, we get

$$\begin{aligned} & 1 + \cot^2 75° = \left(\sqrt{6} - \sqrt{2}\right)^2 \\ \implies & \cot^2 75° = 7 - 4\sqrt{3} \\ \implies & \tan^2 75° = \frac{1}{7 - 4\sqrt{3}}. \end{aligned}$$

We then rationalise the denominator, multiplying top and bottom by its conjugate $7 + 4\sqrt{3}$:

$$\begin{aligned} \tan^2 75° &= \frac{7 + 4\sqrt{3}}{49 - 48} \\ &= 7 + 4\sqrt{3} \\ &= \left(2 + \sqrt{3}\right)^2, \text{ proving the result.} \end{aligned}$$

1907. A set of five data is known to have mean, median and mode 0, and largest value 1. Show that the range R_x of the data must satisfy $\frac{3}{2} < R_x < 3$.

Consider the boundary cases individually, finding the greatest and least value for the lowest value.

Since the median is 0, the data are, in ascending order,

$$\{a, b, 0, c, 1\}.$$

The greatest value of a, and hence the least value of R_x, occurs when $a = b$ and $c = 0$. In this case, the mean gives $a = b = -\frac{1}{2}$ and $R_x = \frac{3}{2}$. The least value of a, and so the greatest value of R_x, occurs when $b = c = 0$, giving $a = -2$ and $R_x = 3$. But neither of these bounds is attainable, as each would have two modes, rather than simply 0 as stated in the question. Hence, the inequalities are strict, giving $\frac{3}{2} < R_x < 3$ as required.

1908. Show that $\dfrac{2^x - 1}{4^x - 1} \equiv \dfrac{1}{2^x + 1}$.

Consider the denominator of the LHS as a difference of two squares.

Writing 4^x as $(2^x)^2$, we can express the left-hand side's denominator as a difference of two squares. It factorises as $4^x - 1 = (2^x - 1)(2^x + 1)$. Dividing top and bottom by $(2^x - 1)$ yields the required result.

1909. In each case, decide which of the symbols $\implies$, $\iff$ should occupy the space.

(a) $x = y \qquad \sin x = \sin y$,

(b) $x = y \qquad \arcsin x = \arcsin y$,

(c) $x = y \qquad |x| = |y|$.

In each case, the rightwards implication obviously holds. So, work out whether there are solutions to the right-hand equations which do not have $x = y$.

In each case, the rightwards implication obviously holds. So, the question is whether the leftwards implication holds or not.

(a) $x = y \implies \sin x = \sin y$. A counterexample to the leftwards implication is $x = 0, y = \pi$.

(b) $x = y \iff \arcsin x = \arcsin y$. Being an inverse function, arcsin is automatically one-to-one.

(c) $x = y \implies |x| = |y|$. A counterexample to the leftwards implication is $x = 1, y = -1$.

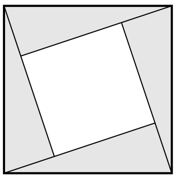
1910. Simplify $\dfrac{3x^2+5xy+2y^2}{6x^2-11xy-10y^2}$.

Factorise top and bottom. If you can't spot the factors directly, solve e.g. $3x^2+5x+2=0$ and use the factor theorem.

We can find the relevant factors either by spotting them or by solving $3x^2+5x+2=0$ and $6x^2-11x-10=0$. This gives

$$\frac{(3x+2y)(x+y)}{(3x+2y)(2x-5y)} \equiv \frac{x+y}{2x-5y}.$$

1911. The figure below consists of four congruent right-angled triangles. By calculating its area in two different ways, prove Pythagoras's theorem.



Use the standard (a,b,c) lengths, with c as the hypotenuse. Calculate the overall area as c^2, and then as the sum of the four shaded triangles and the central square.

Using (a,b,c) with c as the hypotenuse, the whole square has area c^2. This can also be expressed as the sum of four triangles and the central square, giving

$$\begin{aligned} &(b-a)^2+4\times\tfrac{1}{2}ab=c^2 \\ \Longrightarrow\;& b^2-2ab+2ab+a^2=c^2 \\ \Longrightarrow\;& a^2+b^2=c^2. \quad \text{QED.} \end{aligned}$$

1912. Prove or disprove the following statement:

$$P(A \mid B)=P(A) \iff P(A' \mid B)=P(A').$$

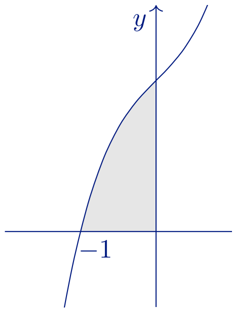
The statement is true. Use the standard formula for conditional probability.

This is true. Both the statements are expressions of the condition of independence. If information about B occurring doesn't affect the probability of A (as the left-hand statement says), then it cannot affect the probability of not-A (as the right-hand statement says). The reverse also holds.

1913. Find the area enclosed by the graph $y=x^3+x+2$ and the coordinate axes.

Solve $x^3+x+2=0$ first, then sketch the graph, then set up the relevant definite integral.

We can factorise to $y=(x+1)(x^2-x+2)=0$. The quadratic has negative discriminant, meaning that the curve has a single root at $x=-1$. Sketching, then, the required area is



So, the required integral is

$$\begin{aligned} &\int_{-1}^{0} x^3+x+2\,dx \\ =\;& \left[\tfrac{1}{4}x^4+\tfrac{1}{2}x^2+2x\right]_{-1}^{0} \\ =\;& -\left(\tfrac{1}{4}+\tfrac{1}{2}-2\right) \\ =\;& \tfrac{5}{4}. \end{aligned}$$

1914. In an attempted proof, a student writes: "If three linear equations in two unknowns have exactly one simultaneous solution, then two of the equations must be identical." Give a counterexample to show that this is not true.

Consider three concurrent lines.

Any set of three distinct, concurrent lines provides a counterexample, such as $y=x$, $y=2x$, $y=3x$. These all meet at a unique point, namely the origin.

1915. Show that $y=x^4-3x^3+3$ has a stationary point of inflection.

Find the relevant stationary point using the first derivative, and then show algebraically that the second derivative changes sign at this point.

Set the first derivative to zero to find stationary points: $4x^3-9x^2=0$. This has solution $x=0$ or $x=\frac{9}{4}$. The second derivative, then, is

$$\frac{d^2y}{dx^2}=12x^2-18x=6x(2x-3).$$

This is zero at $x=0$. Furthermore, there is only a single factor of x, meaning that $\frac{d^2y}{dx^2}$ changes sign at $x=0$. Hence, this stationary point is also a point of inflection.

1916. A_n is a quadratic sequence and B_n is a non-constant arithmetic sequence. Determine whether the following sequences are quadratic, arithmetic or neither:

(a) $u_n = A_n + B_n$,

(b) $u_n = A_n B_n$.

Consider the fact that an arithmetic sequence could also be called a linear sequence.

(a) An arithmetic sequence is linear, and adding a linear function to a quadratic function yields another quadratic function. Hence, $A_n + B_n$ is quadratic.

(b) This is neither. It is cubic, in fact, since the ordinal formulae are polynomials of degree 2 and 1, and are multiplied together.

1917. Two particles move along an x axis. They have displacements given, for $t \geq 0$, by

$$x_1 = \frac{t^2}{t^2+1}, \qquad x_2 = \frac{4t}{4t+1}.$$

(a) Verify that the particles start at the origin.

(b) Find the x location at which they next meet.

(c) What happens in the long-term?

In (a), substitute the value $t = 0$. In (b), equate the positions and solve for t. Then substitute this back into either one of the position formulae.

(a) At $t = 0$, the positions are $x_1 = x_2 = \frac{0}{1} = 0$.

(b) Equating the positions, they are in the same place when

$$\begin{aligned} & \frac{t^2}{t^2+1} = \frac{4t}{4t+1} \\ \Longrightarrow\ & t^2(4t+1) = 4t(t^2+1) \\ \Longrightarrow\ & 4t^3 + t^2 = 4t^3 + 4t \\ \Longrightarrow\ & t^2 - 4t = 0 \\ \Longrightarrow\ & t = 0, 4 \end{aligned}$$

The root $t = 0$ gives the initial positions, so we want $t = 4$. Substituting gives $x_1 = x_2 = \frac{16}{17}$.

(c) In the long-term, both formulae give positions of the form $\frac{n}{n+1}$, which tend towards 1 as n gets large. Hence, both particles approach $x = 1$ as $t \to \infty$.

1918. Determine the number of real roots of

$$(x^2 + 3x + 2)(x^4 + 3x^2 + 2) = 0.$$

Solve fully. It isn't enough to check the discriminants of each factor here, because the right-hand factor is a biquadratic.

We can't use the discriminant here, for either of two reasons: some roots might be repeated across the two brackets, and the right-hand factor is a biquadratic, which may yield non-real roots even if the relevant quadratic has non-negative discriminant. So, we proceed by solving fully.

The equation has roots when either factor is equal to zero. For the LH factor, this gives $x = -1, -2$. For the RH factor, $x^2 = -1, -2$. However, since x^2 is always positive, the biquadratic yields no real roots. Overall, therefore, the equation has 2 real roots.

1919. Find the equation of the tangent to $y = 1 + \sin^2 x$ at $x = \frac{\pi}{4}$, in the form $ay + \pi = bx + c$, for $a, b, c \in \mathbb{Z}$.

Use the chain rule to differentiate. Then, having found $\frac{dy}{dx}$ and y at $x = \frac{\pi}{4}$, substitute the values into $y - y_0 = m(x - x_0)$.

The y coordinate at $x = \frac{\pi}{4}$ is

$$y = 1 - \sin^2 \tfrac{\pi}{4} = 1 + \left(\tfrac{\sqrt{2}}{2}\right)^2 = \tfrac{3}{2}.$$

Next, using the chain rule, we get

$$\frac{dy}{dx} = 2 \sin x \cos x.$$

Evaluating this, we have $m = 1$. This gives the equation of the tangent as

$$y - \tfrac{3}{2} = 1(x - \tfrac{\pi}{4}).$$

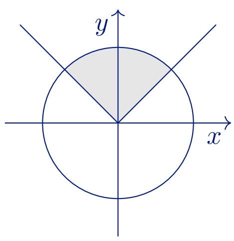
Multiplying by four and rearranging the terms, we get the form required:

$$4y + \pi = 4x + 6.$$

1920. A point (x, y) is randomly chosen inside the circle $x^2 + y^2 = 1$. Find the probability that $y > |x|$.

Consider the interior of the circle as a possibility space. Sketch it, and sketch the region satisfying the inequality.

The interior of the circle is the possibility space here, so we need to calculate the relevant areas. The successful region is



The successful region is a quarter of the possibility space, so $p = \frac{1}{4}$.

1921. Prove that every quadratic graph $y = ax^2 + bx + c$ may be generated from $y = x^2$ by a combination of stretches and translations.

Complete the square on the general quadratic, and then read off the relevant transformations from the resulting expression.

Completing the square, we have

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}. \end{aligned}$$

This is a transformation of the graph $y = x^2$: translation by vector $-\frac{b}{2a}\mathbf{i}$, followed by a stretch in the y direction with scale factor a, followed by a translation by vector $(c - \frac{b^2}{2a})\mathbf{j}$. This proves the result by construction.

1922. Prove the following trigonometric identities, where θ is defined in degrees:

(a) $\sin(90° - \theta) \equiv \cos\theta$,

(b) $\tan(90° + \theta) \equiv -\cot\theta$.

In both (a) and (b), you can use either a symmetry argument, or a compound angle formula.

(a) The definitions of the sin and cos functions as the y coordinate and x coordinates of a point on the unit circle are symmetrical in the line $y = x$, which is $\theta = 45°$. The angles θ and $(90 - \theta)$ are equidistant from $\theta = 45°$. This proves the result.

(b) On a unit circle, adding $90°$ to θ rotates by a right angle, producing a perpendicular. Hence, the new gradient $\tan(90° + \theta)$ is the negative reciprocal of $\tan\theta$. This is $-\cot\theta$, as required.

1923. Functions f and g are defined as $f(x) = 10^x$ and $g(x) = x^2 - 2x - 8$.

(a) Show that $g(x)$ is increasing on $(1, \infty)$,

(b) Write down the set of x values for which $f(g(x))$ is increasing.

In (a), find the vertex of the parabola $y = g(x)$. In (b), consider the fact that $f(x)$ is increasing everywhere.

(a) Completing the square, $g(x) = (x-1)^2 - 9$. Hence, the vertex of the parabola $y = g(x)$ is at $(1, -9)$. Since the quadratic is positive, this means $g(x)$ is increasing on $(1, \infty)$.

(b) The function $f(x) = 10^x$ is exponential growth: it is increasing everywhere. Hence, the function $f(g(x))$ is increasing exactly where $g(x)$ is increasing, i.e. $x \in (1, \infty)$.

1924. Solve the following equation for $x \in [-\pi, \pi]$:

$$\operatorname{cosec} x + \operatorname{cosec}^2 x = 2.$$

This is a quadratic in $\operatorname{cosec} x$. Rearrange to LHS = 0 and factorise. There are three roots in $[-\pi, \pi]$.

This is a quadratic in $\operatorname{cosec} x$:

$$\begin{aligned} & \operatorname{cosec}^2 x + \operatorname{cosec} x - 2 = 0 \\ \Longrightarrow\ & (\operatorname{cosec} x - 1)(\operatorname{cosec} x + 2) = 0 \\ \Longrightarrow\ & \operatorname{cosec} x = 1, -2. \\ \Longrightarrow\ & \sin x = 1, -\tfrac{1}{2}. \end{aligned}$$

There are three roots in total: $x = \frac{\pi}{2}$ from the first equation and $x = -\frac{\pi}{6}, -\frac{5\pi}{6}$ from the second.

1925. Show that the curves $x^2 + 3x + y^2 + 5x = 10$ and $x^2 + y^2 = 1$ do not intersect.

These are two circles. Complete the square and show that the unit circle lies entirely within the other circle.

These are circles. The latter is the unit circle, and the former is

$$\left(x + \tfrac{3}{2}\right)^2 + \left(y + \tfrac{5}{2}\right)^2 = \tfrac{37}{2}.$$

This is centred on $\left(-\frac{3}{2}, -\frac{5}{2}\right)$, which is a distance of $\frac{\sqrt{34}}{2}$ from the origin. Therefore, since

$$\tfrac{\sqrt{34}}{2} = 2.91... < \sqrt{\tfrac{37}{2}} = 4.30...,$$

the origin lies inside the first circle. Furthermore, since the difference is greater than the radius of the unit circle, as in

$$1 + \tfrac{\sqrt{34}}{2} = 3.91... < \sqrt{\tfrac{37}{2}} = 4.30...,$$

the unit circle lies entirely within the first circle. Hence, they do not intersect.

1926. If $\dfrac{d}{dx}\left(x^3 + \dfrac{dy}{dx}\right) = 1$, find $\dfrac{d^2y}{dx^2}$ in terms of x.

Apply the differential operator $\frac{d}{dx}$ to both terms in the bracket. Then rearrange.

We can distribute the differential operator $\frac{d}{dx}$ over the bracket:

$$\begin{aligned} &\frac{d}{dx}\left(x^3+\frac{dy}{dx}\right)=1 \\ \implies\ & 3x^2+\frac{d^2y}{dx^2}=1 \\ \implies\ & \frac{d^2y}{dx^2}=1-3x^2. \end{aligned}$$

1927. On a 2×3 grid, three red counters and three green counters are placed at random.

(a) Show that the probability that the two colours end up in distinct rows is 10%.

(b) Find the probability that no two same-coloured counters are adjacent to each other.

Consider the possibility space as the set of possible locations of the red counters. Hence, use ${}^nC_r$.

Ignoring the green counters, we are choosing three locations for the red counters, giving a possibility space of ${}^6C_3=20$ equally likely outcomes.

(a) There are two outcomes in which the reds (and therefore greens) are in a single row, which gives a probability of $\frac{2}{20}=10\%$.

(b) There are also two outcomes in which the colours are in a chequerboard pattern, so the probability is again 10%.

1928. "The line $y=0$ is a tangent to $y=x^4-x^2-12$." Is this statement true or false?

Differentiate and find the turning points, or else factorise the biquadratic and consider the multiplicity of the roots.

This is false. Factorising the biquadratic, we get

$$y=(x^2-4)(x^2+3)=(x-2)(x+2)(x^2+3).$$

This has no repeated factors, which rules out the possibility that $y=0$ is tangent to the curve.

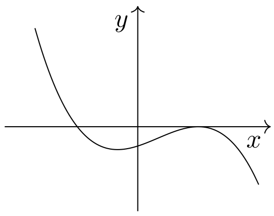
1929. Let the constants a,b,c,d be distinct and non-zero, and let k be a natural number. Find any x values for which the following function is undefined:

$$x\mapsto\frac{(x^{2k}-a^{2k})(x^{2k}+b^{2k})}{(x^{2k}-c^{2k})(x^{2k}+d^{2k})}.$$

Consider the values for which the denominator is zero, recognising that the index $2k$ is even.

The function is undefined when the denominator is zero. Since the index $2k$ is even and d is non-zero, this can only occur when $x^{2k}=c^{2k}$, which gives $x=\pm c$.

1930. A cubic graph $y=f(x)$ is shown below.



State, with a reason, whether the following are true descriptions of the function f:

(a) "one root of f is a single root",

(b) "one root of f is a double root",

(c) "one root of f is a triple root",

(d) "one root of f is a repeated root".

There is a sign change at a single and a triple root, but not at a double root. Both double roots and triple roots are repeated roots, where the x axis is a tangent.

(a) This is true. At the root on the left, the graph is not tangential to the x axis. This root must, therefore, be a single root.

(b) This is true. Since the graph is cubic and the root on the left is a single root, the root on the right, at which there is no sign change, must be a double root.

(c) This is false.

(d) This is true. By definition, a double root is a repeated root.

1931. Give a set of integers that is a counterexample to the claim: "The interquartile range is always smaller than the range."

It is true that the IQR cannot exceed the range. So, find a counterexample in which the IQR is equal to the range.

A trivial counterexample is $\{0\}$, for which IQR and range are zero. A non-trivial counterexample is $\{0,0,0,1,1,1\}$, for which both are 1.

1932. The equation of a straight line, gradient m, passing through the point (a,b) is

$$\frac{y-b}{x-a}=m.$$

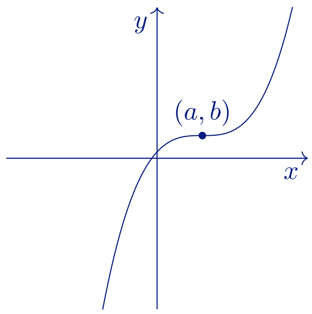
Sketch the following graphs:

(a) $\dfrac{y-b}{(x-a)^3}=m$ (b) $\dfrac{(y-b)^3}{x-a}=m.$

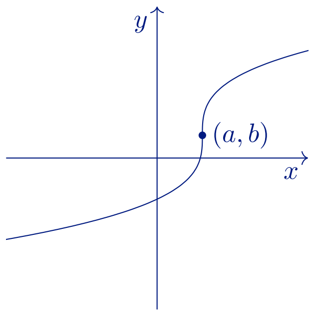
Consider each graph as a cubic centred on the point (a, b).

Each graph is an elementary cubic of the general shape of $y = x^3$, with its point of inflection at (a, b). The constant m determines the steepness of the cubic; this is drawn as $m = 1$ below.

(a) $\dfrac{y - b}{(x - a)^3} = m$



(b) $\dfrac{(y - b)^3}{x - a} = m$



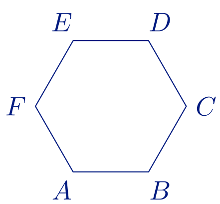
1933. Two of the sides of a regular hexagon $ABCDEF$ are given by vectors $\overrightarrow{AB} = 2\mathbf{i}$ and $\overrightarrow{BC} = \mathbf{i} + \sqrt{3}\mathbf{j}$. Find the following vectors:

(a) $\overrightarrow{CD}$,

(b) $\overrightarrow{AD}$.

Sketch the hexagon, and note that opposite sides have the same vector.

Sketching, the hexagon is



(a) Since $\overrightarrow{AB}$ has no $\mathbf{j}$ component, $\overrightarrow{CD}$ is the same as $\overrightarrow{BC}$ with the $\mathbf{i}$ component negated. This gives $\overrightarrow{CD} = -\mathbf{i} + \sqrt{3}\mathbf{j}$.

(b) Adding top-to-tail,

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \\ &= 2\mathbf{i} + (\mathbf{i} + \sqrt{3}\mathbf{j}) + (-\mathbf{i} + \sqrt{3}\mathbf{j}) \\ &= 2\mathbf{i} + 2\sqrt{3}\mathbf{j}.\end{aligned}$$

1934. If $y = 9^x$, write $(3^x + 1)^2(3^{-x} - 1)^2$ in terms of y.

Multiply the two brackets together first, leaving the squares until afterwards. Then, when squaring, use $(3^x)^2 = 3^{2x} = (3^2)^x = 9^x$.

Multiplying the brackets together before squaring,

$$\begin{aligned}&(3^x + 1)^2(3^{-x} - 1)^2 \\ \equiv\ &(1 + 3^x + 3^{-x} - 1)^2 \\ \equiv\ &(3^x + 3^{-x})^2 \\ \equiv\ &9^x + 2 + 9^{-x} \\ \equiv\ &y + 2 + \frac{1}{y}.\end{aligned}$$

1935. Functions f and g have domains and codomains

$$\begin{aligned}f &: A \mapsto B, \\ g &: C \mapsto D.\end{aligned}$$

State, with a reason, what must be true of sets A, B, C, D if f and g are inverse functions.

Since inverse functions, in order to be invertible, must be one-to-one, the domain of each function must be the same set as the codomain of the other.

Since inverse functions, in order to be invertible, must be one-to-one, the domain of each function must be the same set as the codomain of the other. Hence $A = D$ and $B = C$.

1936. Show that the graph of $y = 2^x$ may be transformed into that of $y = 3^x$ by a stretch in the x direction.

A stretch in the x direction is a replacement of x with kx. Hence, write 3^x as 2^{kx}.

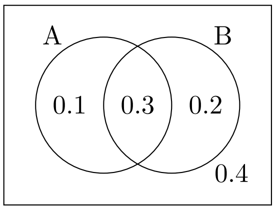
We can rewrite $y = 3^x$ over base 2. This produces $y = 3^x = (2^{\log_2 3})^x = 2^{x \log_2 3}$. Comparing this to $y = 2^x$, we have replaced x by $x \log_2 3$. This is a stretch in the x direction, with scale factor $\frac{1}{\log_2 3} = \log_3 2$.

1937. Prove algebraically that, if a geometric iteration $u_{n+1} = ru_n$ has a fixed point x, then at least one of $x = 0$ or $r = 1$ must hold.

A fixed point satisfies $x = f(x)$. Set this equation up, and solve it.

Since x is a fixed point of the iteration, we know that $rx = x$. This gives $rx - x = 0$, which we can factorise to $x(r-1) = 0$. Hence, $x = 0$ or $r = 1$.

1938. A Venn diagram contains probabilities as below.



Solve for k in the equation $P(A \mid B) = kP(B \mid A)$.

1939. Disprove the following claim: "If $f(a) < 0 < f(b)$, then there must be a root of f between a and b."

1940. Give, by sketching, two different reasons why the Newton-Raphson method may fail to converge to a given root, despite starting close to it.

1941. State, with a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning real numbers x and y:

- $|x| = |y|$,
- $x^2 = y^2$.

1942. You are given that two functions f and g satisfy $f(x) = |g(x)|$ for all $x \in \mathbb{R}$. State, with a reason, whether the following statements are true or false:

(a) $f'(x) = |g'(x)|$ for all $x \in \mathbb{R}$,

(b) $\displaystyle\int_a^b f(x)\,dx = \left|\int_a^b g(x)\,dx\right|$ for all $x \in \mathbb{R}$.

1943. A sequence has ordinal definition $u_n = n^2 + n + 3$. Prove that $u_{n+1} < 2u_n$, for all $n \in \mathbb{N}$.

1944. Write down the equations of the reflections of the following graphs in the line $x = k$:

(a) $y = x - k$,

(b) $y = (x - k)^2$,

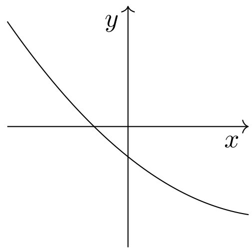
(c) $y = (x - k)^3$.

1945. Explain why one of the following expressions is well-defined and the other is not:

$$\left.\frac{t^3 - a^3}{t^2 - a^2}\right|_{t=a} \qquad \lim_{t \to a} \frac{t^3 - a^3}{t^2 - a^2}$$

1946. A dodecahedron has 12 faces, each of which is a pentagon. Two distinct faces of a dodecahedron are selected at random. Determine the probability that the two faces share a common edge.

1947. A section of the graph $y = f(x)$ is as shown:



State whether, over the domain shown, the terms "increasing", "decreasing", "convex", "concave" can be used to describe the curve.

1948. Show that the sum of the first 100 natural numbers which are not divisible by 5 is 6250.

1949. Given that $x = 3$ is a root, solve

$$\frac{4x}{x+1} - \frac{4x^2}{x-1} + 15 = 0.$$

1950. A gutter is a half-cylinder of radius r. Show that, when water stands to a depth of $\frac{1}{2}r$, it occupies $\frac{3\sqrt{3}}{4\pi}$ of the volume of the gutter.

1951. Write the following in terms of e^x:

(a) e^{3x},

(b) e^{3x-1},

(c) e^{3x-2}.

1952. Prove that $\displaystyle\lim_{x \to \infty} \frac{2x+1}{2x-1} = 1$.

1953. A student says: "The graph $y = 3 - 2|x|$ stays above the x axis, because the modulus function makes everything positive." Explain the error, and sketch the correct graph.

1954. A pair of perpendicular vectors are given as

$$\mathbf{a} = \begin{pmatrix} x \\ 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} x+3 \\ -2 \end{pmatrix}.$$

Find the possible values of x.

1955. The variable Y has a normal distribution. You are given that $P(Y < 0) = P(Y > 4) = 0.2$.

(a) Write down the mean.
(b) Find the standard deviation.

1956. A question is written as follows: "The inequality $ax^2 + bx + c > 0$ has solution set $(\infty, 4) \cup (5, \infty)$. Find the constants a, b, c."

(a) Show that it is not possible to determine a, b, c with this information.
(b) State the possible values of a.

1957. Sketch $4 \geq (x-2)^2 + (y-3)^2 \geq 9$.

1958. A $(3, 4, 5)$ triangle undergoes an enlargement: the perpendicular sides lengthen continuously, at a rate of 1 unit per second. Find the rate at which the hypotenuse is lengthening when the area of the triangle is 15 square units.

1959. $F(x) = x^3 - 12x$ is not invertible over $\mathbb{R}$.

(a) Explain why not.
(b) By considering stationary points, determine the largest $a > 0$ such that F is invertible over the domain $[-a, a]$.
(c) Give the codomain for this version of F.

1960. Find the values $x \in [0, 2\pi)$ for which the function $x \mapsto \sin x + \cos^2 x$ has a local maximum.

1961. The variables Z_1 and Z_2 are independent, and each follows the same normal distribution $Z \sim N(0, 1)$. Find the following probabilities:

(a) $P(Z_1, Z_2 > 0)$,
(b) $P(Z_1 < Z_2)$,
(c) $P(0 < Z_1 < Z_2)$.

1962. Using the generalised binomial expansion, find a quadratic approximation to $(1 + 4x)^{-1}$, giving the domain of validity of the expansion.

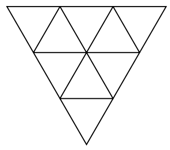
1963. By taking out a factor of $(3x + 2)$, or otherwise, solve the equation $(3x - 2)^3 + 8 = 12x$.

1964. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $\sin^2\theta = \frac{3}{4} \implies \cos\theta = \frac{1}{2}$,
(b) $\sin^2\theta = \frac{3}{4} \impliedby \cos\theta = \frac{1}{2}$.

1965. Give the formula for the speed of the end-point of a radius of length r sweeping out an angle of ω radians per second.

1966. A pattern consisting of nine small triangles is as depicted below.



Determine the number of ways of colouring the pattern if:

(a) two colours are used, and two small triangles sharing an edge cannot be the same colour,
(b) three colours are used, with no restrictions.

1967. Solve the inequality $x^2 - x + 6 > 0$, giving your answer in set notation.

1968. A general recurring decimal is $x = 0.\dot{a}_1 a_2 ... \dot{a}_n$, where $a_1, a_2, ..., a_n$ represent digits. Prove that

$$x = \frac{a_1 a_2 ... a_n}{\underbrace{99...9}_{n \text{ 9's}}}.$$

1969. Show that $y = x^2 - x$ and $x = y^2 - y$ are tangent.

1970. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$|x - 2| < 1, \qquad |y - 3| < 1.$$

1971. A particle moves in one dimension with velocity given by $v = 3t^2 + 2$ for $t \in [0, 4]$. Find the time at which the particle's instantaneous velocity is equal to its average velocity over the time period.

1972. Find the coefficient of x^2 in the expansion of $(1 + 3x + 2x^2)^5$.

1973. Solve, for $x, y \in [0, 360°)$,

$$\begin{aligned} 2\sin x + \cos y &= 3 \\ \sin x - 4\cos y &= 5. \end{aligned}$$

1974. A statistician models the maximum running speed of adult humans with a normal distribution. Give two reasons why this distribution will not give an accurate picture of reality.

1975. Prove that, for $a \neq b$,

$$\frac{a}{b} = \frac{c}{d} \implies \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

1976. A differential equation has led to

$$\int \frac{1}{x^2}\,dx = \int \frac{1}{y^2}\,dy.$$

Show that the solution curves are $y = \dfrac{x}{kx+1}$.

1977. Disprove the following: "A parabola and a cubic must always intersect." (You will need to consider curves that are not of the form $y = f(x)$.)

1978. Consider the function $f(x) = 4 \times 4^x - 5 \times 2^x + 1$, defined over $\mathbb{R}$.

(a) Show that the range is $\{y \in \mathbb{R} : y \geq -9/16\}$.

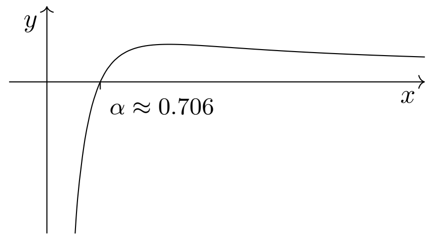
(b) Show that the roots of $f(x)$ differ by 2.

1979. Two concentric circles have the same centre, and radii of lengths 9 cm and 15 cm. A chord, which is tangent to the smaller circle, is drawn inside the larger circle. Determine the length of the chord.

1980. If $\dfrac{d}{dx}\sqrt{x+y} = x$, find $\dfrac{dy}{dx}$ in terms of x and y.

1981. Show that the percentage error in the evaluation of the area enclosed by the curves $y = \sqrt{x}$ and $y = x$, when it is approximated with the trapezium rule using four strips, is approximately 14%.

1982. The graph below shows $y = f(x) = e^{-x} + x^{-1}\ln x$. The equation $f(x) = 0$ has one real root at α, and the graph has one stationary point.



(a) Find $f'\left(\frac{1}{2}(1+\sqrt{5})\right)$.

(b) Hence, verify that, for starting values $x_0 \geq \frac{1}{2}(1+\sqrt{5})$, the Newton-Raphson method will fail to converge to α.

Consider the fact that the gradient in (a) is small and *negative*.

(a) Differentiating by the chain and quotient rules,

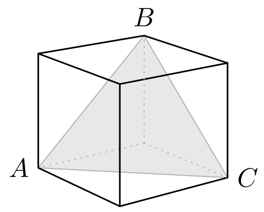
$$f'(x) = -e^{-x} + \frac{\frac{1}{x}\cdot x - \ln x}{x^2}.$$

This gives $f'(\frac{1}{2}(1+\sqrt{5})) = -0.0001287 < 0$.

(b) The gradient is negative, so $x_0 = \frac{1}{2}(1+\sqrt{5})$ is to the right of the stationary point. The tangent at x_0 will therefore cross the x axis at $x_1 > x_0$. Hence, if $x_0 \geq \frac{1}{2}(1+\sqrt{5})$, the N-R iteration will diverge $x_n \to +\infty$, failing to find the root.

1983. Prove that $\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$.

1984. The diagram shows a cube of unit side length.



Find the area of triangle ABC.

1985. If $f(x) = \dfrac{1}{1+x}$, prove that $f^3(x) = \dfrac{x+2}{2x+3}$.

1986. In a game of cards, player A and player B swap a card at random. Before the swap, A has two Jacks and a Queen, and B has two Queens and a King. Find the probability that, after the swap:

(a) A has a pair,

(b) B has at least a pair.

1987. Using a suitable substitution, show that

$$\int_0^1 6x\sin(x^2+4)\,dx = 3\int_4^5 \sin u\,du.$$

1988. You are given that a function g and its derivative g' are related in the following way:

$$2g'(x)(g(x) - 1) = 1.$$

Verify that the function $g(x) = \sqrt{x} + 1$ satisfies this relationship.

1989. Prove the following implication statement:

$$\big(|x| - |y|\big)\big(|x| + |y|\big) = 0 \iff x^2 = y^2.$$

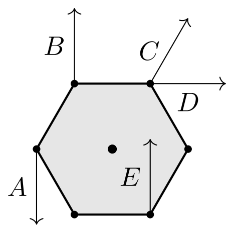
1990. Ramanujan produced the following extraordinary infinite series approximation to $\frac{1}{\pi}$:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}.$$

Using the fact that 0! is defined to be 1, show that the first term of this series gives an approximation of π which is correct to six decimal places.

1991. Using integration, find the average value of the function $f(x) = \sqrt{x}$ on the domain $[0, 81]$.

1992. A merry-go-round is a regular hexagon, which may rotate freely about its centre in a horizontal plane. Five children exert forces as depicted below, all of which have the same magnitude and are parallel or perpendicular to sides of the hexagon.



Determine, showing your reasoning carefully, the direction in which the merry-go-round will rotate.

1993. A variable has distribution $X \sim B(5, p)$. It is given that $P(X < 1 \mid X < 2) = \frac{1}{6}$. Find p.

1994. Determine whether the curve $y = x^3 - 4x^2 + 2x - 1$ is concave, convex or neither as it passes through the point $(3, -4)$.

Evaluate the second derivative at $x = 3$.

Differentiating gives

$$\frac{dy}{dx} = 4x^2 - 8x + 2$$
$$\implies \frac{d^2y}{dx^2} = 8x - 8.$$

At $x = 3$, the second derivative is $16 > 0$. Hence, the curve is convex at this point.

1995. Two sequences are defined ordinally by

$$a_n = 20n - n^2,$$
$$b_n = 500 - 40n + n^2.$$

Show that $|a_n - b_n|$ is minimised at $n = 15$.

Solve to find the vertex of the parabola $y = a_n - b_n$.

The function $a_n - b_n = -500 + 60n - 2n^2$ is a negative quadratic, and has a maximum at $(15, -50)$. Hence, the minimum of $|a_n - b_n|$ is at $(15, 50)$.